WEIGHTED LACUNARY MAXIMAL FUNCTIONS ON CURVES

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ABSTRACT. Let $(\gamma(t) = (t, t^2, \ldots, t^n) + a)$ be a curve in $\mathbb{R}^n$, where $n \geq 2$ and $a \in \mathbb{R}^n$.

We prove $L^p$--$L^q$ estimates for the weighted lacunary maximal function, related to this curve, defined by

$$M_{\gamma} f(x) = \sup_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} f(x - 2^k \gamma(t)) \, dt \right|, \quad f \in C_0^\infty(\mathbb{R}^n).$$

If $n = 2$ or $3$, our results are (nearly) sharp.

Let $n \geq 2$ and fix a vector $a \in \mathbb{R}^n$. Let $(\gamma(t) = (t, t^2, \ldots, t^n) + a)$, for $t \in \mathbb{R}$. Consider the curve $\Gamma = \{\gamma(t) : 0 \leq t \leq 1\} \subset \mathbb{R}^n$, and the measure $\mu$ supported on $\Gamma$ given by $d\mu(\gamma(t)) = dt$. That is, $\mu$ acts on functions $f$ by $\langle \mu, f \rangle = \int_0^1 f(\gamma(t)) \, dt$. For $r > 0$ a dilate $\mu_r$ of $\mu$ is defined by

$$\langle \mu_r, f \rangle = \int_0^1 f(r\gamma(t)) \, dt,$$

or equivalently, $\mu_r$ may be defined by the equation $\hat{\mu_r}(\xi) = \hat{\mu}(r\xi)$. Here $\hat{\sim}$ denotes the Fourier transform in $\mathbb{R}^n$. A dilate of a distribution $\nu$ is defined similarly.

In analogy with the spherical maximal function introduced by E. M. Stein (see [S3]), one may define the maximal function $\mathcal{N}$ associated to the curve $\Gamma$, with $a = (0, \ldots, 0, 1)$ say, by

$$\mathcal{N} f(x) = \sup_{r > 0} |\mu_r * f(x)| = \sup_{r > 0} \left| \int_0^1 f(x - r\gamma(t)) \, dt \right|, \quad f \in C_0^\infty(\mathbb{R}^n).$$

If $n = 2$ this is a variant of the spherical (circular) maximal function and it is known that $\mathcal{N}$ is bounded on $L^p$ if and only if $p > 2$ (see [B], [MSS], [So]). On the other hand if $n \geq 3$, it is at present unknown whether there is some $p < \infty$ for which $\mathcal{N}$ is bounded on $L^p(\mathbb{R}^n)$.

Let us now abbreviate the lacunary dilate $\mu_{2^k}$ as $\mu_k$ ($k \in \mathbb{Z}$). The corresponding lacunary maximal function may then be defined by

$$\mathcal{M} f(x) = \sup_{k \in \mathbb{Z}} |\mu_k * f(x)| = \sup_k \left| \int_0^1 f(x - 2^k \gamma(t)) \, dt \right|, \quad f \in C_0^\infty(\mathbb{R}^n).$$

In contrast to $\mathcal{N}$, it is well known that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$ (see [DR], [S3]; also see [C]).

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The purpose of this note is to study the $L^p-L^q$ mapping properties of a weighted version of the lacunary maximal function:

$$
\mathcal{M}_{p,q} f(x) = \mathcal{M}_{1/p-1/q} f(x) = \sup_{k \in \mathbb{Z}} |2^{kn/p-n/q} \mu_k * f(x)|, \quad f \in C^0_c(\mathbb{R}^n).
$$

(A weighted maximal function (for the sphere) was first considered by Oberlin [O2]. As was noted there, homogeneity implies that $\mathcal{M}_{p,q}$ can only be bounded from $L^r$ to $L^s$ when $1/r - 1/s = 1/p - 1/q$.)

It appears that the mapping properties of $\mathcal{M}_{p,q}$ are closely related to those of the convolution operator $Tf = \mu * f$. Let

$$
\Delta = \Delta_2 = \left\{ (1/p, 1/q) \in [0, 1] \times [0, 1] : 0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n(n+1)}, \frac{1}{q} \geq \frac{n-1}{np}, \frac{1}{q} \leq \frac{n}{(n-1)p} - \frac{1}{n-1} \right\}.
$$

Thus $\Delta$ is the closed trapezoid (triangle when $n = 2$) with vertices $(0, 0), (1, 1), D = \left(\frac{n^2 - n + 2}{n^2 + n}, \frac{n - 1}{n + 1}\right)$, and $D' = \left(\frac{2}{n+1}, \frac{2n - 2}{n^2 + n}\right)$. For $T$ to be bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ it is necessary that $(1/p, 1/q) \in \Delta$ (see e.g. [M]). When $n = 2$ or $3$ the complete mapping properties of $T$ are known: $T$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $(1/p, 1/q) \in \Delta$ (see [O1]). But when $n \geq 4$ the only known sufficient condition is that $T$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if $(1/p, 1/q)$ belongs to the closed triangle with vertices $(0, 0), (1, 1)$, and $E = \left(\frac{n^2 + n + 2}{2n^2 + 2n}, \frac{n^2 + n - 2}{2n^2 + 2n}\right)$, where $E$ is the midpoint of the line segment $DD'$ (see [M]). Thus when $n \geq 4$ there is a large gap between the known necessary and sufficient conditions.

Note that $\mathcal{M}_{p,q}$ may not be bounded unless $(1/p, 1/q) \in \Delta$, since $\mu * f$ is pointwise dominated by $\mathcal{M}_{p,q} f$. We obtain the following positive result for $\mathcal{M}_{p,q}$ in $\mathbb{R}^3$. It affirms a conjecture of Oberlin. The letter $C$ will denote a constant which may not be the same at each occurrence, but always independent of $l \in \mathbb{Z}$ and $f$ (or $\xi$). Let $\Delta^0$ denote the interior of $\Delta$.

**Theorem.** Let $n = 3$. Then

$$
||\mathcal{M}_{p,q} f||_{L^q(\mathbb{R}^n)} \leq C ||f||_{L^p(\mathbb{R}^n)}
$$

if $(\frac{1}{p}, \frac{1}{q}) \in \Delta^0$, or if $p = q \in (1, \infty]$.

When $p = q \in (1, \infty]$ this is the known result about $\mathcal{M}$ mentioned above. Let us give a brief outline of its proof. The $L^2$ estimate follows from the decay of $\mu$ and a Littlewood-Paley decomposition of $f$ (as in Lemma 1 below). The $L^p$ estimates for $1 < p < 2$ (the other values of $p$ being trivial) are then deduced by applying a “bootstrap” argument (an iterated interpolation argument) similar to the one appearing in [NSW] (see also [DR], [S3]). The proof of the estimates (1) for the points in $\Delta^0$ is similar: it may be based on a Littlewood-Paley decomposition of $f$, and certain uniform oscillatory integral estimates.

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due to Oberlin [O3] and McMichael [M] (see Lemma 2 below), and the convolution properties of \( \mu \) in \( \mathbb{R}^3 \) ([O1]; see above), combined with complex interpolation and a bootstrap argument. A similar argument also shows that (1) holds in \( \mathbb{R}^n \) \((n \geq 2)\) whenever \((1/p, 1/q)\) belongs to the open triangle with vertices \((0, 0), (1, 1)\) and \(E\), where \(E\) is as above.

It may be an interesting problem to determine what happens on the boundary of \( \Delta \) (see e.g. [Ch1, Theorem 4]). It might also be worth pointing out that (1) holds independent of the vector \(a\), in particular when \(a = 0\), although there are related maximal functions whose properties when \(a = (0, \ldots, 0, 1)\) and when \(a = 0\), say, are very different.

To prove the theorem we first need to state two lemmas. Fix a nonnegative function \( \phi \in C_0^\infty(\mathbb{R}) \) such that \( \phi \) is supported in the interval \((1/2, 2)\) and \( \sum_{j \in \mathbb{Z}} \phi(2^j t) \equiv 1 \) for \( t > 0 \). For \( j \in \mathbb{Z} \) the Littlewood-Paley operator \( P_j \) is defined by \( P_j f(\xi) = \phi_j(\xi)\hat{f}(\xi) = \phi(2^j |\xi|)\hat{f}(\xi) \), for \( f \in C_0^\infty(\mathbb{R}^n) \), say. Thus \( f = \sum_{j \in \mathbb{Z}} P_j f \).

The following lemma is standard (see [DR]). It follows by Plancherel’s theorem from the hypotheses on the decay of the Fourier transform of \( \nu \) and the support properties of \( \phi_j \).

**Lemma 1.** Suppose that \( \nu \) is a distribution on \( \mathbb{R}^n \) such that for some number \( \delta > 0 \) \( |\hat{\nu}(\xi)| \leq C|\xi|^{-\delta} \), and \( |\nu(\xi)| \leq C|\xi|^0 \) for \( \xi \in \mathbb{R}^n \). Then

\[
\left\| \left( \sum_{k \in \mathbb{Z}} |\nu_k * P_k f|^2 \right)^{1/2} \right\|_2 \leq C 2^{-\delta|\xi|} \|f\|_2.
\]

It follows from the last inequality that

\[
\left\| \left( \sum_{k} |\nu_k * f|^2 \right)^{1/2} \right\|_2 = \left\| \left( \sum_{k} |\nu_k * \left( \sum_{\ell} P_{k+\ell} f \right)|^2 \right)^{1/2} \right\|_2 \leq C \sum_{\ell} 2^{-\delta|\xi|} \|f\|_2 \leq C \|f\|_2.
\]

Certain special cases of the next lemma were proved by Oberlin [O3]. The general version stated below is due to McMichael [M]. Let \( \mathcal{P}_N \) be the space of real-valued polynomials on \( \mathbb{R} \) of degree at most \( N \).

**Lemma 2.** Given a positive integer \( N \), there exists a constant \( C_N \) such that if \( \alpha_1, \ldots, \alpha_N \) are nonnegative real numbers with \( \sum_{j=1}^N \alpha_j = 1 \), then

\[
\left| \int_a^b e^{ip(t)} \left( \prod_{j=1}^N |p^{(j)}(t)|^{\alpha_j} \right)^{1+is} dt \right| \leq C_N (1 + |s|)^s
\]

if \( p \in \mathcal{P}_N \), \( a < b \), and \( s \in \mathbb{R} \), where \( \sigma = \sum_{j=1}^N \alpha_j \).

**Proof of Theorem.** Following Oberlin and McMichael [M] we define an analytic family of operators by

\[
T_f(x) = \frac{1}{\Gamma((z + 1)/2)} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - t - ur(t) - v\gamma''(t) - v\gamma'''(t)) |u|^z |v|^z du dv dt
\]

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(initially by this equation for \( \Re z > -1 \), then for all complex \( z \) by analytic continuation). Then \( T_z f(x) = \mu^z \ast f(x) \), where

\[
\widehat{\mu^z} (\xi) = C_z \int_0^1 e^{\gamma(t) \cdot \xi} |\gamma''(t) \cdot \xi|^{1-z} \cdot |\gamma'''(t) \cdot \xi|^{-1-z} \, dt
\]

(see [GS, p. 359]). If \( \Re z = -6/5 \), it follows from Lemma 2 with \( p(t) = \gamma(t) \cdot \xi, N = 3, \alpha_1 = 0, \) and \( \alpha_2 = \alpha_3 = 1/5, \) that

\[
|\widehat{\mu^z}(\xi)| \leq C_z \quad \forall \xi \in \mathbb{R}^3,
\]

where the constant \( C_z \) has at most exponential growth in \( |\Im z| \).

Now let \( G_\alpha \) be the Bessel kernel of (complex) order \( \alpha \), i.e.,

\[
\widehat{G_\alpha}(\xi) = (1 + |\xi|^2)^{-\alpha/2},
\]

and take \( \nu = G_\alpha \ast \mu^z \), with \( \Re \alpha = \varepsilon \in (0, 2/5) \). Then \( \hat{\nu}(\xi) = \widehat{G_\alpha}(\xi) \hat{\mu}^z(\xi) \). So \( |\hat{\nu}(\xi)| \leq C_z (1 + |\xi|)^{-\varepsilon} \) if \( \Re z = -6/5 \). Notice also that \(|\hat{\nu}(\xi)| \leq C_2 |\xi|^{2/5} \) if \( \Re z = -6/5 \). Therefore by Lemma 1

\[
(2) \quad \left\| \sup_k \| (G_{\varepsilon + is} \ast \mu^z)_k \ast P_{k+\ell} f \|_2 \right\|_2 \leq C_2 \|e^{-\varepsilon \ell\|f\|} \|_2, \quad \text{if } \Re z = -6/5.
\]

We have \( \|G_{\varepsilon + is}\|_1 \leq C |\Gamma((\varepsilon + is)/2)|^{-1} \) (see [S1, p. 132]). And we can see that \( \mu^z \) is bounded (as a function of \( \xi \)) if \( \tau \in \mathbb{R} \), by making the change of variables \((t, u, v) \rightarrow y = (v_1, y_2, y_3) \) given by \( y = \gamma(t) + u\gamma''(t) + v\gamma'''(t) = (t, t^2 + 2u, 1 + t^3 + 6ut + 6v) \) in the integral for \( T_z f(x) = \mu^z \ast f(x) \), and noting that the Jacobian is a constant. Thus

\[
\left\| (G_{\varepsilon + is} \ast \mu^z) \ast f \right\|_\infty \leq \|G_{\varepsilon + is} \ast \mu^z\|_1 \|f\|_1 \leq \|G_{\varepsilon + is}\|_1 \|\mu^z\|_\infty \|f\|_1 \leq C_{\varepsilon, s} C_\tau \|f\|_1,
\]

where the constant \( C_{\varepsilon,s} C_\tau \) has at most exponential growth in \( s \) and \( \tau \). Hence by homogeneity we have

\[
(3) \quad \left\| \sup_k \|2^{k\varepsilon} (G_{\varepsilon + is} \ast \mu^z)_k \ast P_{k+\ell} f \|_\infty \right\|_\infty \leq C_{\varepsilon,s} C_\tau \|f\|_1, \quad \text{if } \Re z = 0.
\]

To interpolate (2) and (3) we consider an analytic family of vector-valued linear operators defined by

\[
S_\varepsilon(f) = \left\{ 2^{k(3+5\varepsilon/2)} (G_{\varepsilon + is} \ast \mu^z)_k \ast P_{k+\ell} f \right\}_{k \in \mathbb{Z}}
\]

(with \( \varepsilon + is \) and \( \ell \) fixed). Observe that (2) may be restated as boundedness of \( S_\varepsilon \) from \( L^2 \) to \( L^2(\ell^\infty) \) (a mixed-norm space):

\[
\left\| S_\varepsilon(f) \right\|_{L^\infty(Z)} \left\|_{L^2(\mathbb{R})} \right\| \leq C 2^{-\ell\|f\|} \|f\|_2, \quad \text{if } \Re z = -6/5;
\]

and (3) as

\[
\left\| S_\varepsilon(f) \right\|_{L^\infty(Z)} \left\|_{L^\infty(\mathbb{R})} \right\| \leq C \|f\|_1, \quad \text{if } \Re z = 0.
\]
Therefore by complex interpolation in the mixed-norm setting (see [BP], [O2]) we obtain

\[
\|\sup_k \left|2^{k/2} (G_{\xi+i\delta} * \mu)_k * P_{k+\xi} f\right|\|_{L_{2/5}} \leq C 2^{-(5/6)|\xi|}\|f\|_{L_{12/7}},
\]

since \(\mu^{-1} = \mu\).

Now fix a number \(\delta \in (0, 1/3)\). By Theorem 2 in [S2, p. 324] we have \(|\tilde{\mu}(\xi)| \leq C(1 + |\xi|)^{-1/3}\). So \(|(G_{-\delta+i\xi} * \mu)^\wedge(\xi)| = |(G_{-\delta+i\xi})^\wedge(\xi)| \cdot |\tilde{\mu}(\xi)| \leq C\). Hence by Plancherel’s theorem

\[
\|\sup_k |(G_{-\delta+i\xi} * \mu)_k * P_{k+\xi} f|\|_{L_2} \leq \left(\sum_k \left|(G_{-\delta+i\xi} * \mu)_k * P_{k+\xi} f\right|^2\right)^{1/2} \leq C\|f\|_2.
\]

We now apply complex interpolation again to the analytic family

\[
S^\alpha(f) = \{2^{k(\alpha+\delta)/2(\epsilon+\delta)}(G_{\alpha} * \mu)_k * P_{k+\xi} f\}_{k \in \mathbb{Z}}.
\]

Since \(G_0 * \mu = \mu\), (4) and (5) thus yield

\[
\|\sup_k \left|2^{k(3/p_0 - 3/q_0)} \mu_k * P_{k+\xi} f\right|\|_{L_{q_0}} \leq C 2^{-\varepsilon(p_0)|\xi|}\|f\|_{L_{p_0}}
\]

for some \(\varepsilon(p_0) > 0\) if \(2 > p_0 > 12/7\) and \(q_0 = p_0'\) (the conjugate exponent of \(p_0\)).

(See [NSW] and [Ch2] for related positivity arguments.) Let \((1/a, 1/b)\) denote the right endpoint of \(L\). (At the left endpoint the argument is simpler and a bootstrap argument is not necessary, since \(a \geq 2\).) It is known from [O1] that

\[
\|\mu * f\|_b \leq C\|f\|_a,
\]

which implies by homogeneity that for \(k \in \mathbb{Z}\) and the same constant \(C\)

\[
\|2^{k(3/a-3/b)} \mu_k * f\|_b \leq C\|f\|_a.
\]

Since \(1 \leq a \leq b\) it is easy to see that

\[
\left\|\left(\sum_k \left|2^{k(3/a-3/b)} \mu_k * f_k\right|^b\right)^{1/b}\right\|_b \leq C \left\|\left(\sum_k |f_k|^a\right)^{1/a}\right\|_a \leq C \sum_k |f_k|_a.
\]
By interpolating (6") and (7) in the mixed-norm setting we get

$$
\left\| \left( \sum_{k} |2^{3k} \mu_k \ast f_k| 2b \right)^{1/2b} \right\|_{q_1} \leq C \left\| \left( \sum_{k} |f_k|^2 \right)^{1/2p_1} \right\|_{p_1},
$$

with $1/p_1 - 1/q_1 = \beta$ and $1/p_1 = (1/p_0 + 1/a)/2$. (Thus $(1/p_1, 1/q_1)$ is the midpoint of the line segment joining $(1/p_0, 1/q_0)$ and $(1/a, 1/b)$.) Taking $f_k = P_{k+\ell}f$ in (8) we obtain

$$
\left\| \sup_k |2^{3k} \mu_k \ast P_{k+\ell}f| \right\|_{q_1} \leq C \left\| \left( \sum_{k} |P_{k+\ell}f|^2 \right)^{1/2} \right\|_{p_1} \leq C \|f\|_{p_1},
$$

where the last inequality follows from a Littlewood-Paley inequality (see e.g. [So, p. 21]). Interpolating (6) and (8') yields

$$
\left\| \sup_k |2^{3k} \mu_k \ast P_{k+\ell}f| \right\|_{q} \leq C 2^{-\epsilon(\beta p)(\ell)} \|f\|_{p},
$$

for all $(1/p, 1/q)$ on $L$ lying strictly between $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$. Hence we have for the same values of $p$ and $q$

$$
\left\| M_{p,q}f \right\|_{q} \leq C \|f\|_{p},
$$

and by the positivity of $\mu$ (as before)

$$
\left\| \sup_k |2^{3k} \mu_k \ast f_k| \right\|_{q} \leq C \|f\|_{p},
$$

We interpolate again with (9") (in place of (6") in the interpolation step above) and (7) to get (1) on the entire open line segment with endpoints $(1/p_0, 1/q_0)$ and $(1/p_2, 1/q_2)$, where the latter is the midpoint of the line segment joining $(1/p_1, 1/q_1)$ and $(1/a, 1/b)$. By repeating this process we obtain (1) for any point $(1/p, 1/q)$ on $L$.

It should also be clear from this proof that in the statement of the theorem (1) may be replaced by the following slightly stronger estimate:

$$
\left\| \left( \sum_{k} |2^{k(3/p-3/q)} \mu_k \ast f_k|^q \right)^{1/q} \right\|_{q} \leq C \|f\|_{p}.
$$

To see this observe that, for instance, the $\sup_k$ on the left hand side of (2) may be replaced by an $\ell^2$ norm, so that (4) actually holds with the $\sup_k$ replaced by an $\ell^{12/5}$ norm.

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