COMPACT HANKEL OPERATORS ON WEIGHTED HARMONIC BERGMAN SPACES
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Abstract. We prove the compactness of certain Hankel operators on weighted Bergman spaces of harmonic functions on the unit ball in \( \mathbb{R}^n \).

1. Introduction. We denote the unit ball in \( \mathbb{R}^n \) by \( B_n \). Let \( w \) be a non-negative integrable function on the interval \([0,1)\), henceforth called a weight function, and consider the weighted Bergman space \( b_2^w(B_n) \) of harmonic functions \( u \) on \( B_n \) for which

\[
\|u\|_w = \left( \int_{B_n} |u(x)|^2 w(|x|) \, dV(x) \right)^{1/2} < \infty,
\]

where \( V \) denotes the usual Lebesgue volume measure. We shall show that under mild conditions on the weight function \( w \) the space \( b_2^w(B_n) \) is a closed linear subspace of \( L_2^w(B_n) \), the space of all square-integrable functions on \( B_n \) with respect to the measure \( w(|x|) \, dV(x) \), so that there exists an orthogonal projection \( Q_w \) of \( L_2^w(B_n) \) onto \( b_2^w(B_n) \). For a function \( f \in L^\infty(B_n) \) define the Hankel operator \( H_f : b_2^w(B_n) \rightarrow L_2^w(B_n) \) by

\[
H_f u = (I - Q_w)(fu), \quad u \in b_2^w(B_n).
\]

The operator \( H_f \) is clearly bounded on \( b_2^w(B_n) \) with \( \|H_f\| \leq \|f\|_\infty \). In this paper we prove that for every \( f \) continuous on the closed unit ball \( \overline{B}_n \) the operator \( H_f \) is compact on \( b_2^w(B_n) \), extending a recent result of M. Jovović [4] to the setting of weighted harmonic Bergman spaces.

In Section 2 we give the preliminaries for the paper. In Section 3 we discuss weighted Bergman spaces and the Bergman projection. In Section 4 we discuss Hankel operators and prove the above mentioned result. We furthermore show that these Hankel operators are in general not Hilbert–Schmidt.

2. Preliminaries. We recall that a twice-continuously differentiable function \( u \) on \( B_n \) is harmonic on \( B_n \) if \( \Delta u = 0 \), where \( \Delta = D_1^2 + \ldots + D_n^2 \) and \( D_j \) denotes the partial derivative with respect to the \( j \)-th coordinate. A polynomial on \( \mathbb{R}^n \) is homogeneous of degree \( m \) (or \( m \)-homogeneous) if it is a finite linear combination of monomials \( x_1^{\alpha_1} \ldots x_n^{\alpha_n} \), where \( \alpha_1, \ldots, \alpha_n \) are nonnegative integers such that \( \alpha_1 + \ldots + \alpha_n = m \). It is easy to show that a polynomial \( p \) on \( \mathbb{R}^n \) is homogeneous of degree \( m \) if and only if \( x \cdot \nabla p(x) = mp(x) \) for all \( x \in \mathbb{R}^n \), where \( \nabla \) denotes the gradient. Every harmonic function \( u \) on \( B_n \) can be decomposed as \( u = \sum_{k=0}^\infty u_k \), where each \( u_k \) is a harmonic homogeneous polynomial of degree \( k \), and the convergence is uniform on compact subsets of \( B_n \). Denote the unit sphere in \( \mathbb{R}^n \) by \( S_n \). The space \( \mathcal{H}_k(S_n) \) of restrictions to \( S_n \) of harmonic homogeneous

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polynomials of degree \( k \), the so-called spherical harmonics of degree \( k \), is a (finite-dimensional) Hilbert space with respect to the usual inner product on \( L^2(S^n,d\sigma) \), where \( \sigma \) denotes the normalized surface-area measure on \( S^n \). For each \( \eta \in S^n \) the linear functional \( p \mapsto p(\eta) \) on the space \( \mathcal{A}_k(S^n) \) is uniquely represented by a harmonic \( k \)-homogeneous polynomial \( Z_k(\cdot, \eta) \), called the zonal harmonic of degree \( k \) at \( \eta \). Extending \( Z_k \) to a function on \( \mathbb{R}^n \times \mathbb{R}^n \) by setting \( Z_k(x,y) = |y|^k Z_k(x,|y|) \), and using the fact that each zonal harmonic \( Z_k(\cdot, \eta) \) is real valued (see pages 78–79 in [1]) we have

\[
\int_{S^n} p(\xi) Z_k(\xi, y) d\sigma(\xi) = p(y),
\]

for every harmonic \( k \)-homogeneous polynomial \( p \). Denoting the dimension of \( \mathcal{A}_k(S^n) \) by \( h_k \), it is easily seen that \( Z_k(\eta, \eta) = h_k \), for all \( \eta \in S^n \), and thus \( Z_k(y, y) = |y|^{2k} h_k \), for all \( y \in \mathbb{R}^n \).

Spherical harmonics of distinct degrees are orthogonal; that is,

\[
\int_{S^n} p \bar{q} d\sigma = 0
\]

if \( p \) and \( q \) are harmonic homogeneous polynomials of distinct degree.

In the sequel the following theorem will play an important role.

**Theorem 2.2.** (Spherical Decomposition Theorem.) If \( p \) is a homogeneous polynomial of degree \( m \), then for each \( k = 0,1, \ldots, \lfloor m/2 \rfloor \) there exist a harmonic homogeneous polynomial \( p_{m-2k} \) of degree \( m - 2k \), such that

\[
p(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} p_{m-2k}(x).
\]

A constructive proof of the above theorem has recently been given in [3]. We observe that another constructive proof may be given as follows. It is elementary to show that for a harmonic \( j \)-homogeneous polynomial \( q \) we have

\[
\Delta[|x|^{2j} q] = 2i(n + 2j + 2i - 2) |x|^{2j-2} q.
\]

Assuming that \( \sum_{k=0}^{\lfloor m/2 \rfloor - 1} |x|^{2k} q_{m-2k-2} \) is the spherical decomposition of \( \Delta p \), it follows with the help of (2.3) that

\[
\Delta \left[ \sum_{k=1}^{\lfloor m/2 \rfloor} |x|^{2k} \frac{q_{m-2k}}{2k(n + 2m - 2k - 2)} \right] = \sum_{k=1}^{\lfloor m/2 \rfloor} |x|^{2k-2} q_{m-2k}
\]

\[
= \sum_{k=0}^{\lfloor m/2 \rfloor - 1} |x|^{2k} q_{m-2k-2} = \Delta p,
\]

so that

\[
p_m = p - \sum_{k=1}^{\lfloor m/2 \rfloor} |x|^{2k} \frac{q_{m-2k}}{2k(n + 2m - 2k - 2)}
\]

is a harmonic \( m \)-homogeneous polynomial, and thus \( p = \sum_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} p_{m-2k} \) is the spherical
decomposition of $p$, where $p_{m-2k} = q_{m-2k}/(2k(n+2m-2k-2))$ for $k \geq 1$. We shall use this idea in Section 4 to find explicit formulae for the norms of the Hankel operators associated with the coordinate functions.

3. Weighted harmonic Bergman spaces. For a weight function $w$ we introduce the moments

$$\hat{w}(k) = \int_{B_n} |x|^k w(|x|) \, dV(x), \quad (k = 0, 1, \ldots).$$

We shall assume that $\hat{w}(k) > 0$, for all $k = 0, 1, \ldots$. If $p$ and $q$ are homogeneous harmonic polynomials of degrees $k$ and $l$ respectively then, integrating in polar coordinates, it is easily seen that

$$\langle p, q \rangle_w = \begin{cases} \hat{w}(2k) \int_{S^n} p \bar{q} \, d\sigma, & \text{if } k = l, \\ 0 & \text{otherwise}. \end{cases} \quad (3.1)$$

If $u \in b_w^2(B_n)$ has decomposition $u = \sum_{k=0}^{\infty} u_k$, where each $u_k$ is an harmonic $k$-homogeneous polynomial, then it follows from (2.1) and (3.1) that

$$u_k(y) = \frac{1}{\hat{w}(2k)} \langle u_k, Z_k(\cdot, y) \rangle_w.$$

In particular,

$$\|Z_k(\cdot, y)\|_w^2 = \langle Z_k(\cdot, y), Z_k(\cdot, y) \rangle_w = \hat{w}(2k) Z_k(y, y) = \hat{w}(2k) h_k |y|^{2k}.$$ 

Applying the Cauchy-Schwarz inequality we obtain

$$|u_k(y)| \leq (1/\hat{w}(2k)) \|u_k\|_w \|Z_k(\cdot, y)\|_w,$$

and it follows that

$$|u(y)| \leq \sum_{k=0}^{\infty} 1/\hat{w}(2k) \|u_k\|_w \|Z_k(\cdot, y)\|_w$$

$$\leq \left( \sum_{k=0}^{\infty} \|u_k\|_w^2 \right)^{1/2} \left( \sum_{k=0}^{\infty} h_k/\hat{w}(2k) |y|^{2k} \right)^{1/2}.$$ 

We conclude that

$$|u(y)| \leq \|u\|_w \left( \sum_{k=0}^{\infty} h_k/\hat{w}(2k) |y|^{2k} \right)^{1/2}. \quad (3.2)$$

The numbers $h_k$ can be expressed in terms of binomial coefficients (see page 82 or 92 in [1]), and it is easily shown that $h_k \approx k^{n-2}$ as $k \to \infty$. The series $\sum_{k=0}^{\infty} (h_k/\hat{w}(2k)) |y|^{2k}$ has radius of convergence equal to 1, and thus converges uniformly for $|y| \leq r < 1$, for each $0 < r < 1$, if

$$\limsup_{k \to \infty} 1/\sqrt[k]{\hat{w}(2k)} = 1. \quad (3.3)$$

It follows from (3.2) that $b_w^2(B_n)$ is a closed subspace of $L_w^2(B_n)$ if the weight function
satisfies (3.3). Using Exercise 3.4 of [5] it is easily shown that condition (3.3) is equivalent to the requirement that, for all $0 < \delta < 1$, the set \( \{ r \in (\delta, 1) : w(r) > 0 \} \) has positive measure. In the sequel we assume that this condition is satisfied, so that \( b^2_w(B_n) \) is a closed linear subspace of \( L^2_w(B_n) \).

Furthermore, by uniform convergence and orthogonality of homogeneous harmonic polynomials of distinct degree, for each $0 < r < 1$ we have

\[
\int_{S_n} |u(r\xi)|^2 \, d\sigma(\xi) = \sum_{k=0}^{\infty} \int_{S_n} |u_k(r\xi)|^2 \, d\sigma(\xi),
\]

and integrating in polar coordinates we obtain

\[
\|u\|^2_w = \sum_{k=0}^{\infty} \|u_k\|^2_w. \tag{3.4}
\]

Applying formula (3.4) to the function \( u - \sum_{k=0}^{m} u_k = \sum_{k=m+1}^{\infty} u_k \) we obtain

\[
\left\| u - \sum_{k=0}^{m} u_k \right\|^2_w = \sum_{k=m+1}^{\infty} \|u_k\|^2_w.
\]

Thus \( \sum_{k=0}^{m} u_k \to u \) in \( b^2_w(B_n) \) as \( m \to \infty \). Hence the harmonic polynomials are dense in \( b^2_w(B_n) \).

Also, if \( p \) and \( q \) are harmonic homogeneous polynomials of degrees \( k \) and \( l \), respectively, then

\[
\langle |x|^2 p, q \rangle_w = nV(B) \int_0^1 r^{n+2k+2j-1} w(r) \, dr \int_{S_n} pq \, d\sigma
\]

\[
= \hat{w}(2k + 2j) \int_{S_n} pq \, d\sigma,
\]

and thus

\[
\langle |x|^2 p, q \rangle_w = \frac{\hat{w}(2k + 2j)}{\hat{w}(2k)} \langle p, q \rangle_w. \tag{3.5}
\]

It follows from (3.5) and the fact that the harmonic polynomials are dense in \( b^2_w(B_n) \) that

\[
Q_w[|x|^2 p] = \frac{\hat{w}(2k + 2j)}{\hat{w}(2k)} p, \tag{3.6}
\]

for every harmonic homogeneous polynomial \( p \) of degree \( k \).

The following result shows that the Bergman projection of a polynomial is a harmonic polynomial of degree less than or equal to that of the original polynomial.

**Theorem 3.7.** If an \( m \)-homogeneous polynomial \( p \) has spherical decomposition given by \( p(x) = \sum_{k=0}^{[m/2]} |x|^{2k} p_{m-2k}(x) \), then

\[
Q_w[p] = \sum_{k=0}^{[m/2]} \frac{\hat{w}(2m - 2k)}{\hat{w}(2m - 4k)} p_{m-2k}.
\]
Proof. If \( p = \sum_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} p_{m-2k} \) is the spherical decomposition of \( p \), then by linearity and (3.6)

\[
Q_w[p] = \sum_{k=0}^{\lfloor m/2 \rfloor} Q_w[|x|^{2k} p_{m-2k}] = \sum_{k=0}^{\lfloor m/2 \rfloor} \hat{w}(2m-2k) \hat{w}(2m-4k) p_{m-2k},
\]

proving the result. \( \square \)

**Corollary 3.8.** Let \( w(r) = (1 - r^2)^\lambda \), where \( -1 < \lambda < \infty \). If an \( m \)-homogeneous polynomial \( p \) has spherical decomposition given by \( p(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} p_{m-2k}(x) \), then the projection \( Q_\lambda[p] \) of \( p \) onto \( b_\lambda^w(B_n) \) is given by

\[
Q_\lambda[p] = \sum_{k=0}^{\lfloor m/2 \rfloor} \prod_{j=1}^{k} \frac{n + 2(m - 2k) + 2j - 2}{n + 2(m - 2k) + 2j + 2\lambda} p_{m-2k}.
\]

**Proof.** An elementary calculation shows that

\[
\hat{w}(2j) = \frac{n}{2} V(B_n) \frac{\Gamma\left(\frac{n}{2} + j\right) \Gamma(\lambda + 1)}{\Gamma\left(j + \frac{n}{2} + \lambda + 1\right)},
\]

and thus

\[
\hat{w}(2j) = \frac{n + 2j - 2}{n + 2j + 2\lambda} \hat{w}(2j - 2),
\]

for \( j \geq 1 \). This implies that

\[
\frac{\hat{w}(2m - 2k)}{\hat{w}(2m - 4k)} = \prod_{j=1}^{k} \frac{\hat{w}(2m - 4k + 2j)}{\hat{w}(2m - 4k + 2j - 2)} = \prod_{j=1}^{k} \frac{n + 2(m - 2k) + 2j - 2}{n + 2(m - 2k) + 2j + 2\lambda}
\]

and the stated result follows from the above theorem. \( \square \)

**Remarks.** 1. Note that as \( \lambda \to -1^+ \), \( Q_\lambda[p] \) converges to the Poisson integral of

\[
p : \sum_{k=0}^{\lfloor m/2 \rfloor} p_{m-2k}.
\]

2. If \( \lambda = 0 \), then

\[
Q_0[p] = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{n + 2m - 4k}{n + 2m - 2k} p_{m-2k},
\]

as in [3].

**4. Hankel operators.** Let \( w \) be a weight function satisfying condition (3.3). We shall consider the Hankel operator \( H_{x_1} \) on \( b_\lambda^w(B_n) \). Let \( p \) be a harmonic \( m \)-homogeneous polynomial on \( \mathbb{R}^n \), where \( m \geq 1 \). Then \( \Delta(x_1 p) = 2D_1 p(x) \). Since \( x_1 p \) is homogeneous of degree \( m + 1 \), it follows that \( x_1 p \) has spherical decomposition given by

\[
x_1 p = p_{m+1} + |x|^2 p_{m-1},
\]
with
\[ p_{m-1}(x) = \frac{1}{n+2m-2} D_1 p(x), \quad \text{and} \quad p_{m+1}(x) = x_1 p(x) - |x|^2 p_{m-1}(x). \]
Consequently
\[ Q_w[x_1 p] = p_{m+1} + \frac{\tilde{\omega}(2m)}{\tilde{\omega}(2m-2)} p_{m-1} \]
\[ = x_1 p - |x|^2 \frac{1}{n+2m-2} D_1 p + \frac{\tilde{\omega}(2m)}{(n+2m-2)\tilde{\omega}(2m-2)} D_1 p. \]
Hence
\[ H_{x_1} p = \frac{1}{n+2m-2} \left\{ |x|^2 D_1 p - \frac{\tilde{\omega}(2m)}{\tilde{\omega}(2m-2)} D_1 p \right\}. \quad (4.1) \]
If \( q \) is a harmonic homogeneous polynomial of degree \( k \), then
\[ \langle H_{x_1} p, H_{x_1} q \rangle_w = \langle H_{x_1} p, x_1 q \rangle_w \]
\[ = \frac{1}{n+2m-2} \left\{ \langle |x|^2 D_1 p, x_1 q \rangle_w - \frac{\tilde{\omega}(2m)}{\tilde{\omega}(2m-2)} \langle D_1 p, x_1 q \rangle_w \right\} \]
\[ = \frac{1}{n+2m-2} \left\{ \langle x_1 D_1 p, |x|^2 q \rangle_w - \frac{\tilde{\omega}(2m)}{\tilde{\omega}(2m-2)} \langle x_1 D_1 p, q \rangle_w \right\}. \]
Similar formulae hold for \( \langle H_{x_1} p, H_{x_j} q \rangle_w, \quad (j = 2, \ldots, n) \). Adding these formulae, and making use of \( \sum_{j=1}^{n} x_j D_j p = mp \), we obtain
\[ \sum_{j=1}^{n} \langle H_{x_j} p, H_{x_j} q \rangle_w = \frac{m}{n+2m-2} \left\{ \langle p, |x|^2 q \rangle_w - \frac{\tilde{\omega}(2m)}{\tilde{\omega}(2m-2)} \langle p, q \rangle_w \right\}. \]
It follows that
\[ \sum_{j=1}^{n} \langle H_{x_j} p, H_{x_j} q \rangle_w = \frac{m}{n+2m-2} \left\{ \langle |x|^2 p, q \rangle_w - \frac{\tilde{\omega}(2m)}{\tilde{\omega}(2m-2)} \langle p, q \rangle_w \right\} \]
\[ = \frac{m}{n+2m-2} \left\{ \frac{\tilde{\omega}(2m+2)}{\tilde{\omega}(2m)} \langle p, q \rangle_w - \frac{\tilde{\omega}(2m)}{\tilde{\omega}(2m-2)} \langle p, q \rangle_w \right\} \]
\[ = \frac{m}{n+2m-2} \left\{ \frac{\tilde{\omega}(2m+2)}{\tilde{\omega}(2m)} - \frac{\tilde{\omega}(2m)}{\tilde{\omega}(2m-2)} \right\} \langle p, q \rangle_w. \]
It is easy to prove that the operators \( H_{x_1}, \ldots, H_{x_n} \) are unitarily equivalent on \( b_w^*(B_n) \). In fact, if \( 1 < j \leq n \) and \( U_j \) is the mapping defined on \( L^2_w(B_n) \) by \( (U_j g)(x) = g(\bar{x}) \), where \( \bar{x} \) is the vector obtained from \( x \) by interchanging its first and \( j \)th coordinate, then \( U_j \) is a unitary operator on \( L^2_w(B_n) \) mapping \( b_w^*(B_n) \) into itself, and \( H_j U_j g = U_j H_{x_j} g \), for all \( g \in b_w^*(B_n) \) (which is easily verified by using (4.1) and the analogous formula for \( H_{x_j} \)). In particular, we have
\[ \| H_{x_1} p \|^2_w = \frac{m}{n(n+2m-2)} \left( \frac{\tilde{\omega}(2m+2)}{\tilde{\omega}(2m)} - \frac{\tilde{\omega}(2m)}{\tilde{\omega}(2m-2)} \right) \| p \|^2_w, \quad (4.2) \]
for every harmonic $m$-homogeneous polynomial $p$ with $m \geq 1$.

Note that (4.2) implies that $\hat{w}(2m+2)/\hat{w}(2m) \geq \hat{w}(2m)/\hat{w}(2m-2)$, which can also be verified directly using the Cauchy-Schwarz inequality: also $\hat{w}(2m)^2 \leq \hat{w}(2m-2)\hat{w}(2m+2)$. It follows from (3.3) that $\lim_{m \to \infty} \frac{\hat{w}(2m+2)}{\hat{w}(2m)} = 1$. That $H_{x_1}$ is compact on $b_w^2(B)$ is proved as follows. Write $V_k$ for the space of all harmonic polynomials of degree at most $k$. Let $S_k$ denote the operators defined on $b_w^2(B_n)$ such that $S_k p = H_{x_1} p$ if $p \in V_k$ and $S_k p = 0$ if $p \in b_w^2(B) \ominus V_m$. We shall estimate $\|H_{x_1} - S_k\|$. Write $u = \sum_{m=0}^\infty u_m$, where each $u_m$ is a harmonic $m$-homogeneous polynomial. Then, using (4.2), Cauchy-Schwarz and (3.4), we have

$$\| (H_{x_1} - S_k) u \|_w \leq \sum_{m=k+1}^\infty \| H_{x_1} u_m \|_w$$

$$\leq \sum_{m=k+1}^\infty \left\{ \frac{m}{n(n+2m-2)} \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2} \| u_m \|_w$$

$$\leq \frac{1}{2} \left\{ \sum_{m=k+1}^\infty \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2} \left\{ \sum_{m=k+1}^\infty \| u_m \|_w \right\}$$

Hence

$$\|H_{x_1} - S_k\| \leq \frac{1}{2} \left\{ \sum_{m=k+1}^\infty \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2}$$

where $\rho_k = \hat{w}(2k+2)/\hat{w}(2k)$, and it follows that $S_k \to H_{x_1}$ as $k \to \infty$. Since each of the $S_k$ is of finite rank, the operator $H_{x_1}$ must be compact on $b_w^2(B_n)$. In fact, we have the following result.

**Theorem 4.3.** Let $w$ be a weight function satisfying (3.3). Then, for every $f$ in $C(\bar{B}_n)$, the Hankel operator $H_f$ is compact on $b_w^2(B_n)$.

**Proof.** That $\mathcal{A} = \{ f \in C(\bar{B}_n) : H_f$ is compact on $b_w^2(B_n) \}$ is a closed algebra can be proved by the same argument as given in [2]. We have just shown that $H_{x_1}$ is compact on $b_w^2(B_n)$ and, since each of the operators $H_{x_j}$ is unitarily equivalent to $H_{x_1}$, we conclude that $x_j \in \mathcal{A}$, for each $j$. This implies that $\mathcal{A}$ contains all polynomials and by the Stone-Weierstrass Theorem $\mathcal{A} = C(\bar{B}_n)$.

It is interesting to note that the Hankel operator $H_{x_1}$ is in general not Hilbert-Schmidt. In fact, we have the following result, similar to the situation on the weighted Bergman spaces of analytic functions on the unit ball in $\mathbb{C}^n$. (See [6].) It shows that for $n > 2$ the Hankel operator $H_{x_1}$ is not Hilbert-Schmidt on $b_w^2(B_n)$ for the indicated weight functions $w$.

**Theorem 4.4.** Let $w(r) = (1 - r^2)^\lambda$, where $-1 < \lambda < \infty$. Then $H_{x_1}$ does not belong to the Schatten $\gamma$-class of $b_w^2(B_n)$ if $\gamma \leq n - 1$. 


Proof. For $2 \leq \gamma < \infty$ we have the inequality
\[
\langle (H_{x_1}^* H_{x_1})^{\gamma/2} p, p \rangle_w \geq \langle H_{x_1}^* H_{x_1}, p, p \rangle_w^{\gamma/2},
\]
for every $p \in b_2^2(B_n)$ of unit norm (by Proposition 6.3.3 in [7]), and it follows from (4.2) that
\[
\langle (H_{x_1}^* H_{x_1})^{\gamma/2} p, p \rangle_w \geq \left\{ \frac{m}{n(n + 2m - 2)} \left( \frac{\hat{w}(2m + 2)}{\hat{w}(2m)} - \hat{w}(2m) \right) \right\}^{\gamma/2},
\]
for every $p \in b_2^2(B_n)$ of unit norm. Summing over an orthonormal set $h_m$ of $m$-homogeneous harmonic polynomials, and subsequently summing over all $m \geq 1$ we obtain
\[
\| H_{x_1} \|_\gamma^\gamma = \text{trace}((H_{x_1}^* H_{x_1})^{\gamma/2}) = \sum_{m=1}^\infty \left\{ \frac{m}{n(n + 2m - 2)} \left( \frac{\hat{w}(2m + 2)}{\hat{w}(2m)} - \hat{w}(2m) \right) \right\}^{\gamma/2} h_m.
\]
Using (3.9) we have
\[
\| H_{x_1} \|_\gamma^\gamma \geq \sum_{m=1}^\infty \left\{ \frac{4(\lambda + 1)m}{n(n + 2m - 2)(n + 2m + 2\lambda + 2)(n + 2m + 2\lambda)} \right\}^{\gamma/2} h_m.
\]
Since $h_m \approx m^{n-2}$, the assumption that $H_{x_1}$ belongs to the Schatten $\gamma$-class, implies that
\[
\sum_{m=1}^\infty m^{n-2-\gamma} < \infty,
\]
and thus $\gamma > n - 1$. \(\square\)

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