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BAER-LEVI SEMIGROUPS OF PARTIAL TRANSFORMATIONS

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Let X be an infinite set and suppose $\aleph_0 \leqslant q \leqslant |X|$. The Baer-Levi semigroup on X is the set of all injective 'total' transformations $\alpha : X \to X$ such that $|X \setminus X\alpha| = q$. It is known to be a right simple, right cancellative semigroup without idempotents, its automorphisms are "inner"; and some of its congruences are restrictions of Malcev congruences on $I(X)$, the symmetric inverse semigroup on X. Here we consider algebraic properties of the semigroup consisting of all injective 'partial' transformations α of X such that $|X \setminus X\alpha| = q$: in particular, we describe the ideals and Green's relations of it and some of its subsemigroups.

1. INTRODUCTION

Throughout this paper, X is an infinite set with cardinal p, and q is a cardinal such that $\aleph_0 \leqslant q \leqslant p$. Let $P(X)$ denote the semigroup (under composition) of all *partial* transformations of X (that is, all mappings $\alpha : A \to B$ where $A, B \subseteq X$). If $\alpha \in P(X)$, we write $\operatorname{dom}\alpha$ for the *domain* of α and $\operatorname{ran}\alpha$ for its *range*. We also write

$$G(\alpha) = X \setminus \operatorname{dom}\alpha, \quad g(\alpha) = |G(\alpha)|,$$
$$D(\alpha) = X \setminus \operatorname{ran}\alpha, \quad d(\alpha) = |D(\alpha)|.$$

and refer to these cardinals as the *gap* and the *defect* of α, respectively.

As usual, $I(X)$ denotes the *symmetric inverse semigroup* on X ([**1**, Vol. 1, p. 29]): namely, the set of all injective mappings in $P(X)$. We write

$$BL(q) = \{\alpha \in I(X) : g(\alpha) = 0,\ d(\alpha) = q\}$$

and call this the *Baer-Levi semigroup* on X: as shown in ([**1**, Vol. 2, Section 8.1]), it is a right simple, right cancellative semigroup without idempotents; and any semigroup with these properties can be embedded in some Baer-Levi semigroup. Note that the ideals and Green's relations on $BL(q)$ are trivial. In addition, every automorphism φ

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of $BL(q)$ is "inner": that is, there exists $g \in G(X)$, the symmetric group on X, such that $\alpha\varphi = g\alpha g^{-1}$ for all $\alpha \in BL(q)$ [**5**]. And some congruences on $BL(q)$ are known to be restrictions of Malcev congruences on $T(X)$, the semigroup consisting of all *total* transformations of X (that is, $\alpha \in P(X)$ such that $\operatorname{dom}\alpha = X$) [**6**].

In this paper, we examine a related semigroup:

$$PS(q) = \{\alpha \in I(X) : d(\alpha) = q\}$$

which we call the *partial Baer-Levi semigroup* on X (as first defined in [**9**, p. 82]). In contrast with $BL(q)$, this semigroup always contains idempotents. In fact, $PS(q)$ always contains an inverse semigroup $R(q) = \{\alpha \in PS(q) : g(\alpha) = q\}$ which, together with $BL(q)$, generates $PS(q)$ in a very specific way. Also Green's relations and ideals are much more complicated. In Sections 4 and 5 we describe the latter for both $PS(q)$ and $R(q)$: this will be the basis for subsequent work regarding the congruences on $PS(q)$.

2. Basic properties

In what follows, $Y = A \dot{\cup} B$ means Y is a *disjoint* union of A and B. Also, $\emptyset$ denotes the empty (one-to-one) mapping which acts as a zero for $P(X)$. In particular, $d(\emptyset) = p$, so $\emptyset \in PS(q)$ precisely when $q = p$. For each non-empty $A \subseteq X$, we write id_A for the identity transformation on A: these mappings constitute all the idempotents in $I(X)$ and belong to $PS(q)$ precisely when $|X \setminus A| = q$.

We adopt the convention introduced in [**1**, Vol. 2, p. 241]: namely, if $\alpha \in P(X)$ is non-zero then we write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $X\alpha = \operatorname{ran}\alpha = \{x_i\}$, $x_i\alpha^{-1} = A_i$ and $\operatorname{dom}\alpha = \bigcup\{A_i : i \in I\}$.

Recall that a semigroup S is *right reductive* if $ax = bx$ for all $x \in S$ implies $a = b$ (and dually for *left reductive*: see [**1**, Vol. 1, p. 9]).

Theorem 1. *If $\aleph_0 \leqslant q \leqslant p$ then $PS(q)$ is a right and left reductive semigroup with idempotents. Moreover, $PS(q)$ contains a zero precisely when $q = p$.*

Proof: If $\alpha, \beta \in PS(q)$, we have

$$\begin{aligned} X \setminus X\alpha\beta &= X \setminus X\beta \cup [X\beta \setminus X\alpha\beta] \\ &= X \setminus X\beta \cup [(X \setminus X\alpha) \cap \operatorname{dom}\beta]\beta \end{aligned}$$

and in the last equation, the first set on the right has cardinal q and the second has cardinal at most q, thus $\alpha\beta \in PS(q)$. Also $PS(q)$ contains idempotents since we can write $X = A \dot{\cup} B$ where $|A| = p, |B| = q$ and then $\mathrm{id}_A \in PS(q)$. In addition, if ζ is a zero for $PS(q)$ then $\zeta = \zeta.\mathrm{id}_A$, hence $\operatorname{ran}\zeta \subseteq A$, for all $A \subseteq X$ such that $|X \setminus A| = q$. In particular, if $x \notin D(\zeta)$ and we choose $B \subseteq X$ such that $x \notin B$ and $\left|X \setminus (B \cup \{x\})\right| = q$ then $D(\zeta)$ contains $B \cup \{x\}$, a contradiction. Thus, every element of X belongs to $D(\zeta)$ and this occurs only when $q = p$.

To show $PS(q)$ is right reductive, suppose $\alpha, \beta \in PS(q)$ and $\alpha\gamma = \beta\gamma$ for all $\gamma \in PS(q)$. If $\alpha, \beta \neq \emptyset$ then $\mathrm{id}_{X\alpha} \in PS(q)$, so $\alpha = \alpha.\mathrm{id}_{X\alpha} = \beta.\mathrm{id}_{X\alpha}$ and this implies $X\alpha \subseteq X\beta$. The reverse inclusion also holds since $\mathrm{id}_{X\beta} \in PS(q)$. Hence $X\alpha = X\beta$ and it follows that $\alpha = \beta$. If (say) $\alpha = \emptyset$ then $q = p$ and $\beta\gamma = \emptyset$ for all $\gamma \in PS(q)$. In particular, $\beta.\mathrm{id}_{\{b\}} = \emptyset$ for all $b \in X\beta$ and thus $\beta = \emptyset$.

Now suppose $\gamma\alpha = \gamma\beta$ for all $\gamma \in PS(q)$. If $\alpha, \beta \neq \emptyset$, let $b \in \operatorname{dom}\alpha$ and write $X = \{b\} \dot{\cup} \{x_i\} \dot{\cup} \{x_j\}$ where $|I| = p, |J| = q$. Then

$$\gamma = \begin{pmatrix} x_i & b \\ x_i & b \end{pmatrix} \in PS(q)$$

and $b \in \operatorname{dom}\gamma\alpha = \operatorname{dom}\gamma\beta$, so $b \in \operatorname{dom}\beta$. Hence, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and the reverse inclusion also holds. It follows that $b\alpha = b\beta$ for all $b \in \operatorname{dom}\alpha = \operatorname{dom}\beta$ and hence $\alpha = \beta$. If (say) $\alpha = \emptyset$ and $x \in X$ then, as before, $\mathrm{id}_{\{x\}} \in PS(q)$, so $\mathrm{id}_{\{x\}}.\beta = \emptyset$ for all $x \in X$ and this implies $\beta = \emptyset$. ∎

EXAMPLE 1. Unlike $BL(q)$, the semigroup $PS(q)$ is not right cancellative nor right simple. For, suppose $X = A \dot{\cup} B$ where $|A| = p$, $|B| = q$, $A = \{a_i\}$ and $b, c \in B$ are distinct. If

$$\alpha = \begin{pmatrix} a_i & b \\ a_i & b \end{pmatrix}, \ \beta = \begin{pmatrix} a_i & b \\ a_i & c \end{pmatrix}$$

then $\alpha, \beta \in PS(q)$ and $\alpha.\mathrm{id}_A = \beta.\mathrm{id}_A$ but $\alpha \neq \beta$. Also, suppose $X = A \dot{\cup} B \dot{\cup} C$ where $|A| = p$ and $|B| = |C| = q$. If $\alpha = \mathrm{id}_{A \cup B}$ and $\beta = \mathrm{id}_{A \cup C}$, both of which are in $PS(q)$, then $C \cap \operatorname{dom}\alpha\gamma = \emptyset$ for each $\gamma \in PS(q)$. Therefore, since $C \subset \operatorname{dom}\beta$, there is no $\gamma \in PS(q)$ such that $\beta = \alpha\gamma$: that is, $PS(q)$ is not right simple.

A subsemigroup S of $P(X)$ is $G(X)$-*normal* if $g\alpha g^{-1} \in S$ for all $\alpha \in S$ and all $g \in G(X)$. Clearly $PS(q)$ is $G(X)$-normal and, if $q = p$, then $PS(q)$ *covers* X: that is, for each $x \in X$, there is an idempotent constant map (namely, $\mathrm{id}_{\{x\}}$) in $PS(q)$ with range $\{x\}$. Hence, by [**9**] Theorem 3, if $q = p$ then every automorphism of $PS(q)$ is 'inner' (as defined in Section 1 above) and moreover $\operatorname{Aut} PS(q)$ is isomorphic to $G(X)$. When $q < p$, $PS(q)$ does not contain any constant maps. Nonetheless, by [**4**, Theorem 3.18], every automorphism of $PS(q)$ is inner in this case also.

We aim to show that $\mathrm{Aut}\, PS(q)$ is also isomorphic to $G(X)$ when $q < p$. For this, we first need to know that if $\varphi \in \mathrm{Aut}\, PS(q)$ then there exists a *unique* $h \in G(X)$ such that $\alpha\varphi = h^{-1}\alpha h$ for all $\alpha \in PS(q)$. In other words, if $h, k \in G(X)$ and $h^{-1}\alpha h = k^{-1}\alpha k$ for all $\alpha \in PS(q)$ then $h = k$. To show this, we use some ideas from [**5**] and let

$$C(p,q) = \{A \subseteq X : |A| = p, |X \setminus A| = q\}.$$

If $A \in C(p,q)$ and α is any bijection from X onto A then $\alpha \in PS(q)$ and $Xh^{-1}\alpha h = Ah$, so $Ah = Ak$ for all $A \in C(p,q)$. Fix $x \in X$ and write $X = A \dot\cup B \dot\cup \{x\}$ where $|A| = p$ and $|B| = q$. Since h and k are injective,

$$(A \cup \{x\})h = Ah \dot\cup \{x\}h \quad \text{and} \quad (A \cup \{x\})k = Ah \dot\cup \{x\}k.$$

Therefore, since $(A \cup \{x\})h = (A \cup \{x\})k$, we find that $xh = xk$ for all $x \in X$, hence $h = k$. We can now prove the following result.

THEOREM 2. *If $q < p$ then $\mathrm{Aut}\, PS(q)$ is isomorphic to $G(X)$.*

PROOF: Let $\theta : \mathrm{Aut}\, PS(q) \to G(X), \varphi \to h_\varphi$, where h_φ is the unique permutation of X such that $\alpha\varphi = h_\varphi^{-1}\alpha h_\varphi$ for all $\alpha \in PS(q)$. To show θ is a morphism, let $\varphi, \psi \in \mathrm{Aut}\, PS(q)$ and note that for all $\alpha \in PS(q)$, we have:

$$\alpha(\varphi\psi) = (h_\varphi h_\psi)^{-1}\alpha(h_\varphi h_\psi),$$

hence $h_{\varphi\psi} = h_\varphi h_\psi$ by uniqueness. Clearly, if $k \in G(X)$ then

$$\varphi : PS(q) \to PS(q), \alpha \to k^{-1}\alpha k,$$

is an automorphism of $PS(q)$ (since $PS(q)$ is $G(X)$-normal). Thus, $h_\varphi = k$ by uniqueness, so θ is onto. Finally, if $h_\varphi = h_\psi$ then $\alpha\varphi = \alpha\psi$ for all $\alpha \in PS(q)$, so $\varphi = \psi$ and θ is one-to-one. ∎

In what follows, we sometimes write $PS(X,p,q)$ or $PS(p,q)$ in place of $PS(q)$ to highlight the underlying set X or its cardinal p.

As might be expected, $PS(X,p,q)$ is isomorphic to $PS(Y,r,s)$ if and only if $p = r$ and $q = s$, and moreover each isomorphism is induced in a natural way by a bijection from X onto Y. To prove this, we need an argument almost identical to that in [**5**]. However, since we are dealing with partial transformations and our argument differs in some important respects, we provide all the details.

LEMMA 1. *If $\alpha, \beta \in PS(p,q)$ then the following are equivalent.*

(a) $\mathrm{ran}\,\alpha \subseteq \mathrm{ran}\,\beta$,

(b) *for each $\gamma \in PS(p,q)$, $\beta\gamma = \beta$ implies $\alpha\gamma = \alpha$.*

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PROOF: If $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$ and $\beta\gamma = \beta$ for some $\gamma \in PS(q)$ then $(x\alpha)\gamma = x\alpha$ for each $x\alpha \in \operatorname{ran}\beta$, so $\alpha\gamma = \alpha$. Conversely, suppose there exists $y = x\alpha \notin \operatorname{ran}\beta = B$ say. Then $\operatorname{id}_B \in PS(q)$ and $\beta \circ \operatorname{id}_B = \beta$ but $y\operatorname{id}_B \neq y$; that is, $\alpha \circ \operatorname{id}_B \neq \alpha$ and hence the condition does not hold. ∎

Suppose $|X| = p \geqslant q \geqslant \aleph_0$ and let $\mathcal{B}(X,q)$ denote the family of all $A \subseteq X$ such that $|X \setminus A| = q$. Note that the poset $(\mathcal{B}(X,q), \subseteq)$ contains a least element if and only if $p = q$, and in this case $\emptyset$ is its least element. For, clearly if $p = q$ then $\emptyset \in \mathcal{B}(X,q)$. And if $q < p$ then each $A \in \mathcal{B}(X,q)$ is non-empty and $A \setminus \{x\} \in \mathcal{B}(X,q)$; that is, $\mathcal{B}(X,q)$ cannot contain a least element in this case. The proof of the next result closely follows the corresponding argument in [**5**].

LEMMA 2. *Suppose $|X| = p \geqslant q \geqslant \aleph_0$ and $|Y| = r \geqslant s \geqslant \aleph_0$. Every order-isomorphism $H : \mathcal{B}(X,q) \to \mathcal{B}(Y,s)$ is induced by a bijection $h : X \to Y$: that is, for each $A \in \mathcal{B}(X,q)$, we have $AH = Ah$, the image of A under h.*

PROOF: Let $A \in \mathcal{B}(X,q)$ and $x \in X \setminus A$. We write $A \cup \{x\}$ as $A \cup x$. Clearly, $A \cup x \in \mathcal{B}(X,q)$ and $A \cup x$ covers A. Hence $(A \cup x)H = AH \cup y$ for some $y \notin AH$. We write $y = xh_A$ and assert that $xh_A = xh_B$ for all $A, B \in \mathcal{B}(X,q)$ not containing x. For, clearly $A \cap B \in \mathcal{B}(X,q)$ and, since H is an order-isomorphism, $(A \cap B)H = AH \cap BH$. Therefore, as in the proof of [**5**, Lemma, p. 493],

$$\begin{aligned}(AH \cap BH) \cup xh_{A\cap B} &= (A \cap B)H \cup xh_{A\cap B} \\ &= \big((A \cap B) \cup x\big)H \\ &= \big((A \cup x) \cap (B \cup x)\big)H \\ &= (A \cup x)H \cap (B \cup x)H \\ &= (AH \cup xh_A) \cap (BH \cup xh_B),\end{aligned}$$

and it follows that

$$\{xh_{A\cap B}\} = \big(AH \cap \{xh_B\}\big) \cup \big(\{xh_A\} \cap BH\big) \cup \big(\{xh_A\} \cap \{xh_B\}\big).$$

Now if $xh_B \in AH$ then $xh_{A\cap B} = xh_B$ and hence

$$\big((A \cap B) \cup x\big)H = (A \cap B)H \cup xh_{A\cap B} = (A \cap B)H \cup xh_B \subseteq AH.$$

This implies $(A \cap B) \cup x \subseteq A$, contradicting $x \notin A$. Therefore, $xh_B \notin AH$ and similarly $xh_A \notin BH$. Hence $\{xh_A\} \cap \{xh_B\} \neq \emptyset$ and this means $xh_A = xh_B$ as asserted.

Now define $h : X \to Y, x \mapsto xh_A$, where $A \in \mathcal{B}(X,q)$ satisfies $x \notin A$. The above argument shows h is well-defined. To see h is injective, suppose $x_1h = x_2h$ and choose

$B \in \mathcal{B}(X,q)$ such that $x_i \notin B$ for $i = 1,2$. Then, by definition, $x_i h = x_i h_B$ for $i = 1,2$. Therefore

$$(B \cup x_1)H = BH \cup x_1 h_B = BH \cup x_2 h_B = (B \cup x_2)H$$

and it follows that $x_1 = x_2$. To show h is surjective, let $y \in Y$ and choose $M \in \mathcal{B}(Y,s)$ such that $y \notin M$. Then $AH = M$ and $BH = M \cup y$ for some $A, B \in \mathcal{B}(X,q)$. Since $M \cup y$ covers M in the poset $\mathcal{B}(Y,s)$, B must cover A in the poset $\mathcal{B}(X,q)$. That is, $B = A \cup x$ for some $x \notin A$. Hence

$$M \cup y = (A \cup x)H = AH \cup xh_A = M \cup xh_A$$

and it follows that $y = xh_A$ and thus $y = xh$ by definition. That is, h is a bijection.

Finally we prove that $AH = Ah$ for each $A \in \mathcal{B}(X,q)$. First recall that the empty map $\emptyset \in PS(q)$ if and only if $p = q$. In this case, the empty set $\emptyset$ is the least element of $\mathcal{B}(X,q)$ and hence $\emptyset H$ is a least element for $\mathcal{B}(Y,s)$. This means $r = s$ and $\emptyset H = \emptyset = \emptyset h$. So we can assume $A \in \mathcal{B}(X,q)$ is non-empty. Now if $y = xh$ for some $x \in A$ then $y = xh_B$ where $x \notin B \in \mathcal{B}(X,q)$. If $y \notin AH$ then $AH \cup y \in \mathcal{B}(Y,s)$ and $AH \cup y = (A \cup z)H$ for some $z \notin A$. Hence $zh_A = y = xh_B$ and, since h is injective, this implies $z = x \in A$, a contradiction. Therefore $y \in AH$ and $Ah \subseteq AH$. Conversely, if $y \in AH$ then AH covers $AH \setminus y$ (this is true even if $AH = \{y\}$, which is possible when $p = q$). Hence $AH \setminus y = (A \setminus x)H$ for some $x \in A$ and so

$$AH = \big((A \setminus x) \cup x\big)H = (A \setminus x)H \cup xh_{A \setminus x}.$$

Therefore, since $y \notin (A \setminus x)H$, we know $y = xh_{A \setminus x}$ and this means $y = xh \in Ah$; that is, $AH \subseteq Ah$ and equality follows. ∎

Recall that $PS(p,q)$ contains a zero element (namely, $\emptyset$) precisely when $p = q$. Consequently, if $PS(X,p,q)$ and $PS(Y,r,s)$ are isomorphic then either $p = q$ and $r = s$, or $p > q$ and $r > s$. In what follows, we need the fact: if $A, B \subseteq X$ and $\alpha \in I(X)$ then $(A \setminus B)\alpha = A\alpha \setminus B\alpha$.

Theorem 3. *The semigroups $PS(X,p,q)$ and $PS(Y,r,s)$ are isomorphic if and only if $p = r$ and $q = s$. Moreover, for each isomorphism φ, there is a bijection $h : X \to Y$ such that $\alpha\varphi = h^{-1}\alpha h$ for each $\alpha \in PS(X,p,q)$.*

Proof: Clearly, if the cardinals are equal as stated, then any bijection from X onto Y will induce an isomorphism between the semigroups. So we assume there is an isomorphism $\varphi : PS(X,p,q) \to PS(Y,r,s)$ and aim to find a bijection $h : X \to Y$. First we observe that φ induces an order-isomorphism from $\mathcal{B}(X,q)$ onto $\mathcal{B}(Y,s)$. Indeed, from Lemma 1 we deduce that, for each $\alpha, \beta \in PS(q)$, $\operatorname{ran}\alpha = \operatorname{ran}\beta$ if and only if $\operatorname{ran}(\alpha\varphi) = \operatorname{ran}(\beta\varphi)$. Also, recall that $\mathrm{id}_A \in PS(q)$ for each $A \in \mathcal{B}(X,q)$. Consequently, there is a well-defined mapping

$$H : \mathcal{B}(X,q) \to \mathcal{B}(Y,s), A \mapsto \operatorname{ran}(\alpha\varphi)$$

where $A = \operatorname{ran}\alpha$ for some $\alpha \in PS(q)$. Note that if $p = q$ and $A = \emptyset = \operatorname{ran}\emptyset$ where $\emptyset \in PS(q)$ then $\emptyset\varphi = \emptyset$ and $\emptyset H = \emptyset$. More generally, if $A, B \in \mathcal{B}(X,q)$ and $A = \operatorname{ran}\alpha, B = \operatorname{ran}\beta$ for some $\alpha, \beta \in PS(q)$ then $AH = \operatorname{ran}(\alpha\varphi), BH = \operatorname{ran}(\beta\varphi)$, and $A \subseteq B$ if and only if $AH \subseteq BH$ by Lemma 1. Also, for each $M \in \mathcal{B}(Y,s)$, there exists $\gamma \in PS(s)$ and $\alpha \in PS(q)$ such that $M = \operatorname{ran}\gamma$ and $\gamma = \alpha\varphi$: that is, $M = (\operatorname{ran}\alpha)H$ where $\operatorname{ran}\alpha \in \mathcal{B}(X,q)$, hence H is surjective.

By Lemma 2, H is induced by a bijection $h : X \to Y$ and now we aim to show $\alpha\varphi = h^{-1}\alpha h$ for each $\alpha \in PS(q)$. Clearly this holds if $p = q$ and $\alpha = \emptyset$. So, suppose $\alpha \neq \emptyset$ and note that $\operatorname{dom}\alpha h = \operatorname{dom}\alpha$ since $\operatorname{dom} h = X$. Let $x \in \operatorname{dom}\alpha$ and $x\alpha = x'$. Choose A, B in $\mathcal{B}(X,q)$ such that $A \subseteq B$ and $B \setminus A = \{x\}$, and consider $\beta, \gamma \in PS(X,q)$ such that $\operatorname{ran}\beta = A$ and $\operatorname{ran}\gamma = B$. Now $\operatorname{ran}\gamma \setminus \operatorname{ran}\beta = \{x\}$ and so

$$\begin{aligned}\operatorname{ran}\big((\gamma\alpha)\varphi\big) \setminus \operatorname{ran}\big((\beta\alpha)\varphi\big) &= \operatorname{ran}\big((\gamma\varphi)(\alpha\varphi)\big) \setminus \operatorname{ran}\big((\beta\varphi)(\alpha\varphi)\big)\\ &= \big(\operatorname{ran}(\gamma\varphi) \setminus \operatorname{ran}(\beta\varphi)\big)(\alpha\varphi)\\ &= (BH \setminus AH)(\alpha\varphi)\\ &= \{xh\}\alpha\varphi.\end{aligned}$$

On the other hand, $\operatorname{ran}(\gamma\alpha) \setminus \operatorname{ran}(\beta\alpha) = (B \setminus A)\alpha = \{x'\}$ and so

$$\begin{aligned}\operatorname{ran}\big((\gamma\alpha)\varphi\big) \setminus \operatorname{ran}\big((\beta\alpha)\varphi\big) &= \big(\operatorname{ran}(\gamma\alpha)\big)H \setminus \big(\operatorname{ran}(\beta\alpha)\big)H\\ &= \big(\operatorname{ran}(\gamma\alpha)\big)h \setminus \big(\operatorname{ran}(\beta\alpha)\big)h\\ &= (\operatorname{ran}\gamma \setminus \operatorname{ran}\beta)\alpha h\\ &= \{x'h\}.\end{aligned}$$

Thus $xh(\alpha\varphi) = x'h = x\alpha h$ for all $x \in \operatorname{dom}\alpha$ and so $\alpha\varphi = h^{-1}\alpha h$. Finally, since $\alpha\varphi \in PS(Y,r,s)$ implies $|Y \setminus Y\alpha\varphi| = s$, whereas $|Y \setminus Yh^{-1}\alpha h| = \big|(X \setminus X\alpha)h\big| = q$ for any bijection $h : X \to Y$, we also have $q = s$. ∎

3. Regular elements

Since $BL(q)$ is idempotent-free, it contains no regular elements (if S is a semigroup, we say $a \in S$ is *regular* if $a = axa$ for some $x \in S$). But $PS(q)$ always contains regular elements.

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THEOREM 4. *If $\aleph_0 \leqslant q \leqslant p$ and $\alpha \in PS(q)$ then the following statements are equivalent.*

(a) α *is regular,*
(b) $g(\alpha) = q$,
(c) $\alpha^{-1} \in PS(q)$.

PROOF: Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in PS(q)$. Then, since α is injective, $x\alpha\beta = x$ for all $x \in \operatorname{dom}\alpha$ and hence $\operatorname{dom}\alpha \subseteq \operatorname{ran}\beta$. Therefore, $q = d(\beta) \leqslant g(\alpha)$. Suppose $g(\alpha) = r > q$. Then $X \setminus \operatorname{dom}\alpha = (\operatorname{ran}\beta \setminus \operatorname{dom}\alpha) \cup (X \setminus X\beta)$ implies $|\operatorname{ran}\beta \setminus \operatorname{dom}\alpha| = r$. That is, if $\operatorname{ran}\beta \setminus \operatorname{dom}\alpha = \{d_k\}$ where $|K| = r$ and $c_k\beta = d_k$ then $\{c_k\} \cap \operatorname{ran}\alpha = \emptyset$ (since $\alpha\beta = \mathrm{id}_{\operatorname{dom}\alpha}$) and so $\{c_k\} \subseteq X \setminus \operatorname{ran}\alpha$ which implies $d(\alpha) \geqslant r > q$, a contradiction. This proves (a) implies (b). If $g(\alpha) = q$ then $d(\alpha^{-1}) = q$, so $\alpha^{-1} \in PS(q)$; and if $\alpha^{-1} \in PS(q)$ then clearly α is a regular element of $PS(q)$. ∎

The set of regular elements in $PS(q)$ plays an important role in what follows, so we let

$$R(q) = \{\alpha \in PS(q) : g(\alpha) = q\}.$$

Clearly any regular subsemigroup of $PS(q)$ is contained in $R(q)$. Therefore, the next result shows that $R(q)$ is the largest regular subsemigroup of $PS(q)$. In fact, since all idempotents of $PS(q)$ have the form id_A for some $A \subseteq X$ and all of these commute, we see that every regular subsemigroup of $PS(q)$ is inverse.

COROLLARY 1. *If $\aleph_0 \leqslant q \leqslant p$ then $R(q)$ is an inverse semigroup.*

PROOF: The idempotents in $PS(q)$ commute and, by the above Theorem, $R(q)$ is regular, so it remains to show $R(q)$ is closed. Suppose $\alpha, \beta \in R(q)$ and note that

$$\operatorname{dom}\alpha\beta = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \subseteq X\alpha^{-1},$$

so

$$\begin{aligned} X \setminus \operatorname{dom}\alpha\beta &= X \setminus X\alpha^{-1} \cup \left[X\alpha^{-1} \setminus (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1}\right] \\ &= X \setminus X\alpha^{-1} \cup \left[X \setminus (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\right]\alpha^{-1} \end{aligned} \tag{1}$$

where the first set on the right of (1) has cardinal q (since $\alpha^{-1} \in PS(q)$ by the Theorem). Also, $X \setminus [\operatorname{ran}\alpha \cap \operatorname{dom}\beta] = (X \setminus \operatorname{ran}\alpha) \cup (X \setminus \operatorname{dom}\beta)$, so the second set on the right of (1) has cardinal at most q (since α^{-1} is injective). Therefore, $g(\alpha\beta) = q$, and we have shown $\alpha\beta \in R(q)$. ∎

REMARK 1. In [**3**], Howie used $R(q) = \{\alpha \in I(X) : d(\alpha) = g(\alpha) = q\}$ to construct a congruence-free inverse semigroup when $p > q$; and in [**10**, Corollary 4], Sullivan showed that $R(p)$ is generated by its nilpotents with index 2: in fact, it equals the subsemigroup of $I(X)$ generated by all the nilpotents in $I(X)$.

For $\aleph_0 \leqslant r \leqslant p$, we write

$$S_r = \{\alpha \in PS(q) : g(\alpha) \leqslant r\}.$$

This is a subsemigroup of $PS(q)$ since if $\alpha, \beta \in S_r$ then

$$g(\alpha\beta) = |X \setminus X\alpha^{-1}| \cup \left|[X \setminus (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)]\alpha^{-1}\right|$$

where $X \setminus X\alpha^{-1} = X \setminus \operatorname{dom}\alpha$, regardless of whether $\alpha^{-1} \in PS(q)$. Hence, $g(\alpha\beta) \leqslant r + (0 + r) = r$, so $\alpha\beta \in S_r$. In particular,

$$BL(q) \cup R(q) \subset S_q$$

and so the two semigroups on the left cannot generate S_r for any $r > q$. In addition, if $\gamma \in PS(q)$ and $\gamma = \alpha\beta$ for some $\alpha \in R(q)$ and $\beta \in BL(q)$ then $g(\gamma) \geqslant g(\alpha)$. Hence $R(q).BL(q)$ is a proper subset of S_q. On the other hand, the next two results show that S_q is generated by $BL(q)$ and $R(q)$ in very specific ways: this will be important when we consider maximal subsemigroups of $PS(q)$ in a subsequent paper.

THEOREM 5. *If $\aleph_0 \leqslant q \leqslant p$ then $S_q = BL(q).R(q)$. In fact, $S_q = \alpha.R(q)$ for each $\alpha \in BL(q)$.*

PROOF: We have already seen that $BL(q).R(q) \subseteq S_q$. For the converse, suppose $\alpha \in S_q$ and note that

$$X \setminus X\alpha = [(X \setminus X\alpha) \cap \operatorname{dom}\alpha] \cup [(X \setminus X\alpha) \cap (X \setminus \operatorname{dom}\alpha)].$$

Hence, if $g(\alpha) < q$ then the second intersection on the right has cardinal less than q, whereas the set on the left of the equation has cardinal equal to q, hence we have:

$$|(X \setminus X\alpha) \cap \operatorname{dom}\alpha| = q.$$

Write $(X \setminus X\alpha) \cap \operatorname{dom}\alpha = \{a_i\} = \{b_i\} \dot\cup \{c_i\} \dot\cup \{d_j\}$ where $|J| = g(\alpha) < q$, and let $\operatorname{dom}\alpha \cap \operatorname{ran}\alpha = \{x_k\}$ and $X \setminus \operatorname{dom}\alpha = \{y_j\}$. Let

$$\lambda = \begin{pmatrix} x_k & a_i & y_j \\ x_k & b_i & d_j \end{pmatrix}, \quad \mu = \begin{pmatrix} x_k & b_i \\ x_k\alpha & a_i\alpha \end{pmatrix}$$

which are well-defined one-to-one maps by construction. Moreover, $\operatorname{dom}\lambda = X$ and $X \setminus X\lambda = \{c_i\} \cup \{y_j\}$: that is, $\lambda \in BL(q)$; and $X \setminus \operatorname{dom}\mu = \{c_i\} \cup \{d_j\} \cup \{y_j\}$ and $X \setminus X\mu = X \setminus X\alpha$: that is, $\mu \in R_q$. And clearly $\alpha = \lambda\mu$.

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If $g(\alpha) = q$, we can write $\operatorname{dom}\alpha = \{u_k\}, X \setminus \operatorname{dom}\alpha = \{y_j\}$ and $X \setminus X\alpha = \{v_j\} \dot\cup \{w_j\}$ where $|J| = q$. Let

$$\lambda = \begin{pmatrix} u_k & y_j \\ u_k\alpha & v_j \end{pmatrix}, \ \mu = \mathrm{id}_{X\alpha} \in R_q.$$

Then λ is a well-defined element of $BL(q)$ and $\lambda\mu = \alpha$ as required.

Finally, suppose $\alpha, \beta \in BL(q)$, let $X = \{x_i\}$ and write

$$\alpha = \begin{pmatrix} x_i \\ a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} x_i \\ b_i \end{pmatrix}, \quad \mu = \begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$

Then $\beta = \alpha\mu$ where $\mu \in R(q)$, so $BL(q) \subseteq \alpha.R(q) \subseteq S_q$. On the other hand, if $\gamma \in S_q$ then the above argument shows $\gamma = \beta\mu$ for some $\beta \in BL(q)$ and some $\mu \in R(q)$, and now we also know $\beta = \alpha\lambda$ for some $\lambda \in R(q)$. Therefore, $\gamma = \alpha(\lambda\mu)$ where $\lambda\mu \in R(q)$ since $R(q)$ is a semigroup; that is, $S_q \subseteq \alpha.R(q)$ and equality follows. ∎

The next result shows that in most cases S_q can be generated in a different way.

Theorem 6. *If $q < p$ then $S_q = BL(q).\mu.BL(q)$ for each $\mu \in R(q)$.*

Proof: Suppose $\gamma \in S_q$ with $g(\gamma) = r$ and let $\mu \in R(q)$. Since $q < p$, both γ and μ have rank p, so we can write

$$\gamma = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \quad \mu = \begin{pmatrix} c_i \\ d_i \end{pmatrix}.$$

Let $X\setminus\{a_i\} = \{a_j\}$ (so $|J| = r$), $X\setminus\{c_i\} = \{y_j\} \dot\cup \{y_k\}$ where $|K| = q$, $X\setminus\{d_i\} = \{d_k\}$ and $X \setminus \{b_i\} = \{u_k\} \dot\cup \{v_k\}$. If

$$\alpha = \begin{pmatrix} a_i & a_j \\ c_i & y_j \end{pmatrix}, \quad \beta = \begin{pmatrix} d_i & d_k \\ b_i & u_k \end{pmatrix}$$

then $\alpha, \beta \in BL(q)$ and $\gamma = \alpha\mu\beta$ (note that if $r = 0$ then $\{a_j\} = \emptyset$ but the conclusion is the same). ∎

In passing we note that if $\gamma \in S_q, \mu \in R(q)$ and $\gamma = \alpha\mu\beta$ for some $\alpha, \beta \in BL(q)$ then $\operatorname{dom}\gamma \subseteq \operatorname{dom}\alpha$, so $(\operatorname{dom}\gamma)\alpha \subseteq \operatorname{dom}\mu$ and hence $|\operatorname{dom}\gamma| \leqslant |\operatorname{dom}\mu| = r(\mu)$. Therefore, if $q = p$ and $r(\mu) < p$ then $g(\gamma) = g(\mu) = p$, so $BL(q).\mu.BL(q)$ is a proper subset of S_q; that is, the above result fails to hold when $q = p$. In addition, it cannot be simplified to read, for example: $S_q = \mu.BL(q)$ for each $\mu \in R(q)$ when $q < p$. For, if $\gamma \in S_q$ then $\gamma \neq \mu\beta$ for each $\mu \in R(q)$ such that $\operatorname{dom}\gamma \not\subseteq \operatorname{dom}\mu$. A similar argument using $\operatorname{ran}\gamma$ shows that also $S_q \neq BL(q).\mu$ for some $\mu \in R(q)$.

4. Green's relations

The semigroup $PS(q)$ is not a regular subsemigroup of $P(X)$, so Hall's Theorem ([**2**, Proposition II.4.5]) cannot be used to describe the $\mathcal{L}$ and $\mathcal{R}$ relations on $PS(q)$ in terms of their well-known characterisation on $P(X)$ (see [**7**, Theorem 10]). Therefore, in this section we first characterise each of the Green's relations on $PS(q)$ and then consider the corresponding problem for S_q and $R(q)$. In fact, for each of these semigroups, S say, we determine when $S^1\alpha \subseteq S^1\beta$ and $\alpha S^1 \subseteq \beta S^1$ for $\alpha, \beta \in S$ (that is, when $\mathcal{L}$ and $\mathcal{R}$ classes are comparable under their usual partial order).

Theorem 7. *If $\alpha, \beta \in PS(q)$ then $\alpha = \beta\mu$ for some $\mu \in PS(q)$ if and only if $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$. Hence $\alpha \mathcal{R} \beta$ in $PS(q)$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$.*

Proof: Clearly, if $\alpha = \beta\mu$ for some $\mu \in PS(q)$ then $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$. Conversely, suppose $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \quad \beta = \begin{pmatrix} a_i & x_j \\ c_i & y_j \end{pmatrix}, \quad \mu = \begin{pmatrix} c_i \\ b_i \end{pmatrix}.$$

Then $\alpha = \beta\mu$ where $\mu \in PS(q)$. ∎

Surprisingly, it is much harder to describe Green's $\mathcal{L}$ relation on $PS(q)$.

Theorem 8. *If $\alpha, \beta \in PS(q)$ then $\alpha = \lambda\beta$ for some $\lambda \in PS(q)$ if and only if $X\alpha \subseteq X\beta$ and*

$$q \leqslant \max\big(g(\beta), |X\beta \setminus X\alpha|\big) \leqslant \max\,(g(\alpha), q). \tag{2}$$

Hence, $\alpha \mathcal{L} \beta$ in $PS(q)$ if and only if

$$(X\alpha = X\beta \text{ and } g(\alpha) = g(\beta) \geqslant q) \text{ or } (\alpha = \beta \text{ and } g(\alpha) < q).$$

Proof: Suppose $\alpha = \lambda\beta$ for some $\lambda \in PS(q)$. Then $X\alpha \subseteq X\beta$ and $\alpha \in PS(q)$ implies

$$\Big|[(X \setminus X\lambda) \cap \operatorname{dom}\beta]\beta\Big| = |(X \setminus X\lambda)\beta| = |X\beta \setminus X\alpha| \leqslant d(\alpha) = q.$$

Also, since β is one-to-one, we have:

$$\begin{aligned} q = |X \setminus X\lambda| &= \Big|[(X \setminus X\lambda) \cap \operatorname{dom}\beta] \cup [(X \setminus X\lambda) \cap (X \setminus \operatorname{dom}\beta)]\Big| \\ &\leqslant |X\beta \setminus X\alpha| + g(\beta) = \max\big(g(\beta), |X\beta \setminus X\alpha|\big). \end{aligned}$$

Since λ is one-to-one and $\alpha = \lambda\beta$, we have

$$(X\lambda \cap \operatorname{dom}\beta)\lambda^{-1} = \operatorname{dom}\alpha \quad \text{and} \quad (X\lambda \cap X \setminus \operatorname{dom}\beta)\lambda^{-1} \subseteq X \setminus \operatorname{dom}\alpha$$

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and hence

$$|X \setminus \operatorname{dom}\beta| = \big|X\lambda \cap (X \setminus \operatorname{dom}\beta)\big| + \big|(X \setminus X\lambda) \cap (X \setminus \operatorname{dom}\beta)\big|$$
$$\leqslant |X \setminus \operatorname{dom}\alpha| + q = \max\big(g(\alpha), q\big).$$

Conversely, suppose $\alpha, \beta \in PS(q)$ and the conditions hold. Write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \ \beta = \begin{pmatrix} x_i & x_k \\ b_i & b_k \end{pmatrix}, \ \lambda = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$$

so that $|K| = |X\beta \setminus X\alpha|$. If $g(\alpha) < q$, the conditions imply $\max\big(g(\beta), |X\beta \setminus X\alpha|\big) = q$ and so $d(\lambda) = \big|\{x_k\} \cup (X \setminus \operatorname{dom}\beta)\big| = q$: that is, $\lambda \in PS(q)$. Suppose $g(\alpha) \geqslant q$. In this case, the conditions imply $g(\beta) \leqslant g(\alpha)$: otherwise, we have

$$|X\beta \setminus X\alpha| \leqslant q \leqslant g(\alpha) < g(\beta)$$

and so

$$\max\big(g(\beta), |X\beta \setminus X\alpha|\big) = g(\beta) > g(\alpha) = \max\ \big(g(\alpha), q\big).$$

We can also assume $q < g(\beta)$: otherwise, $\max\big(g(\beta), |X\beta \setminus X\alpha|\big) = q$ and the result follows as before. Now write $X \setminus \operatorname{dom}\beta = \{x_m\} \mathbin{\dot\cup} \{x_n\}$ where $|M| = g(\beta), |N| = q$ and choose $z_m \in X \setminus \operatorname{dom}\alpha$. Now re-define λ as

$$\lambda = \begin{pmatrix} a_i & z_m \\ x_i & x_m \end{pmatrix}$$

and note that $X \setminus X\lambda = \{x_k\} \cup \{x_n\}$ which has cardinal q. Hence, $\lambda \in PS(q)$ and $\alpha = \lambda\beta$ as required.

It follows that for distinct $\alpha, \beta \in PS(q)$, $\alpha = \lambda\beta$ and $\beta = \lambda'\alpha$ for some $\lambda, \lambda' \in PS(q)$ if and only if $X\alpha = X\beta$ and $g(\alpha) = g(\beta) \geqslant q$. That is, if $\alpha \mathrel{\mathcal{L}} \beta$ in $PS(q)$ and $g(\alpha) \geqslant q$ then $X\alpha = X\beta$ and $g(\alpha) = g(\beta)$, whereas if $g(\alpha) < q$ then $\alpha = \beta$. On the other hand, if one of these events occurs, it is now clear that $\alpha \mathrel{\mathcal{L}} \beta$ in $PS(q)$. ∎

Given that the condition in (2) is so complicated, it is worth noting that it cannot be simplified to read: $g(\beta) \leqslant g(\alpha)$.

Example 2. Let $\alpha, \beta \in PS(q)$ be defined by

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \ \beta = \begin{pmatrix} x_i & x_j \\ b_i & b_j \end{pmatrix}$$

where $g(\beta) \leqslant g(\alpha) < q$ and $|J| < q$. Note that in this case $X\alpha \subseteq X\beta$ and $|I| = p$. Also $\max\left(g(\beta), |X\beta \setminus X\alpha|\right) \not\geqslant q$. If $\alpha = \lambda\beta$ for some $\lambda \in PS(q)$ then $b_i = a_i\alpha = (a_i\lambda)\beta = x_i\beta$ for each i, so $\{x_i\} \subseteq X\lambda$. Therefore

$$d(\lambda) \leqslant \big|X \setminus \{x_i\}\big| = \big|\{x_j\} \cup G(\beta)\big| < q + q = q,$$

a contradiction. That is, for some $\alpha, \beta \in PS(q)$ with $g(\beta) \leqslant g(\alpha)$, there is no $\lambda \in PS(q)$ such that $\alpha = \lambda\beta$.

REMARK 2. From Theorems 7 and 8, we deduce that $\alpha\,\mathcal{H}\,\beta$ in $PS(q)$ if and only if

$$\big(X\alpha = X\beta, \operatorname{dom}\alpha = \operatorname{dom}\beta \text{ and } g(\alpha) \geqslant q\big) \text{ or } \big(\alpha = \beta \text{ and } g(\alpha) < q\big).$$

Recall that each group $\mathcal{H}$-class of $T(X)$ is isomorphic to a symmetric group $G(A)$ for some $A \subseteq X$ ([**1**, Vol. 1, Theorem 2.10]). The corresponding result for $PS(q)$ is even more precise. For, if ε is a non-zero idempotent of $PS(q)$ then $\varepsilon = \mathrm{id}_A$ for some $A \subseteq X$ such that $|X \setminus A| = q$. Consequently, since each $\alpha \in PS(q)$ is injective, we have

$$\begin{aligned} \alpha \in H_\varepsilon &\iff X\alpha = X\varepsilon,\ \operatorname{dom}\alpha = \operatorname{dom}\varepsilon,\\ &\iff \operatorname{ran}\alpha = \operatorname{dom}\alpha = A,\\ &\iff \alpha \in G(A). \end{aligned}$$

That is, $H_\varepsilon = G(A)$ and clearly, when $p = q$, $H_\emptyset = \{\emptyset\}$.

To characterise the $\mathcal{J}$ relation on $PS(q)$, we need two Lemmas. Henceforth, if $\alpha \in P(X)$, we write $r(\alpha) = |\operatorname{ran}\alpha|$ and call this the *rank* of α.

LEMMA 3. *If $q < p$ and $\alpha, \beta \in PS(q)$ then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in PS(q)$ if and only if $g(\alpha) \leqslant q$ or $g(\beta) \geqslant g(\alpha) > q$. Hence, in $PS(q)$ for $q < p$, $\alpha\,\mathcal{J}\,\beta$ if and only if $g(\alpha)$ and $g(\beta)$ are at most q, or $g(\alpha) = g(\beta) > q$.*

PROOF: First note that if $q < p$ then $r(\alpha) = r(\beta) = p$. Suppose $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in PS(q)$ and assume $g(\alpha) = r > q$. Then

$$\big|(X \setminus X\lambda) \cap (X \setminus \operatorname{dom}\alpha)\big| \leqslant q < r$$

and this implies $\big|X\lambda \cap G(\alpha)\big| = r$. That is, there exists $\{a_n\} \subseteq \operatorname{dom}\lambda$ such that $|N| = r$ and $\{a_n\lambda\} \cap \operatorname{dom}\alpha = \emptyset$. Therefore, $\{a_n\} \subseteq G(\beta)$ and $g(\beta) \geqslant r = g(\alpha)$, as required. Conversely, if $g(\alpha) \leqslant q < p$, write

$$\beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$$

where $|I| = p$ and let $\{a_i\} = \{x_i\} \,\dot{\cup}\, \{x_j\}$ where $|J| = q$. Define

$$\lambda = \begin{pmatrix} c_i \\ x_i \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} x_i\alpha \\ d_i \end{pmatrix}$$

and note that $D(\lambda) = \{x_j\} \cup G(\alpha)$, a set with cardinal q. Moreover, $\beta = \lambda\alpha\mu$ where $\lambda, \mu \in PS(q)$. On the other hand, if $g(\beta) \geqslant g(\alpha) = r > q$, choose $n_j \in G(\alpha)$ with $|J| = r$ and $\big|G(\alpha) \setminus \{n_j\}\big| = q$, and choose $m_j \in G(\beta)$ (possible via the assumption). Then, using the same notation for α and β, we see that

$$\lambda = \begin{pmatrix} c_i & m_j \\ a_i & n_j \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} b_i \\ d_i \end{pmatrix}$$

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are elements of $PS(q)$ and $\beta = \lambda\alpha\mu$, as required. ∎

LEMMA 4. *If $q = p$ and $\alpha, \beta \in PS(q)$ then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in PS(q)$ if and only if $r(\beta) \leqslant r(\alpha)$. Hence, in $PS(q)$ for $q = p$, $\alpha\ \mathcal{J}\ \beta$ if and only if $r(\alpha) = r(\beta)$.*

PROOF: Clearly, $\beta = \lambda\alpha\mu$ implies $r(\beta) \leqslant r(\alpha)$. For the converse, write

$$\beta = \begin{pmatrix} c_j \\ d_j \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$

Put $\{a_i\} = \{x_j\} \dot{\cup} \{x_k\}$ (possible since $r(\beta) \leqslant r(\alpha)$) and define

$$\lambda = \begin{pmatrix} c_j \\ x_j \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} x_j\alpha \\ d_j \end{pmatrix}$$

and note that $D(\lambda) = \{x_k\} \cup G(\alpha)$: clearly, this set has cardinal $q = p$ if $g(\alpha) = q$; and if $g(\alpha) < q$ then $|I| = q$, so we can ensure that $|K| = q$. That is, $\lambda, \mu \in PS(q)$ and $\beta = \lambda\alpha\mu$. ∎

Note that $g(\alpha) > q$ can occur only when $q < p$; and if $g(\alpha) \leqslant q < p$ then $r(\alpha) = p$. Also, if $q = p$ then $\max(g(\alpha), g(\beta)) \leqslant q$ is valid for all $\alpha, \beta \in PS(q)$. Hence the last two Lemmas can be combined as follows.

THEOREM 9. *If $\aleph_0 \leqslant q \leqslant p$ then $\alpha\ \mathcal{J}\ \beta$ in $PS(q)$ if and only if*

$$\Big[\max(g(\alpha), g(\beta)) \leqslant q \text{ and } r(\alpha) = r(\beta)\Big] \text{ or } \big[g(\alpha) = g(\beta) > q\big].$$

We now consider the $\mathcal{D}$ relation on $PS(q)$ and find that $\mathcal{D} \neq \mathcal{J}$, unlike the usual situation for other subsemigroups of $P(X)$ (for example, the semigroup generated by the idempotents of $T(X)$ [**8**, Theorem 7], and the semigroup generated by the nilpotents of $P(X)$ [**7**, Theorem 11]).

THEOREM 10. *If $\aleph_0 \leqslant q \leqslant p$ then $\alpha\ \mathcal{D}\ \beta$ in $PS(q)$ if and only if*

$$\big[g(\alpha) < q \text{ and } \operatorname{dom}\alpha = \operatorname{dom}\beta\big] \text{ or } \big[r(\alpha) = r(\beta) \text{ and } g(\alpha) = g(\beta) \geqslant q\big].$$

PROOF: Suppose $\alpha\ \mathcal{L}\ \gamma\ \mathcal{R}\ \beta$ in $PS(q)$. By Theorems 8 and 7, "$\alpha = \gamma$ and $g(\alpha) < q$" or "$X\alpha = X\gamma$ and $g(\gamma) = g(\alpha) \geqslant q$", and $\operatorname{dom}\gamma = \operatorname{dom}\beta$. Since γ and β are one-to-one on their domains, we deduce that

$$\big[g(\alpha) < q \text{ and } \operatorname{dom}\alpha = \operatorname{dom}\beta\big] \text{ or } \big[r(\alpha) = r(\beta) \text{ and } g(\alpha) = g(\beta) \geqslant q\big].$$

Conversely, suppose this condition holds. If $g(\alpha) < q$ and $\operatorname{dom}\alpha = \operatorname{dom}\beta$, then $\alpha\ \mathcal{L}\ \alpha\ \mathcal{R}\ \beta$. On the other hand, if $r(\alpha) = r(\beta)$ and $g(\alpha) = g(\beta) \geqslant q$, we write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \quad \beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} c_i \\ b_i \end{pmatrix}.$$

Then $\gamma \in PS(q)$ and, by Theorems 8 and 7, $\alpha\ \mathcal{L}\ \gamma\ \mathcal{R}\ \beta$ as required. ∎

EXAMPLE 3. Let $\alpha, \beta \in PS(q)$ be defined by

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \ \beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix}$$

where $g(\beta) < g(\alpha) < q$ and $\operatorname{dom}\alpha \neq \operatorname{dom}\beta$. This implies $|I| = p$, so $r(\alpha) = r(\beta)$ and $\max(g(\alpha), g(\beta)) < q$, hence $\alpha \mathcal{J} \beta$ by Theorem 9. Suppose $\alpha \mathcal{L} \gamma \mathcal{R} \beta$ for some $\gamma \in PS(q)$. Then $\operatorname{dom}\gamma = \operatorname{dom}\beta$ by Theorem 7, hence $\alpha \neq \gamma$ (by choice). So Theorem 8 implies $X\alpha = X\gamma$ and $g(\alpha) = g(\gamma) \geqslant q$, contradicting the choice of α. Hence α is not $\mathcal{D}$-related to β in $PS(q)$, and thus $\mathcal{D} \neq \mathcal{J}$.

We now consider Green's relations on S_q. As before, since S_q is not a regular subsemigroup of $PS(q)$, Hall's Theorem cannot be applied to find $\mathcal{R}$ and $\mathcal{L}$ on S_q. Nonetheless, they happen to be the restriction of $\mathcal{R}$ and $\mathcal{L}$ on $PS(q)$.

LEMMA 5. *Let $\alpha, \beta \in S_q$ where $\aleph_0 \leqslant q \leqslant p$. Then*

(a) *$\alpha = \beta\mu$ for some $\mu \in S_q$ if and only if $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$, and*

(b) *$\alpha = \lambda\beta$ for some $\lambda \in S_q$ if and only if $X\alpha \subseteq X\beta$ and $\max(g(\beta), |X\beta \setminus X\alpha|) = q$.*

PROOF: For (a), we simply note that in the proof of Theorem 7, if $\alpha \in S_q$ then $\{x_j\} \subseteq G(\alpha)$, so $|J| \leqslant q$ and $G(\mu) = \{y_j\} \cup D(\beta)$, hence $g(\mu) \leqslant q$.

For (b), observe that if $\alpha = \lambda\beta$ for some $\lambda \in S_q \subseteq PS(q)$ then the condition in Theorem 8 simplifies to the desired result. Conversely, suppose the stated condition holds and write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \ \beta = \begin{pmatrix} x_i & x_j \\ b_i & b_j \end{pmatrix}, \ \lambda = \begin{pmatrix} a_i \\ x_i \end{pmatrix}.$$

Then $|J| \leqslant q$ since $|X\beta \setminus X\alpha| \leqslant d(\alpha) = q$. If $g(\beta) = q$ then $d(\lambda) = g(\beta) + |J| = q$ and clearly $g(\lambda) \leqslant q$, so $\lambda \in S_q$ and $\alpha = \lambda\beta$. On the other hand, if $|X\beta \setminus X\alpha| = q$ then $|J| = q \geqslant g(\beta)$ and again $d(\lambda) = q$, so $\lambda \in S_q$ as required. ∎

COROLLARY 2. *Let $\alpha, \beta \in S_q$ where $\aleph_0 \leqslant q \leqslant p$. Then*

(a) *$\alpha \mathcal{R} \beta$ in S_q if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$, and*

(b) *$\alpha \mathcal{L} \beta$ in S_q if and only if $[X\alpha = X\beta$ and $g(\alpha) = g(\beta) = q]$ or $[\alpha = \beta$ and $g(\alpha) < q]$.*

From Lemma 3 we see that if $q < p$ then S_q forms a $\mathcal{J}$-class in $PS(q)$. Hence we might expect the $\mathcal{J}$ relation on S_q to be universal when $q < p$. In addition, given the last result, we might also expect the $\mathcal{D}$ relation on S_q to be the restriction of $\mathcal{D}$ on $PS(q)$. Both these expectations are correct, as we now show.

THEOREM 11. *Let $\alpha, \beta \in S_q$ where $\aleph_0 \leqslant q \leqslant p$. Then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in S_q$ if and only if $r(\beta) \leqslant r(\alpha)$. Hence*

(a) *$\alpha \mathcal{J} \beta$ in S_q if and only if $r(\alpha) = r(\beta)$, and*

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(b) $\alpha\ \mathcal{D}\ \beta$ *in* S_q *if and only if* $[g(\alpha) < q$ *and* $\operatorname{dom}\alpha = \operatorname{dom}\beta]$ *or* $[r(\alpha) = r(\beta)$ *and* $g(\alpha) = g(\beta) = q]$.

PROOF: Clearly, $\beta = \lambda\alpha\mu$ implies $r(\beta) \leqslant r(\alpha)$. Conversely, if $q < p$ then $r(\alpha) = r(\beta) = p$. Using the same notation as in the proof of Lemma 3, we note that $g(\lambda) = g(\beta) \leqslant q$ and $G(\mu) = D(\alpha) \cup \{x_j\alpha\}$, a set with cardinal q, so $\lambda, \mu \in S_q$ in this case. On the other hand, if $q = p$ and $r(\beta) \leqslant r(\alpha)$ then we observe that the λ, μ defined in the proof of Lemma 4 actually belong to S_q.

It remains to prove (b). If $\alpha\ \mathcal{D}\ \beta$ in S_q then $\alpha\ \mathcal{D}\ \beta$ in $PS(q)$, so Theorem 10 gives the desired result. Conversely, if the condition holds, we note that the converse argument in the proof of Theorem 10 shows in fact that $\gamma \in S_q$ and hence $\alpha\ \mathcal{D}\ \beta$ in S_q. ∎

We now turn to Green's relations on $R(q)$. Since this is a regular subsemigroup of $I(X)$, Hall's Theorem implies that the $\mathcal{L}$ and $\mathcal{R}$ relations on $R(q)$ equal the restriction of the corresponding relations on $I(X)$ to $R(q)$. Hence, $\alpha\ \mathcal{L}\ \beta$ in $R(q)$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$, and $\alpha\ \mathcal{R}\ \beta$ in $R(q)$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$. In fact, the $\mathcal{J}$ and $\mathcal{D}$ relations on $R(q)$ also mimic those on $I(X)$.

THEOREM 12. *If* $\alpha, \beta \in R(q)$ *then* $\beta = \lambda\alpha\mu$ *for some* $\lambda, \mu \in R(q)$ *if and only if* $r(\beta) \leqslant r(\alpha)$. *Hence,* $\alpha\ \mathcal{J}\ \beta$ *in* $R(q)$ *if and only if* $r(\alpha) = r(\beta)$. *Consequently,* $\mathcal{D} = \mathcal{J}$ *in* $R(q)$.

PROOF: As usual, if $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in P(X)$ then $r(\beta) \leqslant r(\alpha)$. Conversely, if this condition holds, we write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \quad \beta = \begin{pmatrix} c_j \\ d_j \end{pmatrix}, \quad \lambda = \begin{pmatrix} c_j \\ x_j \end{pmatrix}, \quad \mu = \begin{pmatrix} x_j\alpha \\ d_j \end{pmatrix}$$

where $\{x_j\} \subseteq \{a_i\}$ (possible since $|J| \leqslant |I|$). Then $\beta = \lambda\alpha\mu$ and $\lambda, \mu \in R(q)$ (note that if $q < p$ then we can assume $I = J$). Finally a standard argument shows that if $r(\alpha) = r(\beta)$ then $\alpha\ \mathcal{D}\ \beta$, so $\mathcal{J} \subseteq \mathcal{D}$ and equality follows. ∎

REMARK 3. From a comment above, we deduce that $\alpha\ \mathcal{H}\ \beta$ in $R(q)$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$ and $\operatorname{dom}\alpha = \operatorname{dom}\beta$. Hence, as in Remark 1 about $PS(q)$, the group $\mathcal{H}$-classes of $R(q)$ are precisely the symmetric groups $G(A)$ where $A \subseteq X$ and $|X \setminus A| = q$. For the group $\mathcal{H}$-classes of S_q, note that no idempotent of $PS(q)$ has gap less than q, hence Corollary 2 shows that $\mathcal{H}$ in S_q can be characterised in the same way as for $R(q)$, and therefore the group $\mathcal{H}$-classes of S_q are also the same as for $R(q)$.

5. TWO-SIDED IDEALS

Recall that for $q \leqslant r \leqslant p$, $S_r = \{\alpha \in PS(q) : g(\alpha) \leqslant r\}$ is a subsemigroup of $PS(q)$. The reverse inequality gives us ideals of $PS(q)$ when $q < p$.

THEOREM 13. *The proper ideals of $PS(q)$ for $q < p$ are precisely the sets:*

$$T_r = \{\alpha \in PS(q) : g(\alpha) \geqslant r\}$$

where $q < r \leqslant p$. Moreover, each T_r is a principal ideal.

PROOF: Let $\alpha \in T_r$ and $\beta \in PS(q)$. Since $\operatorname{dom}\alpha\beta \subseteq \operatorname{dom}\alpha$, we know $g(\alpha\beta) \geqslant g(\alpha)$, so each T_r is a right ideal. Also,

$$X \setminus \operatorname{dom}\beta\alpha = (X \setminus \operatorname{dom}\beta) \cup (\operatorname{dom}\beta \setminus \operatorname{dom}\beta\alpha)$$

and

$$G(\alpha) = [X\beta \cap G(\alpha)] \cup [(X \setminus X\beta) \cap G(\alpha)]$$

where $[X\beta \cap G(\alpha)]\beta^{-1} = \operatorname{dom}\beta \setminus \operatorname{dom}\beta\alpha$ and $d(\beta) = q$. Therefore, $|X\beta \cap G(\alpha)| \geqslant r$ and it follows that $g(\beta\alpha) \geqslant r$. That is, T_r is also a left ideal.

Conversely, suppose A is a proper ideal of $PS(q)$ for $q < p$ and choose $\alpha \in A$ with least gap, r say, so $A \subseteq T_r$. If $r \leqslant q$ then, by Lemma 3, all elements of $PS(q)$ belong to $PS(q)\alpha PS(q)$ which is contained in A: that is, $A = PS(q)$, a contradiction. Therefore $q < r \leqslant p$ and if $\beta \in T_r$ then $g(\beta) \geqslant r = g(\alpha) > q$, so Lemma 3 implies $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in PS(q)$. Hence $\beta \in A$ and equality follows.

Finally, if $\alpha \in T_r$ has gap r where $q < r \leqslant p$ then Lemma 3 implies that, for each $\beta \in T_r$, there exist $\lambda, \mu \in PS(q)$ such that $\beta = \lambda\alpha\mu$ and hence $T_r \subseteq PS(q)^1\alpha PS(q)^1$. Since $\alpha \in T_r$ and T_r is an ideal, the reverse inclusion also holds, and thus each T_r is principal. ∎

In effect, in [**1**, Vol. 2, Lemma 10.54], Clifford and Preston prove that the Rees factor semigroups $I_{\xi'}/I_\xi$ of ideals I_ξ in $T(X)$ are 0-bisimple, and they contain a primitive idempotent precisely when ξ is finite (here ξ' denotes the *successor* of the cardinal ξ). To obtain a corresponding result for the ideals of $PS(q)$, we first observe that if $q < r \leqslant s \leqslant p$ then $q' \leqslant r$ and

$$T_p \subseteq \cdots \subseteq T_s \subseteq T_r \subseteq \cdots \subseteq T_{q'}.$$

Note that if $q < r \leqslant p$ then $G_r = S_r \cap T_r$ is the (non-empty) set of all $\alpha \in PS(q)$ with gap r, and in fact G_r is a semigroup (since it is the intersection of two semigroups). Therefore, if $q < r < p$ then $T_r/T_{r'}$ is essentially G_r with a zero adjoined (note that $G_p = T_p$).

REMARK 4. If α, β are $\mathcal{D}$-related in G_r then they are $\mathcal{D}$-related in $PS(q)$. Conversely, from Theorem 10 we deduce that if α, β are $\mathcal{D}$-related in $PS(q)$ then they have the same gap, r say. Moreover, in this case, $\alpha \;\mathcal{L}\; \gamma \;\mathcal{R}\; \beta$ for some $\gamma \in PS(q)$ with the

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same gap as α (see the proof of Theorem 10). Now by Theorem 8, either $\alpha = \gamma$ or "$X\alpha = X\gamma$ and $g(\alpha) = g(\gamma) \geqslant q$"; and in the latter case, as in the second half of the proof of Theorem 8, we can find λ_1, λ_2 with gap r such that $\alpha = \lambda_1\gamma$ and $\gamma = \lambda_2\alpha$: that is, $\alpha \mathcal{L} \gamma$ in G_r. On the other hand, if $\gamma \mathcal{R} \beta$ in $PS(q)$ then $\operatorname{dom}\gamma = \operatorname{dom}\beta$ by Theorem 7. In addition, if γ and β have gap $r > q$, we can write

$$\gamma = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \quad \beta = \begin{pmatrix} a_i \\ c_i \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} x_i\beta \\ b_i \end{pmatrix},$$

where $\{a_i\} = \{x_i\} \dot\cup \{x_k\}$ and $|K| = r$. Then $g(\mu_1) = \left|\{x_k\beta\} \cup X \setminus \{c_i\}\right| = r$ (since $d(\beta) = q < r$) and $\gamma = \beta\mu_1$. That is, if $\gamma \mathcal{R} \beta$ in $PS(q)$ and $q < r = g(\beta) \leqslant p$, we can show that $\gamma \mathcal{R} \beta$ in G_r. In other words, if α, β are $\mathcal{D}$-related in $PS(q)$ and have gap r where $q < r \leqslant p$ then they are $\mathcal{D}$-related in G_r.

From the above Remark, we deduce that G_r is bisimple if $q < r \leqslant p$. Also, if ε is an idempotent in G_r then $\varepsilon = \mathrm{id}_A$ for some $A \subseteq X$ such that $|A| = p$ and $|X \setminus A| = r > q$, which contradicts $d(\varepsilon) = q$. That is, G_r is idempotent-free.

COROLLARY 3. *If $q < r \leqslant p$ then $G_r = S_r \cap T_r$ is bisimple and idempotent-free.*

When $q = p$, $PS(q)$ contains constant maps, all of which form an ideal of $PS(q)$, so we can expect a more standard description of the ideals in $PS(q)$ in this case: compare [**1**, Vol. 2, Theorem 10.59] for the ideals of $T(X)$.

THEOREM 14. *If $q = p$, the ideals of $PS(q)$ are precisely the sets:*

$$J_r = \{\alpha \in PS(q) : r(\alpha) < r\}$$

where $1 \leqslant r \leqslant p'$. Moreover, J_r is principal precisely when $r = s'$ where $0 \leqslant s \leqslant p$.

PROOF: Clearly each J_r is an ideal of $PS(q)$. Let A be an ideal of $PS(q)$ and let r be the least cardinal greater than $r(\alpha)$ for all $\alpha \in A$. Then $A \subseteq J_r$. Now, for each $\beta \in J_r$, there exists $\alpha \in A$ such that $r(\beta) \leqslant r(\alpha)$ (by the choice of r). Hence Lemma 4 implies $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in PS(q)$, so $\beta \in A$. That is, $J_r \subseteq A$ and equality follows. Moreover, if $r = s'$ then $J_r = \{\alpha \in PS(q) : r(\alpha) \leqslant s\}$. In this case, since $p = q$, Lemma 4 implies $J_r \subseteq PS(q)^1\alpha PS(q)^1$ for each $\alpha \in J_r$ with rank s, and it follows that J_r is principal. Conversely, suppose $J_r = PS(q)^1\alpha PS(q)^1$ for some $\alpha \in J_r$. Let $r(\alpha) = s$ and assume there is a cardinal t such that $s < t < r$. Since $p = q$, there exists $\beta \in PS(q)$ with $r(\beta) = t$ and then $\beta \in J_r$, so $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in PS(q)^1$. But this implies $r(\beta) \leqslant r(\alpha)$, a contradiction. Therefore, t does not exist and thus $r = s'$. ∎

REMARK 5. If non-zero α, β are $\mathcal{D}$-related in $J_{r'}/J_r$ then they are $\mathcal{D}$-related in $PS(q)$. Conversely, from Theorem 10 we deduce that if α, β are $\mathcal{D}$-related in $PS(q)$ then they have the same rank, r say. Moreover, in this case, $\alpha\ \mathcal{L}\ \gamma\ \mathcal{R}\ \beta$ for some $\gamma \in PS(q)$ with the same rank r (see the proof of Theorem 10). Next we observe that, in the proof of Theorem 7, μ has the same rank as α, and this can be used to show that, if elements of $PS(q)$ are $\mathcal{R}$-related in $PS(q)$ and have rank r, then they are $\mathcal{R}$-related in $J_{r'}/J_r$. In addition, if $\alpha\ \mathcal{L}\ \gamma$ in $PS(q)$ then Theorem 8 implies that either $\alpha = \gamma$ or "$X\alpha = X\gamma$ and $g(\alpha) = g(\gamma) \geqslant q$"; and in the latter case, as in the second half of the proof of Theorem 8, we can find λ_1, λ_2 with rank r such that $\alpha = \lambda_1\gamma$ and $\gamma = \lambda_2\alpha$: that is, $\alpha\ \mathcal{L}\ \gamma$ in $J_{r'}/J_r$. In other words, if α, β are $\mathcal{D}$-related in $PS(q)$ and have rank r then they are $\mathcal{D}$-related in $J_{r'}/J_r$.

Now, in Example 2 we found α, β with rank p which are not $\mathcal{D}$-related in $PS(q)$ and so, by the above Remark, $J_{p'}/J_p$ is not 0-bisimple. On the other hand, if $r < p = q$ then all non-zero elements of $J_{r'}/J_r$ have the same rank r and gap p, so Theorem 10 implies they are $\mathcal{D}$-related in $PS(q)$ and hence also in $J_{r'}/J_r$; that is, $J_{r'}/J_r$ is 0-bisimple if $1 \leqslant r < p$. However, if ε is a non-zero idempotent in $J_{r'}/J_r$ then $\varepsilon = \mathrm{id}_A$ for some $A \subseteq X$ such that $|A| = r$ and $|X \setminus A| = q$; and, since $A \setminus \{x\} \subsetneq A$ if $x \in A$, this is primitive precisely when r is finite and positive (see [**1**, Vol. 2, p. 224]). That is, $J_{r'}/J_r$ is completely 0-simple only when $1 \leqslant r < \aleph_0$. Finally, by Theorem 4, if each α in $J_{r'}/J_r$ is regular, we must have $r < q = p$ (since elements with rank p can have gap less than p). In other words, $J_{r'}/J_r$ is inverse precisely when $0 \leqslant r < p$.

COROLLARY 4. *If $1 \leqslant r < p = q$ then $J_{r'}/J_r$ is a 0-bisimple inverse semigroup; it is completely 0-simple only when r is finite.*

Note that if $q < p$ and $\alpha, \beta \in S_q$ then $r(\alpha) = r(\beta) = p$, so $\alpha\ \mathcal{J}\ \beta$ in S_q by Theorem 11(a). Thus, S_q is simple if $q < p$, and of course if $q = p$ then $S_q = PS(q)$. Likewise if $q < p$ then $R(q)$ is simple (in fact, bisimple since $\mathcal{D} = \mathcal{J}$ when $q < p$). And if $q = p$ then $R(q)$ contains constant maps and an argument similar to that in the above proof leads to our last result.

THEOREM 15. *If $q = p$, the ideals of $R(q)$ are precisely the sets $R(q) \cap J_r$ where $1 \leqslant r \leqslant p'$.*

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