

## KAEHLER SUBMANIFOLDS OF THE REAL HYPERBOLIC SPACE

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**Abstract** The local classification of Kaehler submanifolds  $M^{2n}$  of the hyperbolic space  $\mathbb{H}^{2n+p}$  with low codimension  $2 \leq p \leq n-1$  under only intrinsic assumptions remains a wide open problem. The situation is quite different for submanifolds in the round sphere  $\mathbb{S}^{2n+p}$ ,  $2 \leq p \leq n-1$ , since Florit *et al.* [7] have shown that the codimension has to be  $p = n-1$  and then that any submanifold is just part of an extrinsic product of two-dimensional umbilical spheres in  $\mathbb{S}^{3n-1} \subset \mathbb{R}^{3n}$ . The main result of this paper is a version for Kaehler manifolds isometrically immersed into the hyperbolic ambient space of the result in [7] for spherical submanifolds. Besides, we generalize several results obtained by Dajczer and Vlachos [5].

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The study of isometric immersions of Kaehler manifolds  $(M^{2n}, J)$ ,  $n \geq 2$ , into spheres  $\mathbb{S}^{2n+p}$  and hyperbolic spaces  $\mathbb{H}^{2n+p}$  with low codimension  $p$  was initiated by Ryan [8]. He showed that for hypersurfaces there is only  $M^4 = \mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{S}^5 \subset \mathbb{R}^6$  in the sphere and that in the hyperbolic space there is  $M^4 = \mathbb{H}^2 \times \mathbb{S}^2 \subset \mathbb{H}^5 \subset \mathbb{L}^6$  besides the even dimensional horospheres. Later on, Dajczer-Rodríguez [3] proved that, regardless of the codimension, such isometric immersions do not occur if we require minimality.

The possibilities for Kaehler submanifolds with low codimension in spheres are rather restricted. In fact, it was shown by Florit–Hui–Zheng [7] that if we have an isometric immersion into the unit sphere  $f: M^{2n} \rightarrow \mathbb{S}_1^{2n+p}$  with  $p \leq n-1$ , then  $p = n-1$  and  $f(M) \subset \mathbb{S}_1^{3n-1} \subset \mathbb{R}^{3n} = \mathbb{R}^3 \times \dots \times \mathbb{R}^3$  is an open subset of a Riemannian product of umbilical spheres  $\{\mathbb{S}_{c_j}^2\}_{1 \leq j \leq n}$  in  $\mathbb{R}^3$  such that  $1/c_1 + \dots + 1/c_n = 1$ .

In [7], it was observed that for submanifolds in hyperbolic space, a similar result as theirs is not possible due to the presence of the horospheres. In fact, there are the compositions  $f = j \circ g$ , where  $j: \mathbb{R}^{2n+p-1} \rightarrow \mathbb{H}^{2n+p}$  is a horosphere and  $g: M^{2n} \rightarrow \mathbb{R}^{2n+p-1}$

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an isometric immersion. In that respect, we observe that for any codimension, there is an abundance of non-holomorphic Kaehler submanifolds, a class intensively studied in the last 25 years with emphasis on the ones that are minimal. For an account of the basic facts on the subject of real Kaehler submanifolds, we refer to Chapter 15 in [4] and the references listed in [5].

The main goal of this paper is to locally characterize the submanifolds described next.

**Example 1.** Let the Kaehler manifold  $M^{2n}$  be the Riemannian product of a hyperbolic plane and a set of two-dimensional round spheres given by

$$M^{2n} = \mathbb{H}_{c_1}^2 \times \mathbb{S}_{c_2}^2 \times \cdots \times \mathbb{S}_{c_n}^2 \quad \text{with } 1/c_1 + \cdots + 1/c_n = -1.$$

Then let  $f: M^{2n} \rightarrow \mathbb{H}_{-1}^{3n-1}$  be the submanifold defined by  $g = i \circ f: M^{2n} \rightarrow \mathbb{L}^{3n}$ , where  $i: \mathbb{H}^{3n-1} \rightarrow \mathbb{L}^{3n}$  is the inclusion into the flat Lorentzian space  $\mathbb{L}^{3n} = \mathbb{L}^3 \times \mathbb{R}^3 \times \cdots \times \mathbb{R}^3$  and  $g = g_1 \times g_2 \times \cdots \times g_n$  is the extrinsic product of umbilical surfaces  $g_1: \mathbb{H}_{c_1}^2 \rightarrow \mathbb{L}^3$  and  $g_j: \mathbb{S}_{c_j}^2 \rightarrow \mathbb{R}^3, 2 \leq j \leq n$ .

To make our goal feasible, it is necessary to remove the possibility for the submanifold to lay inside a horosphere. Such a task is fulfilled here inspired by the following sharp estimate given as Corollary 15.6 in [4].

*Let  $f: M^{2n} \rightarrow \mathbb{R}^{2n+p}, 1 \leq p \leq n - 1$ , be an isometric immersion of a Kaehler manifold. At any  $x \in M^{2n}$ , there is a complex vector subspace  $L^{2\ell} \subset T_x M$  with  $\ell \geq n - p$  such that the sectional curvature of  $M^{2n}$  satisfies  $K(Z, JZ) \leq 0$  for any  $Z \in L^{2\ell}$ .*

The following is the main result of this paper.

**Theorem 2.** *Let  $f: M^{2n} \rightarrow \mathbb{H}_{-1}^{2n+p}, p \leq n - 1$ , be an isometric immersion of a Kaehler manifold. Assume that at some point  $x_0 \in M^{2n}$ , there is a complex vector subspace  $V^{2m} \subset T_{x_0} M$  with  $m \geq p$  such that the sectional curvature of  $M^{2n}$  satisfies  $K(S, JS) > 0$  for any  $0 \neq S \in V^{2m}$ . Then  $p = n - 1$  and  $f(M)$  is an open subset of the submanifold given by Example 1.*

The remaining of this paper is devoted to generalize several results by Dajczer and Vlachos [5]. The following result improves their Theorem 7 that deals with the size of the dimension of the complex subspaces where the holomorphic sectional curvature is non-positive. In addition, we establish an estimate for the Ricci curvature. Moreover, in both cases, the estimates now obtained are sharp.

**Theorem 3.** *Let  $f: M^{2n} \rightarrow \mathbb{H}_{-1}^{2n+p}, 1 \leq p \leq n - 2$ , be an isometric immersion of a Kaehler manifold. At any point  $x \in M^{2n}$ , there is a complex vector subspace  $V^{2\ell} \subset T_x M$  with  $\ell \geq n - p + 1$  such that for any  $S \in V^{2\ell}$ , the sectional curvature of  $M^{2n}$  satisfies  $K(S, JS) \leq 0$  and the Ricci curvature that  $Ric(S) \leq 0$ .*

For  $p = 1$ , the above estimates follow trivially from the aforementioned result due to Ryan. For codimension  $p = 2$ , they are a consequence of Theorem 1 in [5] and, as is the case for  $p = 1$ , with the stronger assertions  $K(S, JS) = 0 = Ric(S)$ . It is shown in [5] that locally  $f = j \circ g$  is a composition as given above. To reach that conclusion, one has to use the non-flat Kaehler hypersurfaces in Euclidean space, whose classification can be seen in [4] as Theorem 15.14, which have only two non-zero simple principal curvatures. In particular, observe that when the hypersurface has a plane of positive sectional curvature, these examples show that Theorem 3 is sharp already for  $p = 2$ .

From the two results above, we obtain the following generalization of Theorem 3 in [5] given there for codimension  $p \leq n - 2$ .

**Corollary 4.** *A Kaehler manifold  $M^{2n}$ ,  $n \geq 2$ , that at some point possesses positive holomorphic sectional curvature cannot be isometrically immersed in  $\mathbb{H}^{3n-1}$ .*

The next result was obtained in [5] for  $p \leq n - 2$  under the weaker hypothesis that the Omori-Yau weak maximum principle for the Hessian holds on  $M^{2n}$ . Under the assumptions that the Riemannian manifold is complete with sectional curvature bounded from below, we have from Theorem 2.3 in [1] that the Omori-Yau maximum principle for the Hessian holds.

**Theorem 5.** *Let  $f: M^{2n} \rightarrow \mathbb{H}^{3n-1}$  be an isometric immersion of a complete Kaehler manifold with sectional curvature bounded from below. Then  $f(M)$  is unbounded.*

Finally, we consider the case of submanifolds with codimension two.

**Theorem 6.** *Let  $f: M^{2n} \rightarrow \mathbb{H}^{2n+2}$ ,  $n \geq 3$ , be an isometric immersion of a Kaehler manifold which does not contain an open subset of flat points. Then either  $n = 3$  and  $M^6 \subset \mathbb{H}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{H}^8 \subset \mathbb{L}^9$  or there is a composition of isometric immersion  $f = j \circ g$ , where  $g: M^{2n} \rightarrow \mathbb{R}^{2n+1}$  is a real Kaehler hypersurface and  $j: \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^{2n+2}$  is a horosphere.*

In addition, the result extends Theorem 1 in [5] by including the case  $n = 3$ . Moreover, it generalizes this result as well as Theorem 6 in [5] since now it is global and, with respect to the latter, it does not require to assume flat normal bundle.

## 1. Several algebraic considerations

### 1.1. Some general facts

Let  $W^{p,p}$  denote a real vector space of dimension  $2p$  endowed with an indefinite inner product of signature  $(p, p)$ . Thus,  $p$  is the maximal dimension of a vector subspace such that the induced inner product is either positive or negative definite. A vector subspace  $L \subset W^{p,p}$  is called *degenerate* if  $L \cap L^\perp \neq 0$  and *nondegenerate* if otherwise. Moreover, a vector subspace  $L \neq 0$  is called *isotropic* if it satisfies  $L = L \cap L^\perp$ .

The following result is Sublemma 2.3 in [6] and Corollary 4.3 in [4].

**Proposition 7.** *Given a vector subspace  $L \subset W^{p,p}$ , there is a direct sum decomposition  $W^{p,p} = \mathcal{U} \oplus \hat{\mathcal{U}} \oplus \mathcal{V}$ , where  $\mathcal{U} = L \cap L^\perp$  such that the vector subspace  $\hat{\mathcal{U}}$  is isotropic, the vector subspace  $\mathcal{V} = (\mathcal{U} \oplus \hat{\mathcal{U}})^\perp$  is non-degenerate and  $L \subset \mathcal{U} \oplus \mathcal{V}$ .*

Let  $V$  be a finite-dimensional real vector space and  $\varphi: V \times V \rightarrow W^{p,p}$  a bilinear form. Then  $\varphi$  is called a *flat bilinear form* if

$$\langle \varphi(X, Y), \varphi(Z, T) \rangle - \langle \varphi(X, T), \varphi(Z, Y) \rangle = 0$$

for all  $X, Z, Y, T \in V$ . We denote the vector subspace generated by  $\varphi$  by

$$\mathcal{S}(\varphi) = \text{span} \{ \varphi(X, Y) : X, Y \in V \}$$

and say that  $\varphi$  is *surjective* if  $\mathcal{S}(\varphi) = W^{p,p}$ . The (right) kernel  $\varphi$  is defined by

$$\mathcal{N}(\varphi) = \{Y \in V : \varphi(X, Y) = 0 \text{ for all } X \in V\}.$$

If  $V_1, V_2 \subset V$  are vector subspaces, we denote

$$\mathcal{S}(\varphi|_{V_1 \times V_2}) = \text{span}\{\varphi(X, Y) : \text{for all } X \in V_1 \text{ and } Y \in V_2\}.$$

A vector  $X \in V$  is called a (left) *regular element* of  $\varphi$  if  $\dim \varphi_X(V) = r$ , where

$$r = \max\{\dim \varphi_X(V) : X \in V\}$$

and  $\varphi_X : V \rightarrow W^{p,p}$  is the linear transformation defined by

$$\varphi_X Y = \varphi(X, Y).$$

The set  $\text{RE}(\varphi)$  of regular elements of  $\varphi$  is easily seen to be an open dense subset of  $V$ ; for instance, see Proposition 4.4 in [4].

**Proposition 8.** *Let  $\varphi : V \times V \rightarrow W^{p,p}$  be a flat bilinear form. If  $X \in \text{RE}(\varphi)$ , then*

$$\mathcal{S}(\varphi|_{V \times \ker \varphi_X}) \subset \varphi_X(V) \cap (\varphi_X(V))^\perp. \tag{1}$$

**Proof.** See Sublemma 2.4 in [6] or Proposition 4.6 in [4]. □

### 1.2. A general bilinear form

Let  $V^{2n}$  and  $\mathbb{L}^p$ ,  $p \geq 2$ , be real vector spaces such that there is  $J \in \text{Aut}(V)$ , which satisfies  $J^2 = -I$ , and  $\mathbb{L}^p$  is endowed with a Lorentzian inner product  $\langle \cdot, \cdot \rangle$ . Then let  $W^{p,p} = \mathbb{L}^p \oplus \mathbb{L}^p$  be endowed with the inner product of signature  $(p, p)$  defined by

$$\langle\langle (\xi, \bar{\xi}), (\eta, \bar{\eta}) \rangle\rangle = \langle \xi, \eta \rangle - \langle \bar{\xi}, \bar{\eta} \rangle. \tag{2}$$

Let  $\alpha : V^{2n} \times V^{2n} \rightarrow \mathbb{L}^p$  be a symmetric bilinear form, and then let the bilinear form  $\beta : V^{2n} \times V^{2n} \rightarrow W^{p,p}$  be defined by

$$\beta(X, Y) = (\alpha(X, Y) + \alpha(JX, JY), \alpha(X, JY) - \alpha(JX, Y)). \tag{3}$$

Then  $\beta$  satisfies

$$\beta(X, JY) = -\beta(JX, Y) \tag{4}$$

for any  $X, Y \in V^{2n}$ . Notice that  $\beta(X, X) = (\zeta, 0)$  for  $X \in V^{2n}$  and  $\zeta \in \mathbb{L}^p$ . Moreover, if  $\beta(X, Y) = (\xi, \eta)$ , then we have

$$\beta(X, JY) = (\eta, -\xi), \quad \beta(Y, X) = (\xi, -\eta) \quad \text{and} \quad \beta(JY, X) = (\eta, \xi). \tag{5}$$

In particular, it follows that  $\mathcal{N}(\beta)$  is a  $J$ -invariant vector subspace of  $V^{2n}$ .

In the sequel,  $U_0^s \subset \mathbb{L}^p$  is the  $s$ -dimensional vector subspace  $U_0^s = \pi_1(\mathcal{S}(\beta))$ , where  $\pi_1: W^{p,p} \rightarrow \mathbb{L}^p$  denotes the projection onto the first component of  $W^{p,p}$ . Hence,

$$U_0^s = \text{span}\{\alpha(X, Y) + \alpha(JX, JY): X, Y \in V^{2n}\}.$$

**Proposition 9.** *The bilinear form  $\beta: V^{2n} \times V^{2n} \rightarrow W^{p,p}$  satisfies*

$$\mathcal{S}(\beta) = U_0^s \oplus U_0^s. \tag{6}$$

Moreover, if the vector subspace  $\mathcal{S}(\beta)$  is degenerate, then  $1 \leq s \leq p - 1$ , and there is a non-zero light-like vector  $v \in U_0^s$  such that

$$\mathcal{S}(\beta) \cap (\mathcal{S}(\beta))^\perp = \text{span}\{v\} \oplus \text{span}\{v\}. \tag{7}$$

**Proof.** If  $\sum_{j=1}^k \beta(X_j, Y_j) = (\xi, \eta)$ , we obtain from Equation (5) that

$$\sum_{j=1}^k \beta(Y_j, X_j) = (\xi, -\eta), \quad \sum_{j=1}^k \beta(X_j, JY_j) = (\eta, -\xi) \quad \text{and} \quad \sum_{j=1}^k \beta(JY_j, X_j) = (\eta, \xi).$$

Hence,  $(\xi, 0), (0, \xi), (\eta, 0) \in \mathcal{S}(\beta)$  and thus  $\mathcal{S}(\beta) \subset U_0^s \oplus U_0^s$ . If  $(\xi, \eta) \in U_0^s \oplus U_0^s$ , there are  $\delta, \bar{\delta} \in \mathbb{L}^p$  such that  $(\xi, \delta), (\eta, \bar{\delta}) \in \mathcal{S}(\beta)$  and as just seen  $(\xi, \eta) \in \mathcal{S}(\beta)$ .

Since  $\mathcal{S}(\beta) \neq 0$  and Equation (6) is satisfied, then if  $\mathcal{S}(\beta)$  is a degenerate subspace, we have that  $1 \leq s \leq p - 1$ . If  $\mathcal{U} = \mathcal{S}(\beta) \cap (\mathcal{S}(\beta))^\perp$ , we claim that  $\mathcal{U} = U_1 \oplus U_1$ , where  $U_1 = \pi_1(\mathcal{U})$ . We have from Equation (5) that

$$\langle\langle \beta(X, Y), (\eta, -\xi) \rangle\rangle = \langle\langle \beta(X, JY), (\xi, \eta) \rangle\rangle, \quad \langle\langle \beta(X, Y), (\xi, -\eta) \rangle\rangle = \langle\langle \beta(Y, X), (\xi, \eta) \rangle\rangle$$

and

$$\langle\langle \beta(X, Y), (\eta, \xi) \rangle\rangle = -\langle\langle \beta(JY, X), (\xi, \eta) \rangle\rangle.$$

Hence, if  $(\xi, \eta) \in \mathcal{U}$ , then also  $(\eta, -\xi), (\xi, -\eta), (\eta, \xi) \in \mathcal{U}$ . Thus,  $(\xi, 0), (0, \xi), (\eta, 0) \in \mathcal{U}$  and hence  $\mathcal{U} \subset U_1 \oplus U_1$ . If  $(\xi, \eta) \in U_1 \oplus U_1$ , there are  $\delta, \bar{\delta} \in \mathbb{L}^p$  such that  $(\xi, \delta), (\eta, \bar{\delta}) \in \mathcal{U}$  and thus  $(\xi, \eta) \in \mathcal{U}$ , proving the claim. It follows from Equation (2) and the claim that the vector subspace  $U_1 \subset \mathbb{L}^p$  is isotropic and thus Equation (7) holds.  $\square$

Given  $X \in V^{2n}$ , we denote  $N(X) = \ker B_X$ , where  $B_X = \beta_X$ . It follows from Equation (5) that the vector subspace  $N(X)$  is  $J$ -invariant.

**Lemma 10.** *Let the bilinear form  $\beta$  be flat and  $X \in RE(\beta)$  satisfy  $\beta|_{V \times N(X)} \neq 0$ . Then there is a non-zero light-like vector  $v \in \mathbb{L}^p$  such that*

$$\text{span}\{v\} \oplus \text{span}\{v\} \subset \mathcal{S}(\beta|_{V \times N(X)}) \subset B_X(V) \cap (B_X(V))^\perp. \tag{8}$$

Moreover, if  $(v', w') \in B_X(V)$ , where  $v', w'$  are light-like vectors, then  $v', w' \in \text{span}\{v\}$ .

**Proof.** First notice that the second inclusion in Equation (8) is just Equation (1). If  $\beta|_{N(X) \times N(X)} \neq 0$ , then the vector subspace  $\mathcal{S}(\beta|_{N(X) \times N(X)})$  is isotropic. Then by Equation (7), there is  $v \in \mathbb{L}^p$  such that  $\text{span}\{v\} \oplus \text{span}\{v\} = \mathcal{S}(\beta|_{N(X) \times N(X)})$ . Suppose that  $\beta|_{N(X) \times N(X)} = 0$ . By assumption, there are  $Y \in V^{2n}$  and  $Z \in N(X)$  such that  $\beta(Y, Z) = (v, w) \neq 0$ . Since the vector subspace  $\mathcal{S}(\beta|_{V \times N(X)})$  is isotropic, we have

$$0 = \langle\langle \beta(Y, Z), \beta(Y, Z) \rangle\rangle = \|v\|^2 - \|w\|^2,$$

whereas from Equation (5) and the flatness of  $\beta$ , we obtain

$$0 = \langle\langle \beta(Y, Z), \beta(Z, Y) \rangle\rangle = \|v\|^2 + \|w\|^2.$$

Thus, the vectors  $v, w \in \mathbb{L}^p$  are both light-like.

It suffices to argue for  $v \neq 0$  since  $\beta(Y, JZ) = (w, -v)$ . Since  $N(X)$  is  $J$ -invariant and  $\mathcal{S}(\beta|_{V \times N(X)})$  is isotropic, we obtain using Equation (5) that

$$0 = \langle\langle \beta(Y, Z), \beta(Y, JZ) \rangle\rangle = 2\langle v, w \rangle$$

and hence  $w = av$ . Then from

$$\beta(Y, Z + aJZ) = (a^2 + 1)(v, 0) \quad \text{and} \quad \beta(Y, aZ - JZ) = (a^2 + 1)(0, v),$$

we obtain the first inclusion in Equation (8).

Let  $B_X Z = (v', w')$  be as in the statement. By Equation (8), we have

$$\langle v, v' \rangle = \langle\langle (v, 0), (v', w') \rangle\rangle = \langle\langle (v, 0), \beta(X, Z) \rangle\rangle = 0,$$

and thus  $v' \in \text{span}\{v\}$ . Since  $B_X JZ = (w', -v')$ , then also  $w' \in \text{span}\{v\}$ . □

**Proposition 11.** *Let the bilinear form  $\beta: V^{2n} \times V^{2n} \rightarrow W^{p,p}$ ,  $p \leq n$ , be surjective and flat. Then we have*

$$\dim \mathcal{N}(\beta) \geq 2n - 2p. \tag{9}$$

Moreover, if  $\mathcal{N}(\beta) = 0$  and  $X \in RE(\beta)$ , then  $B_X: V^{2n} \rightarrow W^{p,p}$  is an isomorphism.

**Proof.** If  $X \in RE(\beta)$ , we have  $\dim N(X) \geq 2n - 2p$ . Since  $\mathcal{N}(\beta) \subset N(X)$ , if we have that  $N(X) = 0$  for some  $X \in RE(\beta)$ , then the result holds trivially. Thus, it remains to argue when  $N(X) \neq 0$  for any  $X \in RE(\beta)$ . In this case, we also show that  $\mathcal{N}(\beta) \neq 0$ , and this gives the second statement.

If  $\beta|_{V \times N(X)} = 0$  for some  $X \in RE(\beta)$ , then  $N(X) = \mathcal{N}(\beta)$ . Then  $\mathcal{N}(\beta) \neq 0$  and  $\dim \mathcal{N}(\beta) = \dim N(X) \geq 2n - 2p$ . Hence, we assume that  $\beta|_{V \times N(X)} \neq 0$  for any  $X \in RE(\beta)$ . Fix a vector  $X \in RE(\beta)$ . By Lemma 10, there is a non-zero light-like vector  $v \in \mathbb{L}^p$ , such that

$$\text{span}\{v\} \oplus \text{span}\{v\} \subset \mathcal{U}^\tau(X) = B_X(V) \cap (B_X(V))^\perp.$$

On the one hand, since  $\mathcal{S}(\beta) = W^{p,p}$  and the subset  $RE(\beta)$  is dense, there are  $Y \in RE(\beta)$  and  $Z \in V^{2n}$  such that  $\langle\langle \beta(Y, Z), (v, 0) \rangle\rangle \neq 0$ . On the other hand, since  $Y \in RE(\beta)$ ,

then Lemma 10 yields a non-zero light-like vector  $w \in \mathbb{L}^p$  such that  $\text{span}\{w\} \oplus \text{span}\{w\} \subset \mathcal{S}(\beta|_{V \times N(Y)}) \subset B_Y(V) \cap (B_Y(V))^\perp$ , and hence,  $\langle\langle \beta(Y, Z), (w, 0) \rangle\rangle = 0$ . Thus, the vectors  $v$  and  $w$  are linearly independent.

From Equation (1), we obtain that  $B_Y(N(X)) \subset \mathcal{U}^\tau(X)$ . If  $(av, bv) \in B_Y(N(X))$ , then the second part of Lemma 10 gives that  $av, bv \in \text{span}\{w\}$ , and hence,  $a = b = 0$ . Thus,  $\text{span}\{v\} \oplus \text{span}\{v\} \cap B_Y(N(X)) = 0$ . Consequently, for  $B_Y|_{N(X)}: N(X) \rightarrow \mathcal{U}^\tau(X)$  and since  $\text{span}\{v\} \oplus \text{span}\{v\} \subset \mathcal{U}^\tau(X)$ , we have that  $N_1 = \ker B_Y|_{N(X)}$  satisfies

$$\dim N_1 \geq \dim N(X) - \tau + 2. \tag{10}$$

Proposition 7 applied to  $B_X(V) \subset W^{p,p}$  yields a decomposition

$$W^{p,p} = \mathcal{U}^\tau(X) \oplus \hat{\mathcal{U}}^\tau(X) \oplus \mathcal{V}^{p-\tau, p-\tau},$$

verifying that  $B_X(V) \subset \mathcal{U}(X) \oplus \mathcal{V}$  among other properties. Thus,  $\dim B_X(V) \leq 2p - \tau$ , and hence,  $\dim N(X) \geq 2n - 2p + \tau$ . It follows from Equation (10) that  $\dim N_1 \geq 2n - 2p + 2 \geq 2$ .

We prove that  $N_1 = \mathcal{N}(\beta)$ , which gives  $\dim \mathcal{N}(\beta) \geq 2n - 2p + 2 > 0$ , which is even a better estimate than Equation (9). Since  $N_1 = N(X) \cap N(Y)$  by Equation (1), then  $\mathcal{S}(\beta|_{N_1 \times N_1})$  is an isotropic vector subspace unless  $\beta|_{N_1 \times N_1} = 0$ . In the former case, we have from Proposition 9 applied to  $\beta|_{N_1 \times N_1}$  that there is a non-zero light-like vector  $z \in \mathbb{L}^p$ , such that

$$\mathcal{S}(\beta|_{N_1 \times N_1}) = \mathcal{S}(\beta|_{N_1 \times N_1}) \cap (\mathcal{S}(\beta|_{N_1 \times N_1}))^\perp = \text{span}\{z\} \oplus \text{span}\{z\}.$$

Since  $N_1 \subset N(X)$ , we obtain from Equation (8) that  $(z, 0) \in B_X(V)$  and, similarly, we have that  $(z, 0) \in B_Y(V)$ . Then Lemma 10 yields  $z \in \text{span}\{v\} \cap \text{span}\{w\} = 0$ , which is not possible. We conclude that  $\beta|_{N_1 \times N_1} = 0$ .

If  $\beta|_{V \times N_1} \neq 0$ , there are vectors  $Z \in V^{2n}$  and  $T \in N_1$  such that  $\beta(Z, T) = (\xi, \eta) \neq 0$ . Then Equation (5) and the flatness of  $\beta$  give

$$0 = \langle\langle \beta(Z, T), \beta(T, Z) \rangle\rangle = \|\xi\|^2 + \|\eta\|^2.$$

By Equation (1), the vector subspace  $\mathcal{S}(\beta|_{V \times N_1})$  is isotropic and thus

$$0 = \langle\langle \beta(Z, T), \beta(Z, T) \rangle\rangle = \|\xi\|^2 - \|\eta\|^2.$$

Thus, the vectors  $\xi, \eta$  are light-like. Then from Equation (1) and the second part of Lemma 10, we obtain that  $\xi, \eta \in \text{span}\{v\} \cap \text{span}\{w\} = 0$ , and this is a contradiction. Then  $\beta|_{V \times N_1} = 0$  as wished. □

### 1.3. A special flat bilinear form

Throughout this section,  $V^{2n}$  is endowed with a positive definite inner product denoted by  $(,)$  with respect to which  $J \in \text{Aut}(V)$  is an isometry. In addition, we assume that

there exists a time-like of unit length vector  $w \in \mathbb{L}^p$  such that

$$\langle \alpha(X, Y), w \rangle = -(X, Y) \tag{11}$$

for any  $X, Y \in V^{2n}$ .

Under the assumptions above,  $\beta: V^{2n} \times V^{2n} \rightarrow W^{p,p}$  given by Equation (3) satisfies

$$\langle \langle \beta(X, Y), (w, 0) \rangle \rangle = \langle \alpha(X, Y) + \alpha(JX, JY), w \rangle = -2(X, Y). \tag{12}$$

In particular, if  $\beta(X, Y) = 0$ , then  $(X, Y) = 0$  and

$$\beta(X, X) \neq 0 \quad \text{if } 0 \neq X \in V^{2n}; \quad \text{thus } \mathcal{N}(\beta) \neq 0. \tag{13}$$

**Proposition 12.** *Let the bilinear form  $\beta: V^{2n} \times V^{2n} \rightarrow W^{p,p}$  be flat and the vector subspace  $\mathcal{S}(\beta)$  degenerate. If  $v \in U_0^s$  is a light-like vector as in Equation (7), then the plane  $L = \text{span}\{v, w\} \subset \mathbb{L}^p$  is Lorentzian. Moreover, choosing  $v$  such that  $\langle v, w \rangle = -1$  and setting  $\beta_1 = \pi_{L^\perp \times L^\perp} \circ \beta$ , we have*

$$\beta(X, Y) = \beta_1(X, Y) + 2((X, Y)v, (X, JY)v) \tag{14}$$

for any  $X, Y \in V^{2n}$ . Furthermore, if  $s \leq n$ , then

$$\dim \mathcal{N}(\beta_1) \geq 2n - 2s + 2. \tag{15}$$

**Proof.** The plane  $L$  is trivially Lorentzian. We choose  $v$  such that  $v = u + w$ , where  $u$  is a space-like unit vector orthogonal to  $w$ . Then  $L = \text{span}\{u, w\}$ .

We have from Equation (7) that

$$0 = \langle \langle \beta(X, Y), (v, 0) \rangle \rangle = \langle \alpha(X, Y) + \alpha(JX, JY), v \rangle \tag{16}$$

for any  $X, Y \in V^{2n}$ . Then from Equations (12) and (16), we have

$$\langle \alpha(X, Y) + \alpha(JX, JY), u \rangle = 2(X, Y)$$

for any  $X, Y \in V^{2n}$ . From Equation (12), we obtain

$$\begin{aligned} \alpha(X, Y) + \alpha(JX, JY) &= \alpha_{L^\perp}(X, Y) + \alpha_{L^\perp}(JX, JY) + \langle \alpha(X, Y) + \alpha(JX, JY), u \rangle u \\ &\quad - \langle \alpha(X, Y) + \alpha(JX, JY), w \rangle w \\ &= \alpha_{L^\perp}(X, Y) + \alpha_{L^\perp}(JX, JY) + 2(X, Y)v \end{aligned} \tag{17}$$

and then Equation (3) gives Equation (14).

We have seen that  $v$  can be chosen so that  $\langle v, w \rangle = -1$  and that Equation (14) holds. From Equations (6) and (7), we obtain that  $w \notin U_0^s$ . Hence,  $\dim(U_0^s + L^\perp) = p - 1$ .



Then from

$$\dim(U_0^s + L^\perp) = \dim U_0^s + \dim L^\perp - \dim U_0^s \cap L^\perp,$$

we have that  $U_1 = U_0^s \cap L^\perp$  satisfies

$$\dim U_1 = s - 1. \tag{18}$$

Hence,  $\mathcal{S}(\beta_1) = U_1^{s-1} \oplus U_1^{s-1}$  from Equations (6), (7) and (14).

From Equation (14), the bilinear form  $\beta_1: V^{2n} \times V^{2n} \rightarrow L^\perp \oplus L^\perp$  is flat. Let  $X \in RE(\beta_1)$  and set  $N_1(X) = \ker B_{1X}$ , where  $B_{1X}Y = \beta_1(X, Y)$ . To obtain Equation (15), it suffices to show that  $N_1(X) = \mathcal{N}(\beta_1)$ , since then  $\dim \mathcal{N}(\beta_1) = \dim N_1(X) \geq 2n - 2 \dim U_1$ .

Let  $\beta_1(Y, Z) = (\xi, \eta)$ , where  $Y, Z \in N_1(X)$ . From Equations (1) and (5), we have

$$0 = \langle\langle \beta_1(Y, Z), \beta_1(Z, Y) \rangle\rangle = \langle\langle (\xi, \eta), (\xi, -\eta) \rangle\rangle = \|\xi\|^2 + \|\eta\|^2.$$

Thus,  $\beta_1|_{N_1(X) \times N_1(X)} = 0$ , since the inner product induced on  $U_1$  is positive definite. Now let  $\beta_1(Y, Z) = (\delta, \zeta)$ , where  $Y \in V^{2n}$  and  $Z \in N_1(X)$ . The flatness of  $\beta_1$  and Equation (5) yield

$$0 = \langle\langle \beta_1(Y, Z), \beta_1(Z, Y) \rangle\rangle = \langle\langle (\delta, \zeta), (\delta, -\zeta) \rangle\rangle = \|\delta\|^2 + \|\zeta\|^2$$

and hence  $\beta_1|_{V \times N_1(X)} = 0$ . □

**Proposition 13.** *Let the bilinear form  $\beta: V^{2n} \times V^{2n} \rightarrow W^{p,p}$  be flat and the vector subspace  $\mathcal{S}(\beta)$  degenerate. If  $\gamma: V^{2n} \times V^{2n} \rightarrow W^{p,p}$  is the bilinear form defined by*

$$\gamma(X, Y) = (\alpha(X, Y), \alpha(X, JY)), \tag{19}$$

assume that

$$\langle\langle \beta(X, Y), \gamma(Z, T) \rangle\rangle = \langle\langle \beta(X, T), \gamma(Z, Y) \rangle\rangle \tag{20}$$

for any  $X, Y, Z, T \in V^{2n}$ . If  $s \leq n - 1$ , we have the following:

(i) For  $v \in U_0^s$  satisfying Equation (7), we have

$$\langle \alpha(X, Y), v \rangle = 0 \quad \text{for any } X, Y \in V^{2n}. \tag{21}$$

(ii) Choosing  $v \in U_0^s$  satisfying Equation (7) such that  $\langle v, w \rangle = -1$ , then with respect to the Lorentzian plane  $L = \text{span}\{v, w\}$ , we have

$$\alpha(X, Y) = \alpha_{L^\perp}(X, Y) + (X, Y)v \quad \text{for any } X, Y \in V^{2n}. \tag{22}$$

(iii) There is a  $J$ -invariant vector subspace  $P^{2m} \subset V^{2n}$  with  $m \geq n - s + 1$  such that for any  $S \in P^{2m}$ , we have

$$\mathcal{K}(S) = \langle \alpha(S, S), \alpha(JS, JS) \rangle - \|\alpha(S, JS)\|^2 \leq 0$$

and

$$\mathcal{R}(S) = \sum_{i=1}^{2n} (\langle \alpha(X_i, X_i), \alpha(S, S) \rangle - \|\alpha(X_i, S)\|^2) \leq 0,$$

where  $\{X_i\}_{1 \leq i \leq 2n}$  is an orthonormal basis of  $V^{2n}$ .

**Proof.** (i) It suffices to argue for  $v \in U_0^s$  such that  $\langle v, w \rangle = -1$ . From Equations (14) and (19), we obtain

$$\langle\langle \gamma(X, Y), \beta(S, S) \rangle\rangle = 2\langle \alpha(X, Y), v \rangle \tag{23}$$

for any  $S \in \mathcal{N}(\beta_1)$  of unit length and  $X, Y \in V^{2n}$ . Since we have from Equations (5) and (14) that  $\beta(S, Y) = \beta(Y, S) = 0$  for any  $S \in \mathcal{N}(\beta_1)$  and  $Y \in \{S, JS\}^\perp$ , then Equations (20) and (23) give

$$\langle \alpha(X, Y), v \rangle = 0 \tag{24}$$

for any  $X \in V^{2n}$  and  $Y \in \{S, JS\}^\perp$ , where  $S \in \mathcal{N}(\beta_1)$ . Since  $s \leq n - 1$ , then Equation (15) gives  $\dim \mathcal{N}(\beta_1) \geq 4$  and now Equation (21) follows from Equation (24).

(ii) From Proposition 12, the plane  $L = \text{span}\{v, w\}$  is Lorentzian. There is  $u \perp w$  space-like of unit length such that  $v = u + w$ . Hence, Equations (11) and (21) give  $\langle \alpha(X, Y), u \rangle = \langle X, Y \rangle$ . Now since

$$\alpha(X, Y) = \alpha_{L^\perp}(X, Y) + \langle \alpha(X, Y), u \rangle u - \langle \alpha(X, Y), w \rangle w,$$

then Equation (22) follows from Equation (11).

(iii) We choose  $v \in U_0^s$  satisfying Equation (7) such that  $\langle v, w \rangle = -1$ . From Equation (22), we have

$$\gamma(X, Y) = (\alpha_{L^\perp}(X, Y) + (X, Y)v, \alpha_{L^\perp}(X, JY) + (X, JY)v).$$

Set  $P^{2m} = \mathcal{N}(\beta_1)$ . From Equation (15), we have  $2m = \dim \mathcal{N}(\beta_1) \geq 2n - 2s + 2$ . It follows from Equation (14) that  $\beta(Z, S) = 2\langle (Z, S)v, (Z, JS)v \rangle$  for any  $S \in P^{2m}$  and  $Z \in V^{2n}$ . Then Equation (20) gives

$$\langle\langle \gamma(X, S), \beta(Z, Y) \rangle\rangle = \langle\langle \gamma(X, Y), \beta(Z, S) \rangle\rangle = 0$$

for any  $S \in P^{2m}$  and  $X, Y, Z \in V^{2n}$ . Thus, the vector subspaces  $\mathcal{S}(\gamma|_{V \times P})$  and  $\mathcal{S}(\beta)$  are orthogonal. From Equation (18), we have

$$U_0^s = U_1^{s-1} \oplus \text{span}\{v\}, \tag{25}$$

where  $U_1^{s-1} = U_0^s \cap L^\perp$ . Then by Equation (6), the vector subspaces  $\mathcal{S}(\gamma|_{V \times P})$  and  $U_1^{s-1} \oplus U_1^{s-1}$  are orthogonal and therefore

$$\langle \alpha(X, S), \xi \rangle = \langle \langle \gamma(X, S), (\xi, 0) \rangle \rangle = 0$$

for any  $X \in V^{2n}$ ,  $S \in P^{2m}$  and  $\xi \in U_1^{s-1}$ . Since  $U_1^{s-1} \subset L^\perp$ , then

$$\alpha_{U_1}(X, S) = 0 \tag{26}$$

for any  $X \in V^{2n}$  and  $S \in P^{2m}$ .

Let  $U_2^{p-s-1} \subset L^\perp$  be given by the orthogonal decomposition  $L^p = U_1^{s-1} \oplus U_2^{p-s-1} \oplus L$ . By Equations (6) and (25), we have

$$\langle \alpha(X, Y) + \alpha(JX, JY), \xi_2 \rangle = \langle \langle \beta(X, Y), (\xi_2, 0) \rangle \rangle = 0$$

for any  $X, Y \in V^{2n}$  and  $\xi_2 \in U_2^{p-s-1}$ . Thus,

$$\alpha_{U_2}(X, Y) = -\alpha_{U_2}(JX, JY) \tag{27}$$

for any  $X, Y \in V^{2n}$ .

From Equations (22), (26) and (27) and since the inner product induced on  $U_2^{p-s-1}$  is positive definite, we have

$$\mathcal{K}(S) = \langle \alpha_{U_2}(S, S), \alpha_{U_2}(JS, JS) \rangle - \|\alpha_{U_2}(S, JS)\|^2 = -\|\alpha_{U_2}(S, S)\|^2 - \|\alpha_{U_2}(S, JS)\|^2 \leq 0$$

for any  $S \in P^{2m}$ . Also,

$$\begin{aligned} \mathcal{R}(S) &= \sum_{i=1}^n (\langle \alpha_{U_2}(Y_i, Y_i), \alpha_{U_2}(S, S) \rangle - \|\alpha_{U_2}(Y_i, S)\|^2 + \langle \alpha_{U_2}(JY_i, JY_i), \alpha_{U_2}(S, S) \rangle \\ &\quad - \|\alpha_{U_2}(JY_i, S)\|^2) \\ &= -\sum_{i=1}^n (\|\alpha_{U_2}(Y_i, S)\|^2 + \|\alpha_{U_2}(JY_i, S)\|^2) \leq 0 \end{aligned}$$

for any  $S \in P^{2m}$  and an orthonormal basis  $\{Y_j, JY_j\}_{1 \leq j \leq n}$ . □

**Lemma 14.** *Let the bilinear form  $\beta: V^{2n} \times V^{2n} \rightarrow W^{p,p}$  be flat. Assume that the vector subspace  $\mathcal{S}(\beta)$  is non-degenerate. If  $Z \in V^{2n}$  satisfies  $\dim N(Z) = 2n - 2$ , then  $B_Z Z = (\xi, 0) \neq 0$ , the vector subspace  $B_Z(V) = \text{span}\{\xi\} \oplus \text{span}\{\xi\}$  is non-degenerate and the decomposition*

$$\mathcal{S}(\beta) = B_Z(V) \oplus \mathcal{S}(\beta|_{N(Z) \times N(Z)}) \tag{28}$$

is orthogonal.

**Proof.** We have  $0 \neq B_Z Z = (\xi, 0)$  and  $B_Z JZ = -(0, \xi)$ . If  $Y \in N(Z)$ , then

$$0 = \langle\langle B_Z Y, (w, 0) \rangle\rangle = -2(Z, Y) \quad \text{and} \quad 0 = \langle\langle B_Z Y, (0, w) \rangle\rangle = -2(JZ, Y)$$

and thus  $N(Z) \subset \{Z, JZ\}^\perp$ . Since  $\dim N(Z) = 2n - 2$ , then  $N(Z) = \{Z, JZ\}^\perp$  and  $\dim B_Z(V) = 2$  gives  $B_Z(V) = \text{span}\{\xi\} \oplus \text{span}\{J\xi\}$ . From Equations (4) and (5), we obtain

$$\mathcal{S}(\beta) = B_Z(V) + \mathcal{S}(\beta|_{N(Z) \times N(Z)}).$$

The flatness of  $\beta$  gives

$$\langle\langle \beta(Z, Y), \beta(S, T) \rangle\rangle = \langle\langle B_Z T, \beta(S, Y) \rangle\rangle = 0$$

for any  $S, T \in N(Z)$  and  $Y \in V^{2n}$ . Hence, the decomposition (28) is orthogonal and thus the vector subspace  $B_Z(V)$  is non-degenerate.  $\square$

The following is the fundamental result obtained in this first section, although it is not needed for the proof of Theorem 3. We observe that Lemma 7 in [7] is a Riemannian version of the result. For an alternative version of the Riemannian result in the spirit of this paper, see Proposition 2.6 in [2].

**Proposition 15.** *Let the bilinear form  $\beta: V^{2n} \times V^{2n} \rightarrow W^{p,p}$ ,  $s \leq n$ , be flat. Assume that the subspace  $\mathcal{S}(\beta)$  is non-degenerate and Equation (20) holds. For  $p \geq 4$ , assume further that there is no non-trivial  $J$ -invariant vector subspace  $V_1 \subset V^{2n}$  such that the subspace  $\mathcal{S}(\beta|_{V_1 \times V_1})$  is degenerate and that  $\dim \mathcal{S}(\beta|_{V_1 \times V_1}) \leq \dim V_1 - 2$ . Then  $s = n$ , and there is an orthogonal basis  $\{X_i, JX_i\}_{1 \leq i \leq n}$  of  $V^{2n}$  such that*

- (i)  $\beta(Y_i, Y_j) = 0$  if  $i \neq j$  and  $Y_k \in \text{span}\{X_k, JX_k\}$  for  $k=i, j$ .
- (ii) The vectors  $\{\beta(X_j, X_j), \beta(X_j, JX_j)\}_{1 \leq j \leq n}$  form an orthonormal basis of  $\mathcal{S}(\beta)$ .

**Proof.** Since  $\mathcal{N}(\beta) = 0$ , then by Proposition 11, we have that  $s = n$ . Moreover, we have that  $B_X: V^{2n} \rightarrow U_0^n \oplus U_0^n \subset W^{p,p}$  is an isomorphism for any  $X \in \text{RE}(\beta)$ .

*Fact 1.* If  $n \geq 2$ , then there are non-zero vectors  $X, Y \in V^{2n}$ , satisfying  $\beta(X, Y) = 0$ .

We have that  $\text{RE}(\beta)$  is open and dense in  $V^{2n}$  and that  $B_X: V^{2n} \rightarrow U_0^n \oplus U_0^n$  is an isomorphism for any  $X \in \text{RE}(\beta)$ . Hence, there is a basis  $Z_1, \dots, Z_{2n}$  of  $V^{2n}$  with  $Z_2 \notin \text{span}\{Z_1, JZ_1\}$ , such that  $\{\beta(Z_1, Z_j)\}_{1 \leq j \leq 2n}$  and  $\{\beta(Z_2, Z_j)\}_{1 \leq j \leq 2n}$  are both basis of  $U_0^n \oplus U_0^n$ . Let  $A = (a_{ij})$  be the  $2n \times 2n$  matrix given by

$$\beta(Z_2, Z_j) = \sum_{r=1}^{2n} a_{rj} \beta(Z_1, Z_r).$$

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $A$  with corresponding eigenvector  $(v^1, \dots, v^{2n}) \in \mathbb{C}^{2n}$ . Extending  $\beta$  linearly from  $V^{2n} \otimes \mathbb{C}$  to  $W^{p,p} \otimes \mathbb{C}$ , we have

$$\sum_{j=1}^{2n} v^j \beta(Z_2, Z_j) = \lambda \sum_{j=1}^{2n} v^j \beta(Z_1, Z_j).$$

Hence,  $\beta(S, T) = 0$ , where  $S = Z_2 - \lambda Z_1$  and  $T = \sum_{j=1}^{2n} v^j Z_j$ . Setting  $S = S_1 + iS_2$  and  $T = T_1 + iT_2$ , it follows that

$$\beta(S_1, T_1) = \beta(S_2, T_2) \quad \text{and} \quad \beta(S_1, T_2) + \beta(S_2, T_1) = 0.$$

Then, if  $X = S_1 - JS_2$  and  $Y = T_1 + JT_2$ , we obtain using Equations (4) and (5) that

$$\begin{aligned} \beta(X, Y) &= \beta(S_1, T_1) + \beta(S_1, JT_2) - \beta(JS_2, T_1) - \beta(JS_2, JT_2) \\ &= \beta(S_1, T_1) + \beta(S_1, JT_2) + \beta(S_2, JT_1) - \beta(S_2, T_2) = 0. \end{aligned}$$

Similarly, we obtain  $\beta(X', Y') = 0$  for  $X' = S_1 + JS_2$  and  $Y' = T_1 - JT_2$ . The vectors  $X$  and  $X'$  are both non-zero. For instance, if  $X = 0$ , then

$$JS_2 + iS_2 = S = Z_2 - \lambda Z_1.$$

Thus,  $JS_2 = Z_2 - \text{Re}(\lambda)Z_1$  and  $S_2 = -\text{Im}(\lambda)Z_1$ . But then  $Z_2 \in \text{span}\{Z_1, JZ_1\}$ , and this is a contradiction. Finally, if we have that  $Y = Y' = 0$ , then  $T = 0$ , which is also a contradiction.

*Fact 2.* There exists a vector  $Z_0 \in V^{2n}$  satisfying  $\dim N(Z_0) = 2n - 2$ .

This Fact holds for  $n = 1$ . In fact, if  $0 \neq X \in V^2$ , then  $\beta(X, X)$  and  $\beta(X, JX)$  are linearly independent and hence  $N(X) = 0$ .

We argue for  $n \geq 2$ . By Fact 1, there are non-zero vectors  $X, Y \in V^{2n}$  such that  $\beta(X, Y) = 0$ . From Equation (5), we have that  $\dim B_X V = 2r$ , where  $1 \leq r \leq n - 1$  since  $B_X Y = 0$  and  $X \notin N(X)$ . Thus,  $\dim N(X) = 2n - 2r$ . Moreover, we may assume that  $2 \leq r \leq n - 1$  since Fact 2 holds when  $r = 1$  by taking  $Z_0 = X$ . Therefore, we assume that  $n \geq 3$ .

Set  $U_1^t = \pi_1(B_X(V))$ . We claim that  $t = r$  and that

$$B_X(V) = U_1^r \oplus U_1^r = \{\beta(Z, X) : Z \in V^{2n}\}. \tag{29}$$

In order to prove the claim, we first show that

$$B_X(V) + \{\beta(Z, X) : Z \in V^{2n}\} = U_1^t \oplus U_1^t \tag{30}$$

which, in particular, yields  $t \geq r$ . If  $\beta(X, Z_1) = (\xi, \eta)$  and  $\beta(Z_2, X) = (\zeta, \delta)$ , then Equation (5) gives  $\beta(X, JZ_1) = (\eta, -\xi)$  and  $\beta(X, JZ_2) = -(\delta, \zeta)$ . Hence,  $\xi, \zeta, \eta, \delta \in U_1^t$  and thus  $(\xi, \eta), (\zeta, \delta) \in U_1^t \oplus U_1^t$ . Now let  $(\xi_1, \xi_2) \in U_1^t \oplus U_1^t$ . Then there are  $Z_1, Z_2 \in V^{2p}$  such that  $\beta(X, Z_i) = (\xi_i, \eta_i)$ ,  $i = 1, 2$ . Using Equation (5), we obtain

$$(\xi_1, \xi_2) = \frac{1}{2} (\beta(X, Z_1 - JZ_2) + \beta(Z_1 + JZ_2, X)),$$

and Equation (30) has been proved.

We show that  $t = r$ . In the process and for later use, we also prove that

$$\mathcal{S}(\bar{\beta}) = U_2^{n-r} \oplus U_2^{n-r}, \tag{31}$$

where  $\bar{\beta} = \beta|_{N(X) \times N(X)}$  and  $U_2^{n-r} = U_0^n \cap (U_1^r)^\perp$ . The flatness of  $\beta$  gives

$$\langle\langle \beta(X, Z), \beta(S, T) \rangle\rangle = \langle\langle B_X T, \beta(S, Z) \rangle\rangle = 0$$

for any  $S, T \in N(X)$  and  $Z \in V^{2n}$ . From Equation (5), we have  $\beta(S, X) = 0$  and thus

$$\langle\langle \beta(Z, X), \beta(S, T) \rangle\rangle = \langle\langle \beta(Z, T), \beta(S, X) \rangle\rangle = 0$$

for any  $S, T \in N(X)$  and  $Z \in V^{2n}$ . Since  $\mathcal{S}(\bar{\beta}) \subset \mathcal{S}(\beta)$ , then from Equation (30), we have

$$\mathcal{S}(\bar{\beta}) \subset U_2^{n-t} \oplus U_2^{n-t}. \tag{32}$$

Suppose that the vector subspace  $\mathcal{S}(\bar{\beta})$  is non-degenerate. Since  $\dim N(X) = 2n - 2r$ , we have from Equations (9) and (32) that

$$\dim \mathcal{N}(\bar{\beta}) \geq \dim N(X) - \dim \mathcal{S}(\bar{\beta}) \geq 2t - 2r.$$

Since  $\mathcal{N}(\bar{\beta}) = 0$  because  $\beta(Z, Z) \neq 0$  if  $Z \neq 0$  and  $t \geq r$ , then  $t = r$  and  $\dim \mathcal{S}(\bar{\beta}) = 2n - 2r$ . Suppose now that the vector subspace  $\mathcal{S}(\bar{\beta})$  is degenerate. If  $n = 3$ , then  $r = 2$ . Since  $\mathcal{S}(\bar{\beta}) \neq 0$ , then also  $t = 2$  and Equation (31) follows. Then let  $n \geq 4$ . We have from Equation (32) that

$$\dim \mathcal{S}(\bar{\beta}) \leq 2n - 2t \leq 2n - 2r = \dim N(X),$$

and the assumption in the statement yields that  $t = r$  and  $\dim \mathcal{S}(\bar{\beta}) = 2n - 2r$ . Having proved that  $\dim \mathcal{S}(\bar{\beta}) = 2n - 2r$  holds in both cases, then Equation (31) follows from Equation (6). Finally, since  $t = r$ , then Equation (29) follows from Equation (30) using Equation (5), and thus the claim has been proved.

We now claim that in fact the vector subspace  $\mathcal{S}(\bar{\beta})$  is non-degenerate. Assuming otherwise, we have from Equation (14) that

$$\bar{\beta}(S, T) = \bar{\beta}_1(S, T) + 2((S, T)v, (S, JT)v) \tag{33}$$

for any  $S, T \in N(X)$ , where the light-like vector  $v \in U_0^n$  satisfies  $\langle v, w \rangle = -1$  and

$$\mathcal{S}(\bar{\beta}) \cap (\mathcal{S}(\bar{\beta}))^\perp = \text{span}\{v\} \oplus \text{span}\{v\}. \tag{34}$$

Since  $\beta(R, X) = 0$  for  $R \in N(X)$  by Equation (5), then the flatness of  $\beta$  gives

$$\langle\langle \beta(Y, X), \beta(R, Z) \rangle\rangle = \langle\langle \beta(Y, Z), \beta(R, X) \rangle\rangle = 0$$

for any  $Y, Z \in V^{2n}$  and  $R \in N(X)$ . Therefore, for  $R \in N(X)$ , the vector subspaces  $B_R(V)$  and  $\{\beta(Y, X) : Y \in V^{2p}\}$  are orthogonal. From Equations (29) and (31), we obtain that

$B_R(V) \subset U_2^{n-r} \oplus U_2^{n-r} = \mathcal{S}(\bar{\beta})$  if  $R \in N(X)$ . By Equation (15), the vector subspace  $N_0 = \mathcal{N}(\bar{\beta}_1) \subset N(X)$  is non-zero, and from Equations (5) and (33), we have  $B_S(N(X)) = \text{span}\{v\} \oplus \text{span}\{v\}$  for any  $S \in N_0$ . Then from Equation (34) and the flatness of  $\beta$ , it follows that

$$\langle\langle \beta(S, Y), \beta(R, T) \rangle\rangle = \langle\langle B_S T, \beta(R, Y) \rangle\rangle = 0$$

for any  $S \in N_0, Y \in V^{2n}$  and  $R, T \in N(X)$ . Hence,  $B_S(V) \subset \mathcal{S}(\bar{\beta}) \cap (\mathcal{S}(\bar{\beta}))^\perp$  for any  $S \in N_0$ . Then from Equations (33) and (34), it follows that  $B_S(V) = \text{span}\{v\} \oplus \text{span}\{v\}$  for any  $S \in N_0$ . Since  $\dim N(S) = 2n - 2$ , then by Lemma 14, the subspace  $B_S(V)$  should be non-degenerate. This is a contradiction that proves the claim.

If  $Z_0 \in N(X)$ , then Equation (5) gives that also  $\beta(Z_0, X) = 0$ . The flatness of  $\beta$  yields

$$\langle\langle \beta(Z_0, Y), \beta(Z, X) \rangle\rangle = \langle\langle \beta(Z_0, X), \beta(Z, Y) \rangle\rangle = 0$$

for any  $Y, Z \in V^{2n}$ . Then the second equality in Equation (29) gives  $B_{Z_0}(V) \subset U_2^{n-r} \oplus U_2^{n-r}$ . If  $r = n - 1$ , then  $Z_0$  satisfies  $\dim \ker B_{Z_0} = 2n - 2$  as required. Hence, we assume that  $r \leq n - 2$ , and thus  $n \geq 4$  since  $r \geq 2$ .

We conclude the proof of Fact 2 arguing by induction. Assume that it is true for any dimension until  $n - 1$ . We have seen that  $\dim N(X) = 2n - 2r$  with  $2 \leq r \leq n - 2$ . Thus, the assumption of the induction applies to  $\bar{\beta}: N(X) \times N(X) \rightarrow U_2^{n-r} \oplus U_2^{n-r} \subset W^{p,p}$  since the vector subspace  $\mathcal{S}(\bar{\beta}) = U_2^{n-r} \oplus U_2^{n-r}$  is non-degenerate. Hence, there is  $Z_0 \in N(X)$  such that  $\dim \ker \bar{B}_{Z_0} = 2n - 2r - 2$ . Then Lemma 14 applies, and by Equations (28) and (31), we have

$$U_2^{n-r} \oplus U_2^{n-r} = \bar{\beta}_{Z_0}(N(X)) \oplus \mathcal{S}(\bar{\beta}|_{\bar{N}(Z_0) \times \bar{N}(Z_0)}),$$

where  $\bar{N}(Z_0) = \ker \bar{B}_{Z_0}$ . It follows that  $\dim \mathcal{S}(\bar{\beta}|_{\bar{N}(Z_0) \times \bar{N}(Z_0)}) = 2n - 2r - 2$ . By the flatness of  $\beta$ , we have

$$\langle\langle \beta(Z_0, Y), \bar{\beta}(R, T) \rangle\rangle = \langle\langle \bar{\beta}(Z_0, T), \beta(R, Y) \rangle\rangle = 0$$

for any  $R, T \in \bar{N}(Z_0)$  and  $Y \in V^{2n}$ . Hence, the vector subspaces  $\mathcal{S}(\bar{\beta}|_{\bar{N}(Z_0) \times \bar{N}(Z_0)})$  and  $B_{Z_0}(V)$  are orthogonal. Since  $B_{Z_0}(V) \subset U_2^{n-r} \oplus U_2^{n-r}$ , then  $\dim B_{Z_0}(V) = 2$ , and hence Fact 2 has been proved.

We conclude the proof of the proposition by means of a recursive construction. Notice that it suffices to construct an orthogonal basis of  $\mathcal{S}(\beta)$  since it can be normalized. By Fact 2 and Equation (5), there is  $X_1 \in V^{2n}$  such that the  $J$ -invariant vector subspace  $N(X_1)$  satisfies  $\dim N(X_1) = 2n - 2$ . Moreover,  $\beta(X_1, X_1) = (\xi_1, 0) \neq 0$  and  $\beta(X_1, JX_1) = (0, -\xi_1)$ . By Lemma 14, the vector  $\xi_1$  is either space-like or time-like.

By Lemma 14, the decomposition  $\mathcal{S}(\beta) = B_{X_1}(V) \oplus \mathcal{S}(\beta_{N(X_1) \times N(X_1)})$  is orthogonal and the vector subspace  $\mathcal{S}(\beta_{N(X_1) \times N(X_1)})$  is non-degenerate and has dimension  $2n - 2$ . By Fact 2, there is  $X_2 \in N(X_1)$  such that  $\dim \ker B_{X_2} \cap N(X_1) = 2n - 4$ . Again, we have that  $\beta(X_2, X_2) = (\xi_2, 0) \neq 0$  and  $\beta(X_2, JX_2) = (0, -\xi_2)$ , where  $\xi_2$  is either space-like or

time-like, and the vectors  $\xi_1$  and  $\xi_2$  are orthogonal. Since  $N(X_1)$  is  $J$ -invariant, then

$$\beta(X_1, X_2) = 0 = \beta(X_1, JX_2).$$

Then we have that  $X_1, JX_1, X_2, JX_2$  are orthogonal. If we have  $n = 2$ , then the desired basis is  $X_1, JX_1, X_2, JX_2$ , and if  $n \geq 3$ , we have to reiterate the construction.  $\square$

## 2. The proofs

This section is devoted to provide the proofs of the results stated in the introduction.

Let  $f: M^{2n} \rightarrow \mathbb{H}^{2n+p}$  be an isometric immersion of a Kaehler manifold. Then let  $g = i \circ f: M^{2n} \rightarrow \mathbb{L}^{2n+p+1}$ , where  $i: \mathbb{H}^{2n+p} \rightarrow \mathbb{L}^{2n+p+1}$  denotes the inclusion. We have that  $N_g M = i_* N_f M \oplus \text{span}\{g\}$  and

$$\alpha^g(X, Y) = i_* \alpha^f(X, Y) + \langle X, Y \rangle g$$

for any  $X, Y \in \mathfrak{X}(M)$ . At  $x \in M^{2n}$ , let  $\gamma, \beta: T_x M \times T_x M \rightarrow N_g M(x) \oplus N_g M(x)$  be the bilinear forms defined by

$$\gamma(X, Y) = (\alpha^g(X, Y), \alpha^g(X, JY))$$

and

$$\begin{aligned} \beta(X, Y) &= \gamma(X, Y) + \gamma(JX, JY) \\ &= (\alpha^g(X, Y) + \alpha^g(JX, JY), \alpha^g(X, JY) - \alpha^g(JX, Y)). \end{aligned}$$

Then since  $\alpha^g$  verifies the condition (11), it follows that  $\beta$  satisfies Equation (13).

**Proposition 16.** *Let  $N_g M(x) \oplus N_g M(x)$  be endowed with the inner product defined by*

$$\langle\langle (\xi, \bar{\xi}), (\eta, \bar{\eta}) \rangle\rangle = \langle \xi, \eta \rangle - \langle \bar{\xi}, \bar{\eta} \rangle.$$

*Then the bilinear form  $\beta$  is flat, and we have that*

$$\langle\langle \beta(X, Y), \gamma(Z, T) \rangle\rangle = \langle\langle \beta(X, T), \gamma(Z, Y) \rangle\rangle \tag{35}$$

*for any  $X, Y, Z, T \in T_x M$ .*

**Proof.** It is well-known that the curvature tensor of a Kaehler manifold  $M^{2n}$  satisfies that  $R(X, Y)JZ = JR(X, Y)Z$  for any  $X, Y, Z \in T_x M$ . Then this together with the



Gauss equation for  $g$  give

$$\begin{aligned} &\langle\langle\gamma(X, T), \beta(Z, Y)\rangle\rangle \\ &= \langle\alpha(X, T), \alpha(Z, Y)\rangle + \langle\alpha(X, T), \alpha(JZ, JY)\rangle - \langle\alpha(X, JT), \alpha(Z, JY)\rangle \\ &\quad + \langle\alpha(X, JT), \alpha(JZ, Y)\rangle \\ &= \langle R(X, Z)Y, T\rangle + \langle\alpha(X, Y), \alpha(Z, T)\rangle + \langle R(X, JZ)JY, T\rangle \\ &\quad + \langle\alpha(X, JY), \alpha(JZ, T)\rangle - \langle R(X, Z)JY, JT\rangle - \langle\alpha(X, JY), \alpha(Z, JT)\rangle \\ &\quad + \langle R(X, JZ)Y, JT\rangle + \langle\alpha(X, Y), \alpha(JZ, JT)\rangle \\ &= \langle\alpha(X, Y), \alpha(Z, T)\rangle + \langle\alpha(X, JY), \alpha(JZ, T)\rangle - \langle\alpha(X, JY), \alpha(Z, JT)\rangle \\ &\quad + \langle\alpha(X, Y), \alpha(JZ, JT)\rangle \\ &= \langle\langle\gamma(X, Y), \beta(Z, T)\rangle\rangle, \end{aligned}$$

which proves Equation (35). Using Equations (4) and (35), we have

$$\begin{aligned} \langle\langle\gamma(JX, JY), \beta(Z, T)\rangle\rangle &= \langle\langle\gamma(JX, T), \beta(Z, JY)\rangle\rangle = -\langle\langle\gamma(JX, T), \beta(JZ, Y)\rangle\rangle \\ &= -\langle\langle\gamma(JX, Y), \beta(JZ, T)\rangle\rangle = \langle\langle\gamma(JX, Y), \beta(Z, JT)\rangle\rangle = \langle\langle\gamma(JX, JT), \beta(Z, Y)\rangle\rangle. \end{aligned}$$

Then

$$\begin{aligned} \langle\langle\beta(X, Y), \beta(Z, T)\rangle\rangle &= \langle\langle\gamma(X, Y), \beta(Z, T)\rangle\rangle + \langle\langle\gamma(JX, JY), \beta(Z, T)\rangle\rangle \\ &= \langle\langle\gamma(X, T), \beta(Z, Y)\rangle\rangle + \langle\langle\gamma(JX, JT), \beta(Z, Y)\rangle\rangle = \langle\langle\beta(X, T), \beta(Z, Y)\rangle\rangle, \end{aligned}$$

and this concludes the proof. □

**Proof of Theorem 3.** Observe that  $s \leq p + 1 \leq n - 1$ . At any point  $M^{2n}$ , the vector subspace  $\mathcal{S}(\beta)$  is degenerate. In fact, if we have otherwise, we obtain from Equation (9) that  $\dim \mathcal{N}(\beta) \geq 2n - 2p - 2 \geq 2$ , which is a contradiction. We have from Proposition 9 that  $s \leq p$  and the proof now follows from part (iii) of Proposition 13. □

**Remark 17.** For  $f: M^{2n} \rightarrow \mathbb{H}^{2n+p}$  as in Theorem 3, one can obtain from part (i) of Proposition 13 that there is a smooth unit normal vector field  $\eta \in \Gamma(N_fM)$  such that the second fundamental form satisfies  $A_\eta^f = I$ ; see below the argument in the proof of Theorem 6. An elementary proof gives that  $f(M)$  is contained in a horosphere in  $\mathbb{H}^{2n+p}$  if and only if  $\eta$  is parallel in the normal connection.

**Lemma 18.** *Let  $f: M^{2n} \rightarrow \mathbb{H}^{2n+p}$ ,  $p \leq n - 1$ , be an isometric immersion of a Kaehler manifold. At any  $y \in M^{2n}$ , let  $\beta: T_yM \times T_yM \rightarrow W^{p+1,p+1} = N_gM(y) \oplus N_gM(y)$  be surjective. Moreover, assume that there is an orthogonal basis  $\{X_j, JX_j\}_{1 \leq j \leq n}$  of  $T_yM$  such that parts (i) and (ii) in Proposition 15 hold. Then  $p = n - 1$  and  $f(M)$  is an open subset of the submanifold given by Example 1.*

**Proof.** The vectors  $(\xi_j, 0) = \beta(X_j, X_j) \in N_gM(y) \oplus N_gM(y)$ ,  $1 \leq j \leq n$ , in part (ii) of Proposition 15 are the orthonormal basis of  $W^{p+1,p+1}$ . Thus,  $p = n - 1$ . Part (i) and

Equation (35) give that

$$\langle A_{\xi_i} X_j, Y \rangle = \langle \langle \gamma(Y, X_j), \beta(X_i, X_i) \rangle \rangle = \langle \langle \gamma(Y, X_i), \beta(X_i, X_j) \rangle \rangle = 0 \quad \text{if } i \neq j$$

for any  $Y \in T_y M$ . Similarly, we have  $\langle A_{\xi_i} JX_j, Y \rangle = 0$ . Hence,

$$\text{span}\{X_i, JX_i\}_{1 \leq i \neq j \leq n} \subset \ker A_{\xi_j}, \tag{36}$$

and thus the vector subspaces  $\{\text{Im}A_{\xi_i}\}_{1 \leq i \leq n}$  are orthogonal. Thus, any pair of shape operator in  $\{A_{\xi_j}\}_{1 \leq j \leq n}$  commutes, and we conclude from the Ricci equation that  $g$  has flat normal bundle.

From Equation (36), we have  $0 \leq \text{rank } A_{\xi_j} \leq 2, 1 \leq j \leq n$ . In fact,  $\text{rank } A_{\xi_j} = 2, 1 \leq j \leq n$ , for if  $\text{rank } A_{\xi_r} \leq 1$  for some  $1 \leq r \leq n$ , then there is  $0 \neq Y \in \text{span}\{X_r, JX_r\} \cap \ker A_{\xi_r}$ . But then Equation (36) yields  $Y \in \mathcal{N}(\alpha^g)$  in contradiction with  $A_g = -I$ .

At any point  $x \in M^{2n}$ , there exist orthonormal normal vectors  $\{\xi_j(x)\}_{1 \leq j \leq n}$  and orthogonal tangent vectors  $\{X_j(x), JX_j(x)\}_{1 \leq j \leq n}$  such that  $\ker A_{\xi_j(x)} = \bigoplus_{i \neq j} E_i(x)$ , where we denote  $E_j(x) = \text{span}\{X_j(x), JX_j(x)\}$ .

From Equation (36) and since  $f$  has flat normal bundle, then at any  $x \in M^{2n}$ , there is  $r = r(x), 0 \leq r \leq n$ , such that the tangent space decomposes orthogonally as

$$T_x M = F_1(x) \oplus \dots \oplus F_{n+r}(x),$$

where  $F_i(x) \oplus F_{n+i}(x) = E_i(x)$  if  $0 \leq i \leq r$  and  $F_i(x) = E_i(x)$  if  $r+1 \leq i \leq n$ , such that for each  $\xi \in N_f M(x)$ , there exists pairwise distinct  $\lambda_i(\xi) \in \mathbb{R}, 1 \leq i \leq n+r$ , satisfying

$$A_\xi|_{F_i(x)} = \lambda_i(\xi)I.$$

Since the maps  $\xi \mapsto \lambda_i(\xi)$  are linear, then there are unique pairwise distinct vectors  $\eta_i(x) \in N_f M(x), 1 \leq i \leq n+r$ , such that

$$\lambda_i(\xi) = \langle \xi, \eta_i(x) \rangle, \quad 1 \leq i \leq n+r, \tag{37}$$

and

$$F_i(x) = \{X \in T_x M : \alpha(X, Y) = \langle X, Y \rangle \eta_i(x) \quad \text{for all } Y \in T_x M\}.$$

On each connected component  $M_r$  of the open dense subset of  $M^{2n}$ , where the function  $r(x)$  is constant, the maps  $x \in M_r \mapsto \eta_i(x)$  and  $x \in M_r \mapsto F_i(x), 1 \leq i \leq n+r$ , define smooth vector fields and smooth vector bundles, respectively.

If  $F_i(x) \oplus F_{n+i}(x) = E_i(x)$ , let  $Y \in E_i(x)$  satisfy  $A_{\xi_i} Y = \mu Y \neq 0$  and  $A_{\xi_i} JY = \bar{\mu} JY \neq 0$ . Let  $(Z)_F$  denote taking the  $F$ -component of the vector  $Z$ . Then Equation (37) gives

$$\begin{aligned} \mu Y &= A_{\xi_i} Y = A_{\xi_i}(Y)_{F_i} + A_{\xi_i}(Y)_{F_{n+i}} = \langle \xi_i, \eta_i \rangle (Y)_{F_i} + \langle \xi_i, \eta_{n+i} \rangle (Y)_{F_{n+i}}, \\ \bar{\mu} JY &= A_{\xi_i} JY = A_{\xi_i}(JY)_{F_i} + A_{\xi_i}(JY)_{F_{n+i}} = \langle \xi_i, \eta_i \rangle (JY)_{F_i} + \langle \xi_i, \eta_{n+i} \rangle (JY)_{F_{n+i}}, \\ 0 &= A_{\xi_i} X_j = A_{\xi_i}(X_j)_{F_j} + A_{\xi_i}(X_j)_{F_{n+j}} = \langle \xi_i, \eta_j \rangle (X_j)_{F_j} + \langle \xi_i, \eta_{n+j} \rangle (X_j)_{F_{n+j}}, \\ 0 &= A_{\xi_i} JX_j = A_{\xi_i}(JX_j)_{F_j} + A_{\xi_i}(JX_j)_{F_{n+j}} = \langle \xi_i, \eta_j \rangle (JX_j)_{F_j} + \langle \xi_i, \eta_{n+j} \rangle (JX_j)_{F_{n+j}}, \end{aligned}$$

where  $i \neq j$ . Thus, the vectors  $\eta_i, \eta_{n+i}$  are non-zero and satisfy  $\eta_i, \eta_{n+i} \in \text{span}\{\xi_i\}$ . Hence, the vector fields  $\eta_1, \dots, \eta_n$  form an orthogonal frame and  $\xi_i \in \text{span}\{\eta_i\}$ ,  $1 \leq i \leq n$ . It follows that on  $M_r$ , the orthonormal vector fields  $\xi_1, \dots, \xi_n$  are smooth.

Next we argue on  $M_r$ . The Codazzi equation  $(\nabla_S A)(T, \xi_j) = (\nabla_T A)(S, \xi_j)$  gives

$$A_{\xi_j}[S, T] + A_{\nabla_S^\perp \xi_j} T = A_{\nabla_T^\perp \xi_j} S$$

for any  $S, T \in \ker A_{\xi_j}$ . Being the vector subspaces  $\{\text{Im} A_{\xi_j}\}_{1 \leq j \leq n}$  orthogonal, we have

$$\langle \nabla_S^\perp \xi_j, \xi_k \rangle A_{\xi_k} T = \langle \nabla_T^\perp \xi_j, \xi_k \rangle A_{\xi_k} S, \quad 1 \leq j \neq k \leq n$$

for any  $S, T \in \ker A_{\xi_j}$ . Since  $\text{rank } A_{\xi_k} = 2$ , then the  $\xi_j$ 's are parallel in the normal connection along  $\ker A_{\xi_j}$ , and hence along  $TM_r$ .

Let  $Y \in E_j$  satisfy  $A_{\xi_j} Y = \lambda Y$ . Then the above Codazzi equation also gives

$$\lambda \nabla_S Y = A_{\xi_j}[S, Y] - S(\lambda)Y \in E_j$$

for any  $S \in \ker A_{\xi_j}$ ,  $1 \leq j \leq n$ . Thus, the subbundle  $E_j$  is parallel along  $\ker A_{\xi_j}$ . Hence, by Equation (36) and since  $\text{rank } A_{\xi_j} = 2$ , we have

$$\begin{aligned} \langle \nabla_{X_j} Y, X_k \rangle &= -\langle Y, \nabla_{X_j} X_k \rangle = 0, & \langle \nabla_{JX_j} Y, X_k \rangle &= -\langle Y, \nabla_{JX_j} X_k \rangle = 0, \\ \langle \nabla_{X_j} Y, JX_k \rangle &= -\langle Y, \nabla_{X_j} JX_k \rangle = 0, & \langle \nabla_{JX_j} Y, JX_k \rangle &= -\langle Y, \nabla_{JX_j} JX_k \rangle = 0 \quad \text{if } j \neq k, \end{aligned}$$

and therefore the  $E_j$ 's are parallel along  $M_r$ . Then by the de Rham theorem, there is an open subset  $U \subset M_r$ , which decomposes in a Riemannian product  $M_1^2 \times \dots \times M_n^2$  of surfaces. Since the codimension of  $g$  is  $n$  and  $\alpha^g(Y_i, Y_j) = 0$ , if  $Y_i \in E_i$  and  $Y_j \in E_j$ ,  $i \neq j$ , then by Theorem 8.7 in [4], there are isometric immersions  $g_1: M_1^2 \rightarrow \mathbb{L}^3$  and  $g_j: M_j^2 \rightarrow \mathbb{R}^3$ ,  $2 \leq j \leq n$ , such that

$$g|_U(x_1, \dots, x_n) = (g_1(x_1), \dots, g_n(x_n)).$$

Since  $g(M) \subset \mathbb{H}^{3n-1}$ , then  $\langle g, g \rangle = -1$ . Hence,  $\langle g_{j*} X_j, g_j \rangle = \langle g_* X_j, g \rangle = 0$ , and therefore,  $\|g_j\| = r_j$  with  $-r_1^2 + \sum_{j=2}^n r_j^2 = -1$ . □

We are now in conditions to give the remaining proofs of the results stated in the introduction.

**Proof of Theorem 2.** First observe that it suffices to prove that this result holds on some open subset of  $M^{2n}$ . By Theorem 3, we have that  $p = n - 1$ . In an open neighbourhood  $U$  of  $x_0$  in  $M^{2n}$ , there is a complex vector subbundle  $\bar{V} \subset TM$  such that  $\bar{V}(x_0) = V^{2m}$ ,  $m \geq n - 1$ , and that  $K(S, JS) > 0$  for any  $0 \neq S \in \bar{V}$ . Suppose that at  $y \in U$ , there is a  $J$ -invariant vector subspace  $V_1 \subset T_y U$  such that the subspace  $\mathcal{S}(\beta|_{V_1 \times V_1})$  is degenerate and  $2\bar{s} = \dim \mathcal{S}(\beta|_{V_1 \times V_1}) \leq \dim V_1 - 2$ . Then by part (iii) of Proposition 13, there exists a  $J$ -invariant vector  $P^{2m} \subset T_y U$  with  $m \geq n - \bar{s} + 1$  such that  $K(S, JS) \leq 0$  for any  $S \in P^{2m}$ . Now by Proposition 9, we have  $\bar{s} \leq p = n - 1$ .

Hence,  $m \geq 2$ , and this is a contradiction. Thus, the vector subspace  $\mathcal{S}(\beta)$  is non-degenerate. From Proposition 15, we obtain that  $s = n = p + 1$ , that is, that  $\beta$  is surjective. The proof now follows from Proposition 15 and Lemma 18.  $\square$

**Proof of Theorem 5.** Recall that  $\alpha^g(X, Y) = i_*\alpha^f(X, Y) + \langle X, Y \rangle g$ . Let  $U$  be the subset of points of  $M^{2n}$  such that at any  $x \in U$ , there is a  $J$ -invariant vector subspace  $V_1 \subset T_x M$  such that  $\bar{\beta} = \beta|_{V_1 \times V_1}$  satisfies that  $\mathcal{S}(\bar{\beta})$  is a degenerate subspace and  $\dim \mathcal{S}(\bar{\beta}) \leq \dim V_1 - 2$ . If the subspace  $\mathcal{S}(\beta(x))$  is degenerate for  $x \in M^{2n}$ , then  $x \in U$ . In fact, we have for  $s$  in Equation (6) that  $s < p + 1$  since, otherwise,  $\mathcal{S}(\beta(x)) = N_g M(x) \oplus N_g M(x)$ , and therefore this subspace would be non-degenerate. Having that  $p = n - 1$ , then  $\dim \mathcal{S}(\beta) = 2s \leq 2p = 2n - 2 = \dim T_x M - 2$ , and thus  $x \in U$ .

Under the above conditions, Proposition 13 applies to  $\bar{\beta}(x)$  for  $x \in U$ . Then by part (ii), there are a unit space-like vector  $i_*\eta(x) \perp g(x)$  defined by  $v = i_*\eta(x) + g(x)$  and the Lorentzian plane  $L = \text{span} \{i_*\eta(x), g(x)\}$  such that for  $P \subset N_f M(x)$  given by  $i_*P = L^\perp$ , we have

$$\alpha^f(X, Y)(x) = \alpha_P^f(X, Y)(x) + \langle X, Y \rangle \eta(x), \tag{38}$$

where  $i_*\alpha_P^f(X, Y) = \alpha_{L^\perp}^g(X, Y)$ . From Equation (15) applied to  $\bar{\beta}$ , we obtain that  $\bar{\beta}_1 = \pi_{L^\perp \times L^\perp} \circ \bar{\beta}$  satisfies  $\dim \mathcal{N}(\bar{\beta}_1(x)) \geq 4$ . If  $S \in \mathcal{N}(\bar{\beta}_1(x))$ , then Equation (14) gives

$$0 = \bar{\beta}_1(S, S) = (\alpha_{L^\perp}^g(S, S) + \alpha_{L^\perp}^g(JS, JS), 0).$$

It now follows from Equation (38) that for any  $x \in U$ , we have

$$\alpha^f(S, S) = \alpha_P^f(S, S) + \|S\|^2 \eta(x) \quad \text{and} \quad \alpha^f(JS, JS) = -\alpha_P^f(S, S) + \|S\|^2 \eta(x). \tag{39}$$

Suppose that there exists an open subset  $U_0 \subset M^{2n}$  such that  $U \cap U_0 = \emptyset$ . We have by Proposition 15, Lemma 18 and Theorem 2 that  $g = i \circ f: M^{2n} \rightarrow \mathbb{L}^{3n}$  is as given by the latter result. Then  $g(M)$  is unbounded due to completeness of  $M^{2n}$  and the presence of the hyperbolic factor, and hence also  $f(M)$  is unbounded.

By the above, it remains to consider the case when  $U$  is dense in  $M^{2n}$ . We denote by  $r: \mathbb{H}^{3n-1} \rightarrow [0, +\infty)$  the distance function in  $\mathbb{H}^{3n-1}$  to a reference point and set  $h = \cosh(r \circ f)$ . Then Equation (14) in [5] states that

$$\text{Hess } h(x)(X, X) = \cosh(r(f(x)))\|X\|^2 + \sinh(r(f(x)))\langle \text{grad } r(f(x)), \alpha^f(X, X) \rangle \tag{40}$$

for any  $X \in T_x M$ . On the other hand, we have from Equation (39) that

$$\langle \text{grad } r(f(x)), \alpha^f(S, S) \rangle = \langle \text{grad } r(f(x)), \alpha_P^f(S, S) \rangle + \langle \text{grad } r(f(x)), \eta(x) \rangle \|S\|^2$$

and

$$\langle \text{grad } r(f(x)), \alpha^f(JS, JS) \rangle = -\langle \text{grad } r(f(x)), \alpha_P^f(S, S) \rangle + \langle \text{grad } r(f(x)), \eta(x) \rangle \|S\|^2$$

for any  $x \in U$  and  $S \in \mathcal{N}(\bar{\beta}_1(x))$ . Then Equation (40) yields

$$\begin{aligned} \text{Hess } h(x)(S, S) + \text{Hess } h(x)(JS, JS) & \tag{41} \\ & = 2 \cosh(r(f(x))) + 2 \sinh(r(f(x))) \langle \text{grad } r(f(x)), \eta(x) \rangle \|S\|^2 \end{aligned}$$

for any  $x \in U$  and  $S \in \mathcal{N}(\bar{\beta}_1(x))$ .

If  $f(M)$  is bounded, by the Omori-Yau maximal principle for the Hessian, there exists a sequence  $\{x_k\}_{k \in \mathbb{N}}$  in  $M^{2n}$  such that

$$\lim h(x_k) = \sup\{h\} < +\infty \quad \text{and} \quad \text{Hess } h(x_k)(X_k, X_k) \leq \frac{1}{k} \|X_k\|^2 \quad \text{for any } X_k \in T_{x_k}M.$$

Let  $U_k \subset M^{2n}$  be an open neighbourhood containing  $x_k$  such that

$$|h(x_k) - h(y)| < \frac{1}{k} \quad \text{and} \quad \text{Hess } h(y)(X, X) \leq \frac{2}{k} \|X\|^2$$

for any  $y \in U_k$  and  $X \in T_yM$ . By the above, we have that  $U \cap U_k \neq \emptyset$  for any  $k \in \mathbb{N}$ . Then there exists a sequence  $y_k \in U_k \cap U$  such that  $r^* = \lim r(f(y_k)) > 0$  when  $k \mapsto +\infty$ . We obtain from Equation (41) that

$$\frac{2}{k} \geq \cosh r(f(y_k)) + \sinh r(f(y_k)) \langle \text{grad } r(f(y_k)), \eta(y_k) \rangle$$

Taking the limit as  $k \mapsto +\infty$  and using that  $\|\text{grad } r\| = 1$  gives

$$0 \geq \cosh r^* - \sinh r^* > 0,$$

and this is a contradiction. □

**Proof of Theorem 6.** Proposition 15 applies on an open subset  $U \subset M^{2n}$  where  $\mathcal{S}(\beta)$  is non-degenerate. We obtain that  $n = 3$  and that  $g(U)$  is as described by Lemma 18. Then the result follows from Theorem 2. Therefore, we may assume that  $\mathcal{S}(\beta)$  is degenerate at any point of  $M^{2n}$ . By Equation (7), the subspaces  $\mathcal{S}(\beta) \cap (\mathcal{S}(\beta))^\perp$  have dimension two and form a smooth vector subbundle. Hence, there is a smooth normal vector field  $v \in \Gamma(N_gM)$  such that  $\langle v, g \rangle = -1$  and Equation (21) holds. Let  $\eta \in \Gamma(N_fM)$  be the smooth unit vector field such that  $v = i_*\eta + g$ . Then Equation (21) gives  $A_\eta^f = I$ . The Codazzi equation yields that  $\eta$  is parallel in the normal connection along the open dense subset of non-flat points of  $M^{2n}$  and then on all of  $M^{2n}$ . Hence,  $f(M)$  is contained in a horosphere. □

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## References

- (1) L. Alías, P. Mastrolia and M. Rigoli, *Maximum principles and geometric applications* (Springer Monographs in Mathematics, 2016).
- (2) S. Chion and M. Dajczer, Minimal real Kähler submanifolds, *Mat. Contemp.* **49** (2022), no. 1, 236–250, doi:[10.21711/231766362022/rmc499](https://doi.org/10.21711/231766362022/rmc499).
- (3) M. Dajczer and L. Rodríguez, Rigidity of real Kähler submanifolds, *Duke Math. J.* **53** (1986), no. 1, 211–220, doi:[10.1215/S0012-7094-86-05314-7](https://doi.org/10.1215/S0012-7094-86-05314-7).
- (4) M. Dajczer and R. Tojeiro, *Submanifold theory: Beyond an introduction* (Springer, 2019).
- (5) M. Dajczer and T. Vlachos, Kähler submanifolds of hyperbolic space, *Proc. Amer. Math. Soc.* **148** no. 9, 4015–4024, doi:[10.1090/proc/15007](https://doi.org/10.1090/proc/15007).
- (6) M. do Carmo and M. Dajczer, Conformal Rigidity, *Amer. J. Math.* **109** (1987), no. 5, 963–985, doi:[10.2307/2374496](https://doi.org/10.2307/2374496).
- (7) L. Florit, W. Hui and F. Zheng, On real Kähler Euclidean submanifolds with non-negative Ricci curvature, *J. Eur. Math. Soc.* **7** (2005), no. 1, 1–11, doi:[10.4171/JEMS/19](https://doi.org/10.4171/JEMS/19).
- (8) P. Ryan, Kähler manifolds as real hypersurfaces, *Duke Math. J.* **40** (1973), no. 1, 207–213, doi:[10.1215/S0012-7094-73-04019-2](https://doi.org/10.1215/S0012-7094-73-04019-2).