COMPACT HANKEL OPERATORS ON WEIGHTED HARMONIC BERGMAN SPACES

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Abstract. We prove the compactness of certain Hankel operators on weighted Bergman spaces of harmonic functions on the unit ball in $\mathbb{R}^n$.

1. Introduction. We denote the unit ball in $\mathbb{R}^n$ by $B_n$. Let $w$ be a non-negative integrable function on the interval $[0, 1)$, henceforth called a weight function, and consider the weighted Bergman space $b^2_w(B_n)$ of harmonic functions $u$ on $B_n$ for which

$$\|u\|_w = \left( \int_{B_n} |u(x)|^2 w(|x|) \, dV(x) \right)^{1/2} < \infty,$$

where $V$ denotes the usual Lebesgue volume measure. We shall show that under mild conditions on the weight function $w$ the space $b^2_w(B_n)$ is a closed linear subspace of $L^2_w(B_n)$, the space of all square-integrable functions on $B_n$ with respect to the measure $w(|x|) \, dV(x)$, so that there exists an orthogonal projection $Q_w$ of $L^2_w(B_n)$ onto $b^2_w(B_n)$. For a function $f \in L^\infty(B_n)$ define the Hankel operator $H_f : b^2_w(B_n) \to L^2_w(B_n)$ by

$$H_f u = (I - Q_w)(fu), \quad u \in b^2_w(B_n).$$

The operator $H_f$ is clearly bounded on $b^2_w(B_n)$ with $\|H_f\| \leq \|f\|_\infty$. In this paper we prove that for every $f$ continuous on the closed unit ball $\overline{B}_n$ the operator $H_f$ is compact on $b^2_w(B_n)$, extending a recent result of M. Jovović [4] to the setting of weighted harmonic Bergman spaces.

In Section 2 we give the preliminaries for the paper. In Section 3 we discuss weighted Bergman spaces and the Bergman projection. In Section 4 we discuss Hankel operators and prove the above mentioned result. We furthermore show that these Hankel operators are in general not Hilbert–Schmidt.

2. Preliminaries. We recall that a twice-continuously differentiable function $u$ on $B_n$ is harmonic on $B_n$ if $\Delta u = 0$, where $\Delta = D_1^2 + \ldots + D_n^2$ and $D_j$ denotes the partial derivative with respect to the $j$-th coordinate. A polynomial on $\mathbb{R}^n$ is homogeneous of degree $m$ (or $m$-homogeneous) if it is a finite linear combination of monomials $x_1^{a_1} \ldots x_n^{a_n}$, where $a_1, \ldots, a_n$ are nonnegative integers such that $a_1 + \ldots + a_n = m$. It is easy to show that a polynomial $p$ on $\mathbb{R}^n$ is homogeneous of degree $m$ if and only if $x \cdot \nabla p(x) = mp(x)$ for all $x \in \mathbb{R}^n$, where $\nabla$ denotes the gradient. Every harmonic function $u$ on $B_n$ can be decomposed as $u = \sum_{k=0}^\infty u_k$, where each $u_k$ is a harmonic homogeneous polynomial of degree $k$, and the convergence is uniform on compact subsets of $B_n$. Denote the unit sphere in $\mathbb{R}^n$ by $S_n$. The space $\mathcal{H}_k(S_n)$ of restrictions to $S_n$ of harmonic homogeneous

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polynomials of degree $k$, the so-called *spherical harmonics* of degree $k$, is a (finite-dimensional) Hilbert space with respect to the usual inner product on $L^2(S_n, d\sigma)$, where $\sigma$ denotes the normalized surface-area measure on $S_n$. For each $\eta \in S_n$ the linear functional $p \mapsto p(\eta)$ on the space $\mathcal{H}_k(S_n)$ is uniquely represented by a harmonic $k$-homogeneous polynomial $Z_k(\cdot, \eta)$, called the *zonal harmonic of degree $k$ at $\eta$*. Extending $Z_k$ to a function on $\mathbb{R}^n \times \mathbb{R}^n$ by setting $Z_k(x, y) = |y|^k Z_k(x, y/|y|)$, and using the fact that each zonal harmonic $Z_k(\cdot, \eta)$ is real valued (see pages 78–79 in [1]) we have

$$\int_{S_n} p(\xi)Z_k(\xi, y) d\sigma(\xi) = p(y), \quad (2.1)$$

for every harmonic $k$-homogeneous polynomial $p$. Denoting the dimension of $\mathcal{H}_k(S_n)$ by $h_k$, it is easily seen that $Z_k(\eta, \eta) = h_k$, for all $\eta \in S_n$, and thus $Z_k(y, y) = |y|^{2k} h_k$, for all $y \in \mathbb{R}^n$.

Spherical harmonics of distinct degrees are orthogonal; that is,

$$\int_{S_n} p \overline{q} d\sigma = 0$$

if $p$ and $q$ are harmonic homogeneous polynomials of distinct degree.

In the sequel the following theorem will play an important role.

**Theorem 2.2. (Spherical Decomposition Theorem.)** If $p$ is a homogeneous polynomial of degree $m$, then for each $k = 0, 1, \ldots, \lfloor m/2 \rfloor$ there exist a harmonic homogeneous polynomial $p_{m-2k}$ of degree $m-2k$, such that

$$p(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} p_{m-2k}(x).$$

A constructive proof of the above theorem has recently been given in [3]. We observe that another constructive proof may be given as follows. It is elementary to show that for a harmonic $j$-homogeneous polynomial $q$ we have

$$\Delta[|x|^{2j} q] = 2i(n + 2j + 2i - 2) |x|^{2j-2} q. \quad (2.3)$$

Assuming that $\sum_{k=0}^{\lfloor m/2 \rfloor-1} |x|^{2k} q_{m-2k-2}$ is the spherical decomposition of $\Delta p$, it follows with the help of (2.3) that

$$\Delta \left[ \sum_{k=1}^{\lfloor m/2 \rfloor} |x|^{2k} \frac{q_{m-2k}}{2k(n + 2m - 2k - 2)} \right] = \sum_{k=1}^{\lfloor m/2 \rfloor} |x|^{2k-2} q_{m-2k}$$

$$= \sum_{k=0}^{\lfloor m/2 \rfloor-1} |x|^{2k} q_{m-2k-2} = \Delta p,$$

so that

$$p_m = p - \sum_{k=1}^{\lfloor m/2 \rfloor} |x|^{2k} \frac{q_{m-2k}}{2k(n + 2m - 2k - 2)}$$

is a harmonic $m$-homogeneous polynomial, and thus $p = \sum_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} p_{m-2k}$ is the spherical
decomposition of $p$, where $p_{m-2k} = q_{m-2k}/(2k(n + 2m - 2k - 2))$ for $k \geq 1$. We shall use this idea in Section 4 to find explicit formulae for the norms of the Hankel operators associated with the coordinate functions.

3. Weighted harmonic Bergman spaces. For a weight function $w$ we introduce the moments

$$
\hat{w}(k) = \int_{B_n} |x|^k w(|x|) \, dV(x), \quad (k = 0, 1, \ldots).
$$

We shall assume that $\hat{w}(k) > 0$, for all $k = 0, 1, \ldots$. If $p$ and $q$ are homogeneous harmonic polynomials of degrees $k$ and $l$ respectively then, integrating in polar coordinates, it is easily seen that

$$
\langle p, q \rangle_w = \begin{cases} 
\hat{w}(2k) \int_S p \bar{q} \, d\sigma, & \text{if } k = l, \\
0, & \text{otherwise.}
\end{cases} \quad (3.1)
$$

If $u \in b^2_w(B_n)$ has decomposition $u = \sum_{k=0}^{\infty} u_k$, where each $u_k$ is an harmonic $k$-homogeneous polynomial, then it follows from (2.1) and (3.1) that

$$
u_k(y) = \frac{1}{\hat{w}(2k)} \langle u_k, Z_k(\cdot, y) \rangle_w.
$$

In particular,

$$
\|Z_k(\cdot, y)\|_w^2 = \langle Z_k(\cdot, y), Z_k(\cdot, y) \rangle_w = \hat{w}(2k) Z_k(y, y) = \hat{w}(2k) h_k |y|^{2k}.
$$

Applying the Cauchy-Schwarz inequality we obtain

$$|u_k(y)| \leq (1/\hat{w}(2k)) \|u_k\|_w \|Z_k(\cdot, y)\|_w,$$

and it follows that

$$|u(y)| \leq \sum_{k=0}^{\infty} \frac{1}{\hat{w}(2k)} \|u_k\|_w \|Z_k(\cdot, y)\|_w
$$

$$\leq \left( \sum_{k=0}^{\infty} \|u_k\|_w^2 \right)^{1/2} \left( \sum_{k=0}^{\infty} \frac{h_k}{\hat{w}(2k)} |y|^{2k} \right)^{1/2}.
$$

We conclude that

$$|u(y)| \leq \|u\|_w \left( \sum_{k=0}^{\infty} \frac{h_k}{\hat{w}(2k)} |y|^{2k} \right)^{1/2}. \quad (3.2)
$$

The numbers $h_k$ can be expressed in terms of binomial coefficients (see page 82 or 92 in [1]), and it is easily shown that $h_k \approx k^{n-2}$ as $k \to \infty$. The series $\sum_{k=0}^{\infty} (h_k/\hat{w}(2k)) |y|^{2k}$ has radius of convergence equal to 1, and thus converges uniformly for $|y| \leq r < 1$, for each $0 < r < 1$, if

$$\limsup_{k \to \infty} \sqrt[k]{\hat{w}(2k)} = 1. \quad (3.3)
$$

It follows from (3.2) that $b^2_w(B_n)$ is a closed subspace of $L^2_w(B_n)$ if the weight function
satisfies (3.3). Using Exercise 3.4 of [5] it is easily shown that condition (3.3) is equivalent to the requirement that, for all \( 0 < \delta < 1 \), the set \( \{ r \in (\delta, 1) : w(r) > 0 \} \) has positive measure. In the sequel we assume that this condition is satisfied, so that \( b_w^2(B_n) \) is a closed linear subspace of \( L_w^2(B_n) \).

Furthermore, by uniform convergence and orthogonality of homogeneous harmonic polynomials of distinct degree, for each \( 0 < r < 1 \) we have

\[
\int_{S_n} |u(r\xi)|^2 d\sigma(\xi) = \sum_{k=0}^{\infty} \int_{S_n} |u_k(r\xi)|^2 d\sigma(\xi),
\]

and integrating in polar coordinates we obtain

\[
\|u\|^2_w = \sum_{k=0}^{\infty} \|u_k\|^2_w. \tag{3.4}
\]

Applying formula (3.4) to the function \( u - \sum_{k=0}^{m} u_k = \sum_{k=m+1}^{\infty} u_k \) we obtain

\[
\left\| u - \sum_{k=0}^{m} u_k \right\|^2_w = \sum_{k=m+1}^{\infty} \|u_k\|^2_w.
\]

Thus \( \sum_{k=0}^{m} u_k \to u \) in \( b_w^2(B_n) \) as \( m \to \infty \). Hence the harmonic polynomials are dense in \( b_w^2(B_n) \).

Also, if \( p \) and \( q \) are harmonic homogeneous polynomials of degrees \( k \) and \( l \), respectively, then

\[
\langle |x|^{2j} p, q \rangle_w = n V(B) \int_0^1 r^{n+2k+2l-1} w(r) \, dr \int_{S_n} p \bar{q} \, d\sigma
\]

\[
= \hat{w}(2k + 2j) \int_{S_n} p \bar{q} \, d\sigma,
\]

and thus

\[
\langle |x|^{2j} p, q \rangle_w = \frac{\hat{w}(2k + 2j)}{\hat{w}(2k)} \langle p, q \rangle_w. \tag{3.5}
\]

It follows from (3.5) and the fact that the harmonic polynomials are dense in \( b_w^2(B_n) \) that

\[
Q_w[|x|^{2j} p] = \frac{\hat{w}(2k + 2j)}{\hat{w}(2k)} p,
\]

for every harmonic homogeneous polynomial \( p \) of degree \( k \).

The following result shows that the Bergman projection of a polynomial is a harmonic polynomial of degree less than or equal to that of the original polynomial.

**Theorem 3.7.** If an \( m \)-homogeneous polynomial \( p \) has spherical decomposition given by \( p(x) = \sum_{k=0}^{[m/2]} |x|^{2k} p_{m-2k}(x) \), then

\[
Q_w[p] = \sum_{k=0}^{[m/2]} \frac{\hat{w}(2m - 2k)}{\hat{w}(2m - 4k)} p_{m-2k}.
\]
Proof. If \( p = \sum_{k=0}^{[m/2]} |x|^{2k} p_{m-2k} \) is the spherical decomposition of \( p \), then by linearity and (3.6)
\[
Q_w[p] = \sum_{k=0}^{[m/2]} Q_w[|x|^{2k} p_{m-2k}] = \sum_{k=0}^{[m/2]} \hat{w}(2m-2k) p_{m-2k},
\]
proving the result. \( \square \)

Corollary 3.8. Let \( w(r) = (1 - r^2)^\lambda \), where \(-1 < \lambda < \infty\). If an \( m \)-homogeneous polynomial \( p \) has spherical decomposition given by \( p(x) = \sum_{k=0}^{[m/2]} |x|^{2k} p_{m-2k}(x) \), then the projection \( Q_\lambda[p] \) of \( p \) onto \( b^2(B_n) \) is given by
\[
Q_\lambda[p] = \sum_{k=0}^{[m/2]} \prod_{j=1}^k \frac{n + 2(m - 2k) + 2j - 2}{n + 2(m - 2k) + 2j + 2\lambda} p_{m-2k}.
\]

Proof. An elementary calculation shows that
\[
\hat{w}(2j) = \frac{n}{2} \cdot \frac{\Gamma \left( n + \frac{j}{2} + 1 \right) \Gamma (\lambda + 1)}{\Gamma \left( j + \frac{n}{2} + \lambda + 1 \right)},
\]
and thus
\[
\hat{w}(2j) = \frac{n + 2j - 2}{n + 2j + 2\lambda} \hat{w}(2j - 2), \quad (3.9)
\]
for \( j \geq 1 \). This implies that
\[
\frac{\hat{w}(2m - 2k)}{\hat{w}(2m - 4k)} = \prod_{j=1}^k \frac{\hat{w}(2m - 4k + 2j)}{\hat{w}(2m - 4k + 2j - 2)} = \prod_{j=1}^k \frac{n + 2(m - 2k) + 2j - 2}{n + 2(m - 2k) + 2j + 2\lambda}.
\]
and the stated result follows from the above theorem. \( \square \)

Remarks. 1. Note that as \( \lambda \to -1^+ \), \( Q_\lambda[p] \) converges to the Poisson integral of \( p : \sum_{k=0}^{[m/2]} p_{m-2k} \).

2. If \( \lambda = 0 \), then
\[
Q_0[p] = \sum_{k=0}^{[m/2]} \frac{n + 2m - 4k}{n + 2m - 2k} p_{m-2k},
\]
as in [3].

4. Hankel operators. Let \( w \) be a weight function satisfying condition (3.3). We shall consider the Hankel operator \( H_{x_1} \) on \( b^2(B_n) \). Let \( p \) be a harmonic \( m \)-homogeneous polynomial on \( \mathbb{R}^n \), where \( m \geq 1 \). Then \( \Delta(x_1,p) = 2D_1 p(x) \). Since \( x_1 p \) is homogeneous of degree \( m + 1 \), it follows that \( x_1 p \) has spherical decomposition given by
\[
x_1 p = p_{m+1} + |x|^2 p_{m-1},
\]
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with
\[ p_{m-1}(x) = \frac{1}{n + 2m - 2} D_1 p(x), \quad \text{and} \quad p_{m+1}(x) = x^1 p(x) - |x|^2 p_{m-1}(x). \]

Consequently
\[ Q_w[x_1 p] = p_{m+1} + \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} p_{m-1} \]
\[ = x_1 p - |x|^2 \frac{1}{n + 2m - 2} D_1 p + \frac{\hat{\omega}(2m)}{(n + 2m - 2) \hat{\omega}(2m - 2)} D_1 p. \]

Hence
\[ H_s, p = \frac{1}{n + 2m - 2} \left\{ |x|^2 D_1 p - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} D_1 p \right\}. \tag{4.1} \]

If \( q \) is a harmonic homogeneous polynomial of degree \( k \), then
\[ \langle H_s, p, H_s, q \rangle_w = \langle H_s, p, x_1 q \rangle_w \]
\[ = \frac{1}{n + 2m - 2} \left\{ \langle |x|^2 D_1 p, x_1 q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle D_1 p, x_1 q \rangle_w \right\} \]
\[ = \frac{1}{n + 2m - 2} \left\{ \langle x_1 D_1 p, |x|^2 q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle x_1 D_1 p, q \rangle_w \right\}. \]

Similar formulae hold for \( \langle H_s, p, H_s, q \rangle_w, \ (j = 2, \ldots, n) \). Adding these formulae, and making use of \( \sum_{j=1}^n x_j D_j p = mp \), we obtain
\[ \sum_{j=1}^n \langle H_s, p, H_s, q \rangle_w = \frac{m}{n + 2m - 2} \left\{ \langle p, |x|^2 q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle p, q \rangle_w \right\}. \]

It follows that
\[ \sum_{j=1}^n \langle H_s, p, H_s, q \rangle_w = \frac{m}{n + 2m - 2} \left\{ \langle |x|^2 p, q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle p, q \rangle_w \right\} \]
\[ = \frac{m}{n + 2m - 2} \left\{ \frac{\hat{\omega}(2m + 2)}{\hat{\omega}(2m)} \langle p, q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle p, q \rangle_w \right\} \]
\[ = \frac{m}{n + 2m - 2} \left\{ \frac{\hat{\omega}(2m + 2)}{\hat{\omega}(2m)} \langle p, q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle p, q \rangle_w \right\}. \]

It is easy to prove that the operators \( H_{x_1}, \ldots, H_{x_n} \) are unitarily equivalent on \( b^2_w(B_n) \). In fact, if \( 1 < j \leq n \) and \( U_j \) is the mapping defined on \( L^2_w(B_n) \) by \((U_j g)(x) = g(x)\), where \( \bar{x} \) is the vector obtained from \( x \) by interchanging its first and \( j \)th coordinate, then \( U_j \) is a unitary operator on \( L^2_w(B_n) \) mapping \( b^2_w(B_n) \) into itself, and \( H_s, U_j g = U_j H_s, g \), for all \( g \in b^2_w(B_n) \) (which is easily verified by using (4.1) and the analogous formula for \( H_s \)). In particular, we have
\[ \| H_s, p \|_w^2 = \frac{m}{n(n + 2m - 2)} \left( \frac{\hat{\omega}(2m + 2)}{\hat{\omega}(2m)} - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \right) \| p \|_w^2. \tag{4.2} \]
for every harmonic $m$-homogeneous polynomial $p$ with $m \geq 1$.

Note that (4.2) implies that $\hat{w}(2m+2)/\hat{w}(2m) \geq \hat{w}(2m)/\hat{w}(2m-2)$, which can also be verified directly using the Cauchy-Schwarz inequality: also $\hat{w}(2m)^2 \leq \hat{w}(2m-2)\hat{w}(2m+2)$. It follows from (3.3) that $\lim_{m \to \infty} \hat{w}(2m+2)/\hat{w}(2m) = 1$. That $H_{x_1}$ is compact on $b_w^2(B)$ is proved as follows. Write $\mathcal{V}_k$ for the space of all harmonic polynomials of degree at most $k$. Let $S_k$ denote the operators defined on $b_w^2(B_n)$ such that $S_k p = H_{x_1} p$ if $p \in \mathcal{V}_k$ and $S_k p = 0$ if $p \in b_w^2(B) \ominus \mathcal{V}_k$. We shall estimate $\|H_{x_1} - S_k\|$. Write

$$u = \sum_{m=0}^\infty u_m,$$

where each $u_m$ is a harmonic $m$-homogeneous polynomial. Then, using (4.2), Cauchy-Schwarz and (3.4), we have

$$\|H_{x_1} - S_k\| = \sum_{m=0}^\infty \|H_{x_1} u_m\|_w$$

$$\leq \sum_{m=k+1}^\infty \left\{ \frac{m}{n(n+2m-2)} \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2} \|u_m\|_w$$

$$\leq \frac{1}{2} \left\{ \sum_{m=k+1}^\infty \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2} \left\{ \sum_{m=k+1}^\infty \|u_m\|_w^2 \right\}^{1/2}$$

$$\leq \frac{1}{2} \left\{ \sum_{m=k+1}^\infty \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2} \|u\|_w$$

Hence

$$\|H_{x_1} - S_k\| \leq \frac{1}{2} \left\{ \sum_{m=k+1}^\infty \left( \frac{\hat{w}(2m+2)}{\hat{w}(2m)} - \frac{\hat{w}(2m)}{\hat{w}(2m-2)} \right) \right\}^{1/2}$$

$$\leq (1 - \rho_k)^{1/2},$$

where $\rho_k = \hat{w}(2k+2)/\hat{w}(2k)$, and it follows that $S_k \to H_{x_1}$ as $k \to \infty$. Since each of the $S_k$ is of finite rank, the operator $H_{x_1}$ must be compact on $b_w^2(B_n)$. In fact, we have the following result.

**Theorem 4.3.** Let $w$ be a weight function satisfying (3.3). Then, for every $f$ in $C(\B_n)$, the Hankel operator $H_f$ is compact on $b_w^2(B_n)$.

**Proof.** That $\mathcal{A} = \{ f \in C(\B_n) : H_f$ is compact on $b_w^2(B_n)\}$ is a closed algebra can be proved by the same argument as given in [2]. We have just shown that $H_{x_1}$ is compact on $b_w^2(B_n)$ and, since each of the operators $H_{x_j}$ is unitarily equivalent to $H_{x_1}$, we conclude that $x_j \in \mathcal{A}$, for each $j$. This implies that $\mathcal{A}$ contains all polynomials and by the Stone-Weierstrass Theorem $\mathcal{A} = C(\B_n)$. \qed

It is interesting to note that the Hankel operator $H_{x_1}$ is in general not Hilbert-Schmidt. In fact, we have the following result, similar to the situation on the weighted Bergman spaces of analytic functions on the unit ball in $\mathbb{C}^n$. (See [6].) It shows that for $n > 2$ the Hankel operator $H_{x_1}$ is not Hilbert-Schmidt on $b_w^2(B_n)$ for the indicated weight functions $w$.

**Theorem 4.4.** Let $w(r) = (1 - r^2)^\lambda$, where $-1 < \lambda < \infty$. Then $H_{x_1}$ does not belong to the Schatten $\gamma$-class of $b_w^2(B_n)$ if $\gamma \leq n - 1$. 


Proof. For $2 \leq \gamma < \infty$ we have the inequality

$$
\langle (H_{x_1}^* H_{x_1})^{\gamma/2} p, p \rangle_w \geq \langle H_{x_1}^* H_{x_1}, p, p \rangle_w^{\gamma/2},
$$

for every $p \in b_w^2(B_n)$ of unit norm (by Proposition 6.3.3 in [7]), and it follows from (4.2) that

$$
\langle (H_{x_1}^* H_{x_1})^{\gamma/2} p, p \rangle_w \geq \left\{ \frac{m}{n(n+2m-2)} \left( \frac{\dot{w}(2m+2)}{\dot{w}(2m)} - \frac{\dot{w}(2m)}{\dot{w}(2m-2)} \right) \right\}^{\gamma/2},
$$

for every $p \in b_w^2(B_n)$ of unit norm. Summing over an orthonormal set $h_m$ of $m$-homogeneous harmonic polynomials, and subsequently summing over all $m \geq 1$ we obtain

$$
\| H_{x_1} \|_\gamma^\gamma = \text{trace}((H_{x_1}^* H_{x_1})^{\gamma/2}) \\
\geq \sum_{m=1}^\infty \left\{ \frac{m}{n(n+2m-2)} \left( \frac{\dot{w}(2m+2)}{\dot{w}(2m)} - \frac{\dot{w}(2m)}{\dot{w}(2m-2)} \right) \right\}^{\gamma/2} h_m.
$$

Using (3.9) we have

$$
\| H_{x_1} \|_\gamma^\gamma \geq \sum_{m=1}^\infty \left\{ \frac{4(\lambda+1)m}{n(n+2m-2)(n+2m+2\lambda+2)(n+2m+2\lambda)} \right\}^{\gamma/2} h_m.
$$

Since $h_m \approx m^{n-2}$, the assumption that $H_{x_1}$ belongs to the Schatten $\gamma$-class, implies that

$$
\sum_{m=1}^\infty m^{n-2-\gamma} < \infty,
$$

and thus $\gamma > n - 1$. 

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