We also study a more general notion than AECs: μ-AECs, which are only required to be closed under μ-directed (rather than $\aleph_0$-directed) colimits. We generalize some basic arguments from the theory of AECs and show that μ-AECs are exactly the accessible categories whose morphisms are monomorphisms (this is joint work with Will Boney, Rami Grossberg, Michael Lieberman, and Jiří Rosický).

Finally, the thesis contains a chapter on simple first-order theories. We present a new proof of the existence of Morley sequences in such theories which avoids using the Erdős-Rado theorem and instead uses only Ramsey's theorem and compactness. The proof shows that the basic theory of forking in simple theories can be developed using only principles from "ordinary mathematics", answering a question of Grossberg, Iovino, and Lessmann, as well as a question of Baldwin.

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ALESSANDRO VIGNATI, *Logic and* C*-*algebras***: *Set Theoretical Dichotomies in the Theory of Continuous Quotients*, York University, Toronto, Canada, 2017. Supervised by Ilijas Farah. MSC: Primary 03E50, 46L05. Keywords: C*-algebras, forcing axioms, automorphisms of quotients.

Abstract

This thesis focuses on interactions between logic and C*-algebras (Banach self-adjoint subalgebras of $\mathcal{B}(H)$, the algebra of bounded operators on a complex Hilbert space). As abelian C*-algebras are dual to locally compact spaces, the study of C*-algebras is considered as noncommutative topology. Our interest is primarily on corona C*-algebras, nonabelian analogs of Čech-Stone remainders (i.e., $\beta X \setminus X$), and on how set theory has an influence on their automorphisms. The motivation comes from two sources: (1) the search for outer automorphisms of the Calkin algebra; (2) the study of homeomorphisms of $\beta X \setminus X$.

(1) The seminal work in [1] developed extension theory and defined analytic K-homology for C*-algebras, to make a major breakthrough in extending Weylvon Neumann theory and understanding index theory for normal[1] elements of the Calkin algebra $\mathcal{C}(H)$ (if H is separable, $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$, $\mathcal{K}(H)$ being the ideal of compact operators). Few questions were left open. Since, for $S, T \in \mathcal{B}(H)$, $\dot{S}$ and $\dot{T}$ (their images in $\mathcal{C}(H)$) are unitarily equivalent if and only if there is an automorphism of $\mathcal{C}(H)$ mapping $\dot{S}$ to $\dot{T}$, can the same be said if S and T are essentially normal (S is essentially normal if $\dot{S}$ is normal)? In particular, is it possible to map the image of the unilateral shift to its adjoint via an automorphism of $\mathcal{C}(H)$? Such an automorphism cannot be inner (induced by a unitary), but it was not known whether an outer automorphism of $\mathcal{C}(H)$ would exist at all. Set theoretic axioms entered play: under CH outer automorphisms exist ([7]) while under OCA don't ([5]). It is still open if there can be an automorphism mapping the shift to its adjoint.

(2) In studying the homogeneity properties of spaces of the form $\beta X \setminus X$, the following question arose: is every homeomorphism of $\beta \mathbf{N} \setminus \mathbf{N}$ induced by an almost permutation? CH gives a negative answer ([8]), while a positive answer is consistent ([9]), and in fact follows from Forcing Axioms ([10]). This question was generalized in two, linked, ways. First, replacing the notion of almost permutation with "permutation up to compact subsets of X", one can ask about homeomorphisms of $\beta X \setminus X$ for a metrizable locally compact X. Second, as homeomorphisms of $\beta \mathbf{N} \setminus \mathbf{N}$ correspond to automorphisms of $\mathcal{P}(\mathbf{N})/\text{Fin}$, the theory of automorphisms and relative embeddings of quotients of the form $\mathcal{P}(\mathbf{N})/\mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{P}(\mathbf{N})$ is an analytic ideal, was extensively studied in [4].

The goal is to extend the results in (1) and (2) to corona C*-algebras. In the same way $\mathcal{K}(H)$ is related to $\mathcal{B}(H)$ and $\mathcal{C}(H)$, to a nonunital C*-algebra A one can associate its

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[1] S is normal if $SS^* = S^*S$.

multiplier algebra $\mathcal{M}(A)$ and its corona $\mathcal{M}(A)/A$.[2] Since there are coronas admitting outer automorphisms in ZFC, the notion of triviality required has to be weaker. In case A is separable, the unit ball of $\mathcal{M}(A)$ carries a Polish topology: the strict topology. We say that an automorphism of $\mathcal{M}(A)/A$ is trivial if its graph in the unit ball of $\mathcal{M}(A)$ is Borel in the strict topology.

CONJECTURE ([2]). *Let A be a separable nonunital* C*-*algebra. The existence of nontrivial automorphisms of* $\mathcal{M}(A)/A$ *is independent of ZFC.*

Instances of the conjecture have been established in both the direction of constructing nontrivial automorphisms under CH and in showing that Forcing Axioms imply rigidity. We add to this list:

THEOREM. • *Assume CH and let X be a locally compact noncompact metrizable manifold. Then* $\beta X \setminus X$ *has nontrivial homeomorphisms;*

• *Assume PFA and let A be an amenable separable unital* C*-*algebra. Let* $B = A \otimes \mathcal{K}(H)$. *Then all automorphisms of* $\mathcal{M}(B)/B$ *are trivial.*

Other results are contained in the thesis. Chapter 3 concerns with applications of model theory to C*-algebras, and in particular with the study of different degrees of countable saturation ([3]). Chapter 4 focuses on perturbation theory for C*-algebras. We prove a stability result for approximate maps with AF domain ([6]). In Chapter 5 we provide independency results for embeddings of reduced products of the form $\prod A_i / \bigoplus_{\mathcal{I}} A_i$, for an analytic ideal $\mathcal{I}$ and algebras A_i, following the intuition of [4] for quotients of discrete structures.

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[2]If $A = C_0(X)$ for some locally compact X, then $\mathcal{M}(A) = C(\beta X)$. Automorphisms of $\mathcal{M}(A)/A$ correspond to homeomorphisms of $\beta X \setminus X$.