On the semigroup of all continuous linear mappings on a Banach space

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It is well-known that every ring automorphism of the ring of all linear transformations of a real vector space into itself is inner. We shall show that, if this ring is regarded as a semigroup with respect to composition and the dimension of the vector space is not less than 2, every semigroup automorphism is inner.

Let $E$ be a real Banach space and $L(E)$ be the set of all continuous linear mappings of $E$ into itself. With pointwise addition and composition, $L(E)$ is a ring, and, with the usual upper bound norm, it is a Banach algebra. In the sequel, we assume that the dimension of $E$ is not 1.

M. Eidelheitt [1] has shown that every algebraic automorphism of this ring is inner; in other words, if $\phi$ is a one-to-one mapping of $L(E)$ onto itself such that

1. $\phi(f+g) = \phi(f) + \phi(g)$ for all $f, g \in L(E)$

and

2. $\phi(fg) = \phi(f) \cdot \phi(g)$ for all $f, g \in L(E)$,

then there exists an invertible $h \in L(E)$ such that

3. $\phi(f) = hfh^{-1}$ for every $f \in L(E)$.

Eidelheitt also proved in the same paper that, if $L(E)$ is regarded as a semigroup with respect to the composition, then every continuous semigroup automorphism is inner. He has done so by showing that the continuity and the condition (2) imply (1).

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In this note, we shall show that the continuity is not necessary, that is, every semigroup automorphism of $L(E)$ is a ring automorphism.

**THEOREM.** Every automorphism of the semigroup $L(E)$ is inner.

Proof. Eidelheit proved that, if $\phi$ is a semigroup automorphism, there exists a one-to-one mapping $h$ of $E$ onto itself such that the condition (3) is satisfied and, moreover,

$$h(\xi x) = \mu(\xi)h(x)$$

for any $x \in E$ and real number $\xi$, where $\mu$ is a one-to-one mapping of $R$ (the set of all real numbers) onto $R$ such that $\mu(1) = 1$ and $\mu(\xi \eta) = \mu(\xi) \cdot \mu(\eta)$ for any $\xi, \eta \in R$, and

$$h(x+y) = h(x) + h(y)$$

if $x$ and $y$ are linearly independent.

(Then he used the continuity of $\phi$ to show that $h$ is homogeneous and hence $h \in L(E)$.)

Now, we shall show that from (3), (4) and (5) it follows that $h$ is additive.

Let $x$ be an arbitrary non-zero element and take another element $y$ such that $x$ and $y$ are linearly independent. For arbitrary $\xi, \eta \in R$, there exist continuous linear functionals $\bar{x}$ and $\bar{y}$ such that

$$(x, \bar{x}) = \xi, \quad (y, \bar{y}) = \eta, \quad (y, \bar{x}) = 0 \quad \text{and} \quad (x, \bar{y}) = 0,$$

where, for instance, $(x, \bar{x})$ denotes the value of $\bar{x}$ at $x$. Then, put

$$f = x \otimes \bar{x} + x \otimes \bar{y},$$

where, for instance, $x \otimes \bar{x}$ is a one-dimensional mapping defined by

$$(x \otimes \bar{x})(z) = (z, \bar{x})x \quad \text{for every} \quad z \in E.$$

Then, $f \in L(E)$, $f(x) = \xi x$ and $f(y) = \eta x$. Now, since $\phi(f) \in L(E)$, it follows from (5) that

$$\phi(f)h(x+y) = \phi(f)h(x) + \phi(f)h(y).$$

On the other hand, it follows from (3) that $\phi(f)h = hf$. Therefore,

$$h((\xi + \eta)x) = hf(x+y) = hf(x) + hf(y) = h(\xi x) + h(\eta x),$$

and, by (4), we have

$$\mu(\xi + \eta) = \mu(\xi) + \mu(\eta).$$
which, together with (4), implies that \( u(\xi) = \xi \) for every \( \xi \in \mathbb{R} \).
Therefore, \( h \) is homogeneous.

Now, for arbitrary \( x \) and \( y \), if these are linearly dependent, \( y = \alpha x \) for some \( \alpha \), and hence
\[
    h(x+y) = h((1+\alpha)x) = (1+\alpha)h(x) = h(x) + \alpha h(x)
\]
This fact and (5) imply that \( h \) is additive. Therefore, from (3) it follows that \( \phi \) is a ring automorphism.

**Remark 1.** As Eidelheit mentioned, the theorem is not true for one-dimensional spaces. For instance, \( \phi(\xi) = \xi^3 \) is a semigroup automorphism of \( \mathbb{R} \) which is not inner.

**Remark 2.** As is easily seen, the same theorem holds for the multiplicative semigroup of all linear mappings of a real vector space whose dimension is not less than 2.

Reference