COMPACT HANKEL OPERATORS ON WEIGHTED HARMONIC
BERGMAN SPACES

by KAREL STROETHOFF†

(Received 10 August, 1995)

Abstract. We prove the compactness of certain Hankel operators on weighted
Bergman spaces of harmonic functions on the unit ball in \( \mathbb{R}^n \).

1. Introduction. We denote the unit ball in \( \mathbb{R}^n \) by \( B_n \). Let \( w \) be a non-negative
integrable function on the interval [0,1), henceforth called a weight function,
and consider
the weighted Bergman space \( b^2_w(B_n) \) of harmonic functions \( u \) on \( B_n \) for which
\[
\|u\|_w = \left( \int_{B_n} |u(x)|^2 w(|x|) \ dV(x) \right)^{1/2} < \infty,
\]
where \( V \) denotes the usual Lebesgue volume measure. We shall show that under mild
conditions on the weight function \( w \) the space \( b^2_w(B_n) \) is a closed linear subspace of
\( L^2_w(B_n) \), the space of all square-integrable functions on \( B_n \) with respect to the measure
\( w(|x|) \ dV(x) \), so that there exists an orthogonal projection \( Q_w \) of \( L^2_w(B_n) \) onto \( b^2_w(B_n) \). For
a function \( f \in L^\infty(B_n) \) define the Hankel operator \( H_f : b^2_w(B_n) \rightarrow L^2_w(B_n) \) by
\[
H_f u = (I - Q_w)(fu), \quad u \in b^2_w(B_n).
\]
The operator \( H_f \) is clearly bounded on \( b^2_w(B_n) \) with \( \|H_f\| \leq \|f\|_\infty \). In this paper we prove
that for every \( f \) continuous on the closed unit ball \( \bar{B}_n \) the operator \( H_f \) is compact on
\( b^2_w(B_n) \), extending a recent result of M. Jovović [4] to the setting of weighted harmonic
Bergman spaces.

In Section 2 we give the preliminaries for the paper. In Section 3 we discuss weighted
Bergman spaces and the Bergman projection. In Section 4 we discuss Hankel operators
and prove the above mentioned result. We furthermore show that these Hankel operators
are in general not Hilbert–Schmidt.

2. Preliminaries. We recall that a twice-continuously differentiable function \( u \) on \( B_n \)
is harmonic on \( B_n \) if \( \Delta u = 0 \), where \( \Delta = D_1^2 + \ldots + D_n^2 \) and \( D_j \) denotes the partial
derivative with respect to the \( j \)-th coordinate. A polynomial on \( \mathbb{R}^n \) is homogeneous of
degree \( m \) (or \( m \)-homogeneous) if it is a finite linear combination of monomials \( x_1^{a_1} \ldots x_n^{a_n} \),
where \( \alpha_1, \ldots, \alpha_n \) are nonnegative integers such that \( \alpha_1 + \ldots + \alpha_n = m \). It is easy to show
that a polynomial \( p \) on \( \mathbb{R}^n \) is homogeneous of degree \( m \) if and only if \( x \cdot \nabla p(x) = mp(x) \)
for all \( x \in \mathbb{R}^n \), where \( \nabla \) denotes the gradient. Every harmonic function \( u \) on \( B_n \) can be
decomposed as \( u = \sum_{k=0}^\infty u_k \), where each \( u_k \) is a harmonic homogeneous polynomial of
degree \( k \), and the convergence is uniform on compact subsets of \( B_n \). Denote the unit
sphere in \( \mathbb{R}^n \) by \( S_n \). The space \( \mathcal{H}_k(S_n) \) of restrictions to \( S_n \) of harmonic homogeneous

† The author was partially supported by grants from the Montana University System and the University of
Montana.

polynomials of degree \( k \), the so-called spherical harmonics of degree \( k \), is a (finite-
dimensional) Hilbert space with respect to the usual inner product on \( L^2(S_n, d\sigma) \), where \( \sigma \) denotes the normalized surface-area measure on \( S_n \). For each \( \eta \in S_n \) the linear functional \( p \mapsto p(\eta) \) on the space \( H_k(S_n) \) is uniquely represented by a harmonic \( k \)-homogeneous polynomial \( Z_k(\cdot, \eta) \), called the zonal harmonic of degree \( k \) at \( \eta \). Extending \( Z_k \) to a function on \( \mathbb{R}^n \times \mathbb{R}^n \) by setting \( Z_k(x, y) = |y|^k Z_k(x, y/|y|) \), and using the fact that each zonal harmonic \( Z_k(\cdot, \eta) \) is real valued (see pages 78–79 in [1]) we have

\[
\int_{S_n} p(\xi) Z_k(\xi, y) d\sigma(\xi) = p(y),
\]

for every harmonic \( k \)-homogeneous polynomial \( p \). Denoting the dimension of \( H_k(S_n) \) by \( h_k \), it is easily seen that \( Z_k(\eta, \eta) = h_k \), for all \( \eta \in S_n \), and thus \( Z_k(y, y) = |y|^{2k} h_k \), for all \( y \in \mathbb{R}^n \).

Spherical harmonics of distinct degrees are orthogonal; that is,

\[
\int_{S_n} p q d\sigma = 0
\]

if \( p \) and \( q \) are harmonic homogeneous polynomials of distinct degree.

In the sequel the following theorem will play an important role.

**Theorem 2.2.** (Spherical Decomposition Theorem.) If \( p \) is a homogeneous polynomial of degree \( m \), then for each \( k = 0, 1, \ldots, \lfloor m/2 \rfloor \) there exist a harmonic homogeneous polynomial \( p_{m-2k} \) of degree \( m - 2k \), such that

\[
p(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} p_{m-2k}(x).
\]

A constructive proof of the above theorem has recently been given in [3]. We observe that another constructive proof may be given as follows. It is elementary to show that for a harmonic \( j \)-homogeneous polynomial \( q \) we have

\[
\Delta[|x|^{2j} q] = 2i(n + 2j + 2i - 2) |x|^{2j-2} q.
\]

Assuming that \( \sum_{k=0}^{\lfloor m/2 \rfloor-1} |x|^{2k} q_{m-2k-2} \) is the spherical decomposition of \( \Delta p \), it follows with the help of (2.3) that

\[
\Delta \left[ \sum_{k=1}^{\lfloor m/2 \rfloor} |x|^{2k} \frac{q_{m-2k}}{2k(n+2m-2k-2)} \right] = \sum_{k=1}^{\lfloor m/2 \rfloor} |x|^{2k-2} q_{m-2k}
\]

\[
= \sum_{k=0}^{\lfloor m/2 \rfloor-1} |x|^{2k} q_{m-2k-2} = \Delta p,
\]

so that

\[
p_m = p - \sum_{k=1}^{\lfloor m/2 \rfloor} |x|^{2k} \frac{q_{m-2k}}{2k(n+2m-2k-2)}
\]

is a harmonic \( m \)-homogeneous polynomial, and thus \( p = \sum_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} p_{m-2k} \) is the spherical
decomposition of $p$, where
\[ p_{m-2k} = q_{m-2k}/(2k(n + 2m - 2k - 2)) \]
for $k \geq 1$. We shall use this idea in Section 4 to find explicit formulae for the norms of the Hankel operators associated with the coordinate functions.

3. Weighted harmonic Bergman spaces. For a weight function $w$ we introduce the moments
\[ \hat{w}(k) = \int_{B_n} |x|^k w(|x|) \, dV(x), \quad (k = 0, 1, \ldots). \]
We shall assume that $\hat{w}(k) > 0$, for all $k = 0, 1, \ldots$. If $p$ and $q$ are homogeneous harmonic polynomials of degrees $k$ and $l$ respectively then, integrating in polar coordinates, it is easily seen that
\[ \langle p, q \rangle_w = \begin{cases} \hat{w}(2k) \int_{S_n} p \tilde{q} \, d\sigma, & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases} \quad (3.1) \]
If $u \in b^2_w(B_n)$ has decomposition $u = \sum_{k=0}^{\infty} u_k$, where each $u_k$ is an harmonic $k$-homogeneous polynomial, then it follows from (2.1) and (3.1) that
\[ u_k(y) = \frac{1}{\hat{w}(2k)} \langle u_k, Z_k(\cdot, y) \rangle_w. \]
In particular,
\[ \|Z_k(\cdot, y)\|_w^2 = \langle Z_k(\cdot, y), Z_k(\cdot, y) \rangle_w = \hat{w}(2k)Z_k(y, y) = \hat{w}(2k)h_k |y|^{2k}. \]
Applying the Cauchy-Schwarz inequality we obtain
\[ |u_k(y)| \leq (1/\hat{w}(2k)) \|u_k\|_w \|Z_k(\cdot, y)\|_w, \]
and it follows that
\[ |u(y)| \leq \sum_{k=0}^{\infty} \frac{1}{\hat{w}(2k)} \|u_k\|_w \|Z_k(\cdot, y)\|_w \]
\[ \leq \left( \sum_{k=0}^{\infty} \|u_k\|_w^2 \right)^{1/2} \left( \sum_{k=0}^{\infty} \frac{h_k}{\hat{w}(2k)} |y|^{2k} \right)^{1/2}. \]
We conclude that
\[ |u(y)| \leq \|u\|_w \left( \sum_{k=0}^{\infty} \frac{h_k}{\hat{w}(2k)} |y|^{2k} \right)^{1/2}. \quad (3.2) \]
The numbers $h_k$ can be expressed in terms of binomial coefficients (see page 82 or 92 in [1]), and it is easily shown that $h_k \approx k^{n-2}$ as $k \to \infty$. The series $\sum_{k=0}^{\infty} (h_k/\hat{w}(2k)) |y|^{2k}$ has radius of convergence equal to 1, and thus converges uniformly for $|y| \leq r < 1$, for each $0 < r < 1$, if
\[ \limsup_{k \to \infty} 1/\sqrt{k} = 1. \quad (3.3) \]
It follows from (3.2) that $b^2_w(B_n)$ is a closed subspace of $L^2_w(B_n)$ if the weight function
satisfies (3.3). Using Exercise 3.4 of [5] it is easily shown that condition (3.3) is equivalent to the requirement that, for all \(0 < \delta < 1\), the set \(\{r \in (\delta, 1) : w(r) > 0\}\) has positive measure. In the sequel we assume that this condition is satisfied, so that \(b_w^2(B_n)\) is a closed linear subspace of \(L_w^2(B_n)\).

Furthermore, by uniform convergence and orthogonality of homogeneous harmonic polynomials of distinct degree, for each \(0 < r < 1\) we have

\[
\int_{S_n} |u(r\xi)|^2 d\sigma(\xi) = \sum_{k=0}^{\infty} \int_{S_n} |u_k(r\xi)|^2 d\sigma(\xi),
\]

and integrating in polar coordinates we obtain

\[
\|u\|^2 = \sum_{k=0}^{\infty} \|u_k\|^2_w. \tag{3.4}
\]

Applying formula (3.4) to the function \(u - \sum_{k=0}^{m} u_k = \sum_{k=m+1}^{\infty} u_k\) we obtain

\[
\left\| u - \sum_{k=0}^{m} u_k \right\|^2_w = \sum_{k=m+1}^{\infty} \|u_k\|^2_w.
\]

Thus \(\sum_{k=0}^{m} u_k \to u\) in \(b_w^2(B_n)\) as \(m \to \infty\). Hence the harmonic polynomials are dense in \(b_w^2(B_n)\).

Also, if \(p\) and \(q\) are harmonic homogeneous polynomials of degrees \(k\) and \(l\), respectively, then

\[
\langle |x|^{2j} p, q \rangle_w = n V(B) \int_0^1 r^{n+2k+2l-1} w(r) \, dr \int_{S_n} p\bar{q} \, d\sigma
\]

\[
= \hat{w}(2k+2j) \int_{S_n} p\bar{q} \, d\sigma,
\]

and thus

\[
\langle |x|^{2j} p, q \rangle_w = \frac{\hat{w}(2k+2j)}{\hat{w}(2k)} \langle p, q \rangle_w. \tag{3.5}
\]

It follows from (3.5) and the fact that the harmonic polynomials are dense in \(b_w^2(B_n)\) that

\[
Q_w[|x|^{2j} p] = \frac{\hat{w}(2k+2j)}{\hat{w}(2k)} p, \tag{3.6}
\]

for every harmonic homogeneous polynomial \(p\) of degree \(k\).

The following result shows that the Bergman projection of a polynomial is a harmonic polynomial of degree less than or equal to that of the original polynomial.

**Theorem 3.7.** If an \(m\)-homogeneous polynomial \(p\) has spherical decomposition given by \(p(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} p_{m-2k}(x)\), then

\[
Q_w[p] = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{\hat{w}(2m-2k)}{\hat{w}(2m-4k)} p_{m-2k}.
\]
COMPACT HANKEL OPERATORS

Proof. If \( p = \sum_{k=0}^{m/2} |x|^{2k} p_{m-2k} \) is the spherical decomposition of \( p \), then by linearity and (3.6)

\[
Q_w[p] = \sum_{k=0}^{m/2} Q_w[|x|^{2k} p_{m-2k}] = \sum_{k=0}^{m/2} \tilde{w}(2m - 2k) p_{m-2k},
\]

proving the result. \( \square \)

**Corollary 3.8.** Let \( w(r) = (1-r^2)^\lambda \), where \(-1 < \lambda < \infty\). If an \( m \)-homogeneous polynomial \( p \) has spherical decomposition given by \( p(x) = \sum_{k=0}^{m/2} |x|^{2k} p_{m-2k}(x) \), then the projection \( Q_\lambda[p] \) of \( p \) onto \( b_w^*(B_n) \) is given by

\[
Q_\lambda[p] = \sum_{k=0}^{m/2} \prod_{j=1}^{k} \frac{n + 2(m - 2k) + 2j - 1}{n + 2(m - 2k) + 2j + 2\lambda} p_{m-2k}.
\]

**Proof.** An elementary calculation shows that

\[
\tilde{w}(2j) = \frac{n}{2} V(B_n) \frac{\Gamma\left(\frac{n}{2} + j\right) \Gamma(\lambda + 1)}{\Gamma\left(\frac{n}{2} + \lambda + 1\right)},
\]

and thus

\[
\tilde{w}(2j) = \frac{n + 2j - 2}{n + 2j + 2\lambda} \tilde{w}(2j - 2), \tag{3.9}
\]

for \( j \geq 1 \). This implies that

\[
\frac{\tilde{w}(2m - 2k)}{\tilde{w}(2m - 4k)} = \prod_{j=1}^{k} \frac{\tilde{w}(2m - 4k + 2j)}{\tilde{w}(2m - 4k + 2j - 2)} = \prod_{j=1}^{k} \frac{n + 2(m - 2k) + 2j - 1}{n + 2(m - 2k) + 2j + 2\lambda},
\]

and the stated result follows from the above theorem. \( \square \)

**Remarks.** 1. Note that as \( \lambda \to -1^+ \), \( Q_\lambda[p] \) converges to the Poisson integral of \( p = \sum_{k=0}^{m/2} p_{m-2k} \).

2. If \( \lambda = 0 \), then

\[
Q_0[p] = \sum_{k=0}^{m/2} \frac{n + 2m - 4k}{n + 2m - 2k} p_{m-2k},
\]

as in [3].

**4. Hankel operators.** Let \( w \) be a weight function satisfying condition (3.3). We shall consider the Hankel operator \( H_{x_1} \) on \( b_w^*(B_n) \). Let \( p \) be a harmonic \( m \)-homogeneous polynomial on \( \mathbb{R}^n \), where \( m \geq 1 \). Then \( \Delta(x_1, p) = 2D_1 p(x) \). Since \( x_1 p \) is homogeneous of degree \( m + 1 \), it follows that \( x_1 p \) has spherical decomposition given by

\[
x_1 p = p_{m+1} + |x|^2 p_{m-1},
\]
with

\[ p_{m-1}(x) = \frac{1}{n + 2m - 2} D_1 p(x), \quad \text{and} \quad p_{m+1}(x) = x_1 p(x) - |x|^2 p_{m-1}(x). \]

Consequently

\[ Q_w[x_1 p] = p_{m+1} + \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} p_{m-1} \]

\[ = x_1 p - |x|^2 \left( \frac{1}{n + 2m - 2} D_1 p + \frac{\hat{\omega}(2m)}{(n + 2m - 2)\hat{\omega}(2m - 2)} D_1 p. \right) \]

Hence

\[ H_{x_1} p = \frac{1}{n + 2m - 2} \left\{ |x|^2 D_1 p - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} D_1 p \right\}. \quad (4.1) \]

If \( q \) is a harmonic homogeneous polynomial of degree \( k \), then

\[ \langle H_{x_1} p, H_{x_1} q \rangle_w = \langle H_{x_1} p, x_1 q \rangle_w \]

\[ = \frac{1}{n + 2m - 2} \left\{ \langle |x|^2 D_1 p, x_1 q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle D_1 p, x_1 q \rangle_w \right\} \]

\[ = \frac{1}{n + 2m - 2} \left\{ \langle x_1 D_1 p, |x|^2 q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle x_1 D_1 p, q \rangle_w \right\}. \]

Similar formulae hold for \( \langle H_{x_1} p, H_{x_1} q \rangle_w \), \( (j = 2, \ldots, n) \). Adding these formulae, and making use of \( \sum_{j=1}^{n} x_j D_j p = mp \), we obtain

\[ \sum_{j=1}^{n} \langle H_{x_1} p, H_{x_1} q \rangle_w = \frac{m}{n + 2m - 2} \left\{ \langle p, |x|^2 q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle p, q \rangle_w \right\}. \]

It follows that

\[ \sum_{j=1}^{n} \langle H_{x_1} p, H_{x_1} q \rangle_w = \frac{m}{n + 2m - 2} \left\{ \langle |x|^2 p, q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle p, q \rangle_w \right\} \]

\[ = \frac{m}{n + 2m - 2} \left\{ \frac{\hat{\omega}(2m + 2)}{\hat{\omega}(2m)} \langle p, q \rangle_w - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \langle p, q \rangle_w \right\} \]

\[ = \frac{m}{n + 2m - 2} \left\{ \frac{\hat{\omega}(2m + 2)}{\hat{\omega}(2m)} - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \right\} \langle p, q \rangle_w. \]

It is easy to prove that the operators \( H_{x_1}, \ldots, H_{x_n} \) are unitarily equivalent on \( b^2_w(B_n) \). In fact, if \( 1 < j \leq n \) and \( U_j \) is the mapping defined on \( L^2_w(B_n) \) by \( (U_j g)(x) = g(\bar{x}) \), where \( \bar{x} \) is the vector obtained from \( x \) by interchanging its first and \( j \)th coordinate, then \( U_j \) is a unitary operator on \( L^2_w(B_n) \) mapping \( b^2_w(B_n) \) into itself, and \( H_j U_j g = U_j H_j g \), for all \( g \in b^2_w(B_n) \) (which is easily verified by using (4.1) and the analogous formula for \( H_{x_j} \)). In particular, we have

\[ \| H_{x_1} p \|^2_w = \frac{m}{n(n + 2m - 2)} \left( \frac{\hat{\omega}(2m + 2)}{\hat{\omega}(2m)} - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m - 2)} \right) \| p \|^2_w, \quad (4.2) \]
for every harmonic $m$-homogeneous polynomial $p$ with $m \geq 1$.

Note that (4.2) implies that $\hat{\omega}(2m + 2)/\hat{\omega}(2m) \geq \hat{\omega}(2m - 2)/\hat{\omega}(2m - 2)$, which can also be verified directly using the Cauchy-Schwarz inequality: also $\hat{\omega}(2m)^2 \leq \hat{\omega}(2m - 2)\hat{\omega}(2m - 2)$. It follows from (3.3) that $\lim_{m \to \infty} \hat{\omega}(2m + 2)/\hat{\omega}(2m) = 1$. That $H_{x_j}$ is compact on $b^2_w(B)$ is proved as follows. Write $\mathcal{V}_k$ for the space of all harmonic polynomials of degree at most $k$. Let $S_k$ denote the operators defined on $b^2_w(B_n)$ such that $S_k p = H_{x_j} p$ if $p \in \mathcal{V}_k$ and $S_k p = 0$ if $p \in b^2_w(B) \ominus \mathcal{V}_m$. We shall estimate $\|H_{x_j} - S_k\|$. Write $u = \sum_{m=0}^{\infty} u_m$, where each $u_m$ is a harmonic $m$-homogeneous polynomial. Then, using (4.2), Cauchy-Schwarz and (3.4), we have

$$\|H_{x_j} - S_k\| = \sum_{m=k+1}^{\infty} \|H_{x_j} u_m\|_w \leq \frac{1}{2} \left\{ \sum_{m=k+1}^{\infty} \left( \frac{\hat{\omega}(2m + 2)}{\hat{\omega}(2m)} - \frac{\hat{\omega}(2m - 2)}{\hat{\omega}(2m)} \right) \right\} \left\{ \sum_{m=k+1}^{\infty} \|u_m\|_w \right\}^{1/2} \leq \frac{1}{2} \left( \frac{\hat{\omega}(2m + 2)}{\hat{\omega}(2m)} - \frac{\hat{\omega}(2m - 2)}{\hat{\omega}(2m)} \right) \leq 1 - \rho_k$$

where $\rho_k = \hat{\omega}(2k + 2)/\hat{\omega}(2k)$, and it follows that $S_k \to H_{x_j}$ as $k \to \infty$. Since each of the $S_k$ is of finite rank, the operator $H_{x_j}$ must be compact on $b^2_w(B_n)$. In fact, we have the following result.

**Theorem 4.3.** Let $w$ be a weight function satisfying (3.3). Then, for every $f$ in $C(\overline{B}_n)$, the Hankel operator $H_f$ is compact on $b^2_w(B_n)$.

**Proof.** That $\mathcal{A} = \{ f \in C(\overline{B}_n) : H_f$ is compact on $b^2_w(B_n) \}$ is a closed algebra can be proved by the same argument as given in [2]. We have just shown that $H_{x_j}$ is compact on $b^2_w(B_n)$ and, since each of the operators $H_{x_j}$ is unitarily equivalent to $H_{x_j}$, we conclude that $x_j \in \mathcal{A}$, for each $j$. This implies that $\mathcal{A}$ contains all polynomials and by the Stone-Weierstrass Theorem $\mathcal{A} = C(\overline{B}_n)$.

It is interesting to note that the Hankel operator $H_{x_j}$ is in general not Hilbert-Schmidt. In fact, we have the following result, similar to the situation on the weighted Bergman spaces of analytic functions on the unit ball in $\mathbb{C}^n$. (See [6].) It shows that for $n > 2$ the Hankel operator $H_{x_j}$ is not Hilbert-Schmidt on $b^2_w(B_n)$ for the indicated weight functions $w$.

**Theorem 4.4.** Let $w(r) = (1 - r^2)^\lambda$, where $-1 < \lambda < \infty$. Then $H_{x_j}$ does not belong to the Schatten $\gamma$-class of $b^2_w(B_n)$ if $\gamma \leq n - 1$. 

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 19 Aug 2021 at 04:11:39, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089500031931
Proof. For $2 \leq \gamma < \infty$ we have the inequality
\[
\langle (H_{x_1}^* H_{x_1})^{\gamma/2} p, p \rangle_w \geq \langle H_{x_1}^* H_{x_1}, p, p \rangle_w^{\gamma/2},
\]
for every $p \in b_{\gamma}^w(B_n)$ of unit norm (by Proposition 6.3.3 in [7]), and it follows from (4.2) that
\[
\langle (H_{x_1}^* H_{x_1})^{\gamma/2} p, p \rangle_w \geq \left\{ \frac{m}{n(n+2m-2)} \left( \frac{\hat{\omega}(2m+2)}{\hat{\omega}(2m)} - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m-2)} \right) \right\}^{\gamma/2},
\]
for every $p \in b_{\gamma}^w(B_n)$ of unit norm. Summing over an orthonormal set $h_m$ of $m$-homogeneous harmonic polynomials, and subsequently summing over all $m \geq 1$ we obtain
\[
\|H_{x_1}\|_{\gamma}^\gamma = \text{trace}( (H_{x_1}^* H_{x_1})^{\gamma/2} )
\geq \sum_{m=1}^\infty \left\{ \frac{m}{n(n+2m-2)} \left( \frac{\hat{\omega}(2m+2)}{\hat{\omega}(2m)} - \frac{\hat{\omega}(2m)}{\hat{\omega}(2m-2)} \right) \right\}^{\gamma/2} h_m.
\]
Using (3.9) we have
\[
\|H_{x_1}\|_{\gamma}^\gamma \geq \sum_{m=1}^\infty \left\{ \frac{4(\lambda+1)m}{n(n+2m-2)(n+2m+2\lambda+2)(n+2m+2\lambda)} \right\}^{\gamma/2} h_m.
\]
Since $h_m = m^{-\lambda-1}$, the assumption that $H_{x_1}$ belongs to the Schatten $\gamma$-class, implies that
\[
\sum_{m=1}^\infty m^{-\lambda-1-\gamma} < \infty,
\]
and thus $\gamma > n - 1$.

Acknowledgements. This article was written while visiting the Free University, Amsterdam, The Netherlands, on sabbatical leave. I thank the University of Montana for awarding me a sabbatical and the Mathematics Department of the Free University for its hospitality and support.

REFERENCES


Department of Mathematical Sciences
University of Montana
Missoula
MT 59812-1032, USA
e-mail: ma_kms@selway.umt.edu