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ATOMS, PRIMES AND IMPLICATIVE LATTICES

BY

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ABSTRACT. Let L be an a-implicative semilattice. We obtain a characterization of those elements which cover a. This gives a characterization of atoms in pseudocomplemented semilattices, and leads to various results on primes and irreducibles in semilattices. As an application, we prove that in a complete, atomistic lattice L, the following are equivalent (i) L is implicative (ii) L is $(2, \infty)$ meet distributive (iii) each element of L is a meet of primes.

§1. **Introduction.** In [8], D. P. Smith obtained various results regarding atoms and meet-irreducible elements in implicative lattices. She also considered the situation in the MacNeille completion. It is clear that she actually proved most of her results for implicative meet semilattices. We shall formulate and prove more general results for arbitrary meet semilattices, which reduce to her results in the case of implicative semilattices. We shall also consider in considerable detail the prime elements in a semilattice. Finally, we shall apply these results to show that in a complete atomistic lattice L the following are equivalent (i) L is implicative (ii) L is $(2, \infty)$ meet distributive (iii) each element of L is a meet of primes. We thank the referee for suggestions which have improved the presentation of the paper.

§2. **Preliminaries.** Let L be a meet semilattice. We shall denote the binary operation by juxtaposition, so that ab will then also denote the greatest lower bound of two elements a, b of L. For a set A, the greatest lower bound shall be denoted by ΛA, and the least upper bound by ΣA. if they exist. Given elements a, b of L, we shall denote by $[a, b]$ the closed interval $\{x \in L \mid a \leq x \leq b\}$. We denote by $[a)$ the principal filter $\{x \in L \mid x \geq a\}$ generated by a and by $(a]$ the principal ideal $\{x \in L \mid x \leq a\}$ generated by a. Let $a \in L$. Then an element $x \in L$ is an a-cover if $a < x$, and if $a \leq y \leq x$ implies that $y = a$ or $y = x$. In a semilattice with 0, an element x is an atom if $x \neq 0$, and if $y \leq x$ implies that $y = 0$ or $y = x$. Clearly, a 0-cover is an atom. Because of a slight confusion in terminology, we shall, for definiteness, follow Grätzer [5] and define the term atomistic as follows. A lattice with 0 is atomistic if every element $\neq 0$ is the join of atoms.

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An element a of a semilattice L is irreducible if $a = bc$ implies that $a = b$ or $a = c$. It is prime if $a \geq bc$ implies that $a \geq b$ or $a \geq c$. Clearly, a prime is irreducible, and the two notions coincide in case L is distributive. Recall that a semilattice L is distributive if $z \geq xy$ (where $x, y, z \in L$) implies the existence in L of elements x_1, y_1 such that $x_1 \geq x$, $y_1 \geq y$ and $z = x_1 y_1$. L is modular if whenever $x \geq z \geq xy$, we can find $y_1 \geq y$ such that $z = xy_1$. A distributive semilattice is modular. L is $(2, \infty)$ meet distributive if whenever Σx_i exists for some family $\{x_i \mid i \in I\}$ of elements of L, then for each $x \in L$, Σxx_i exists and equals $x\Sigma x_i$.

If x, a are elements of a semilattice L, then by the annihilator $\langle x, a\rangle$ we mean $\{y \in L \mid xy \leq a\}$. If for a fixed $a \in L$, the annihilator $\langle x, a\rangle$ is a principal ideal for each $x \in L$, then L is said to be a-implicative (see [9]). In that case, we shall denote the principal ideal by $(x * a]$ for each $x \in L$, where $a \in L$ is fixed. More generally, if for any two elements x, a of L, the annihilator $\langle x, a\rangle$ is a principal ideal, then we shall also write $x * a$ for the generator of this ideal, even in the case L is not a-implicative. The semilattice L is implicative if and only if it is a-implicative for all $a \in L$. If L has a least element 0, then 0-implicative means pseudocomplemented, and it is customary in that case to write x^* for $x * 0$. An implicative semilattice is distributive. We shall denote by $\mathcal{I}(L)$ the set of all elements $a \in L$ such that L is a-implicative. It is a subsemilattice of L (see [6], [9]). We refer the reader to [4], [6], [7], [8] and [9] for lists of properties satisfied by $*$ in implicative and a-implicative semilattices.

An element x of an a-implicative semilattice is a-closed if $(x * a) * a = x$. The set of a-closed elements will be denoted by C_a. It is a subsemilattice of L. In [6], we showed that it is a Boolean algebra. If $a = 0$, 0-closed means closed. If a is an element of a semilattice L, then $x \in L$ is a-dense if $\langle x, a\rangle = (a]$. The set of all a-dense elements of L will be denoted by D_a. If $a = 0$, 0-dense means dense. D_a is either empty or is a filter. If L is a-implicative, then $x \in D_a$ if and only if $x * a = a$, or equivalently, $(x * a) * a = 1$. In [9] Varlet showed that an element $a \neq 1$ of a semilattice L is prime if and only if $D_a = L - (a]$.

§3. **Atoms and primes.** We begin by proving some results regarding irreducible elements, in the same spirit as Varlet's result on primes and Smith's result on meet irreducible elements in implicative lattices. In the rest of the paper, L shall denote a general semilattice unless we qualify it with various adjectives.

PROPOSITION 3.1. *Let L be a modular semilattice. Then $a \in L$ is irreducible if and only if $\{x \mid x > a\} \subset D_a$.*

Proof. Let $\{x \mid x > a\} \subset D_a$, and suppose that $a = xy$. Suppose that $a < x$. Then $x \in D_a$, that is $\langle x, a\rangle \subset (a]$. Since $xy = a$, we have $y \in \langle x, a\rangle \subset (a]$, and thus $y \leq a$. But $a \leq y$. Hence $a = y$. Conversely, suppose that a is irreducible and that $x > a$. Let $y \in \langle x, a\rangle$, that is $xy \leq a$. Now $xy \leq a < x$. Since L is modular, we

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can write $a = xy_1$ for some $y_1 \geq y$. Since a is irreducible, it follows that $a = x$ or $a = y_1$. Since $x > a$, we have $a = y_1$. Thus $a \geq y$, that is, $y \in (a]$.

If we do not assume modularity, then the situation is as described in the next few results.

LEMMA 3.2. *If $a \in \mathcal{I}(L)$ and $x \in C_a$, then $D_a \subset D_x$.*

Proof. We observe that if $a \in \mathcal{I}(L)$ and $x \in C_a$, then $x \in \mathcal{I}(L)$ also. In fact, for each $y \in L$, we can define $y * x = \{y(x * a)\} * a$. This is because $z \leq \{y(x * a)\} * a \Leftrightarrow zy(x * a) \leq a \Leftrightarrow zy \leq (x * a) * a \Leftrightarrow zy \leq x$. Thus the principal ideal generated by $\{y(x * a)\} * a$ is $\{z \mid z \leq \{y(x * a)\} * a\} = \{z \mid zy \leq x\} = \langle y, x\rangle$. This is true for each $y \in L$ if $x \in C_a$. Thus $x \in \mathcal{I}(L)$. Now suppose that $y \in D_a$. Then $y * x = y * \{(x * a) * a\} = \{y(x * a)\} * a = (x * a) * (y * a) = (x * a) * a = x$, that is, $y \in D_x$.

LEMMA 3.3. *Let $a \in \mathcal{I}(L)$ and let $x \in C_a$. Then $\{y \mid y > x\} \cap D_a = \{y \mid y > x\} \cap D_x$.*

Proof. By Lemma 3.2, $\{y \mid y > x\} \cap D_a \subset \{y \mid y > x\} \cap D_x$. Conversely, suppose that $y > x$ and $y \in D_x$. Then we have $y * x = x = (x * a) * a$. Hence $(x * a)(y * x) = (x * a)[(x * a) * a] = a$. But $(x * a)(y * x) = (x * a)[y * \{(x * a) * a\}] = (x * a)[\{y(x * a)\} * a] = (x * a)[(x * a) * (y * a)] = (x * a)(y * a)$. Thus $(x * a)(y * a) = a$. Hence $1 = a * a = \{(x * a)(y * a)\} * a = (y * a) * \{(x * a) * a\} = (y * a) * x$. Thus $y * a \leq x < y$, and hence $y(y * a) = y * a$. Thus $a \leq y * a = y(y * a) \leq a$, that is, $y * a = a$, which means that $y \in D_a$.

COROLLARY 3.4. *Let $a \in \mathcal{I}(L)$ and $x \in C_a$. Then x is irreducible if and only if $\{y \mid y > x\} \subset D_x$.*

Proof. Suppose that x is irreducible and that $y > x$. Then $y(y * x) = yx = x$. Since x is irreducible, we have $x = y * x$. Conversely, suppose that $\{y \mid y > x\} \subset D_x$. Let $x = st$ and suppose that $x < s$. Then $s \in D_x$, and $1 = x * x = (st) * x = t * (s * x) = t * x$. Thus $t \leq x$. But $x \leq t$. Hence $x = t$.

PROPOSITION 3.5. *Let $x \in \mathcal{I}(L)$. Then x is irreducible if and only if $\{y \mid y > x\} \subset D_x$.*

Proof. In Corollary 3.4, let $x = a$.

THEOREM 3.6. *Let $a \in \mathcal{I}(L)$. Then x is an a-cover if and only if $x > a$ and $\langle x, y\rangle \cap [a) = [a, x * a]$ for each $y \in (L - [x)) \cap [a)$.*

Proof. Suppose that x is an a-cover, and that $y \not\geq x$, $y \geq a$. Let $a \leq t \leq x * a$. Then $xt \leq x(x * a) = xa \leq a \leq y$. Hence $t \in \langle x, y\rangle \cap [a)$. Thus $[a, x * a] \subset \langle x, y\rangle \cap [a)$. On the other hand, suppose that $z \in \langle x, y\rangle \cap [a)$, that is, $xz \leq y$ and $z \geq a$. Then $a \leq xa \leq xy \leq x$. Since x is an a-cover, it follows that $xy = a$ or $xy = x$. But $xy = x$ means that $x \leq y$, a contradiction. Hence $xy = a$. Thus

$a \leq xz \leq xy = a$, that is, $xz = a$. Thus $a \leq z \leq x * a$, that is, $z \in [a, x * a]$. Conversely, suppose that $a \leq z < x$, that is, $x \nleq z$. Thus $z \in (L-[x)) \cap [a)$. Hence $\langle x, z\rangle \cap [a) = [a, x * a]$. Since $z \in \langle x, z\rangle \cap [a)$, it follows that $a \leq z \leq x * a$. But $z \leq x$. Hence $a \leq z \leq x(x * a) = xa = a$. Hence $z = a$.

COROLLARY 3.7. *Let L be pseudocomplemented. Then x is an atom if and only if $x \neq 0$ and $\langle x, y\rangle = (x^*]$ for all $y \in L-[x)$.*

PROPOSITION 3.8. *Let $a \in \mathcal{I}(L)$. If x is an a-cover, then $x * a$ is prime.*

Proof. Clearly $x * a \neq 1$ since otherwise we would have $x \leq a$. According to Varlet's result (Theorem 2.7 [9]), we need only show that $L-(x * a] \subset D_{x*a}$ since it is always true in general that $D_z \subset L-(z]$ for any element $z \neq 1$. Let $y \in L-(x * a]$, that is, $y \nleq x * a$. Then $x \nleq y * a$, for if $x \leq y * a$, then $y \leq (y * a) * a \leq x * a$. Also $y * a \geq a$. Thus $y * a \in (L-[x)) \cap [a)$. Since x is an a-cover, we have $\langle x, y * a\rangle \cap [a) = [a, x * a]$. But it is easily checked that $\langle x, y * a\rangle = \langle y, x * a\rangle$. Thus $\langle y, x * a\rangle \cap [a) = [a, x * a]$. On the other hand, $y\{y * (x * a)\} = y(x * a) \leq x * a$, and $y * (x * a) \geq x * a \geq a$. Thus $y * (x * a) \in \langle y, x * a\rangle \cap [a) = [a, x * a]$. Hence $a \leq y * (x * a) \leq x * a$. But $y * (x * a) \geq x * a$. Thus $y * (x * a) = x * a$, that is, $y \in D_{x*a}$.

PROPOSITION 3.9. *Let $a \in \mathcal{I}(L)$ and let $x \geq a$. Suppose that x is prime. If $y \geq x$, then either $y = x$ or $y \in D_a$.*

Proof. We may assume that $x \neq 1$ since otherwise we would have $y = 1 \in D_a$. Suppose that $y \neq x$. Then $y \nleq x$, that is, $y \in L-(x]$. But $L-(x] = D_x$ since x is prime. Thus $y \in D_x$, that is $\langle y, x\rangle \subset (x]$. Now, $y(y * a) = ya \leq a \leq x$. Hence $y * a \in \langle y, x\rangle \subset (x]$, that is, $y * a \leq x$. It follows that $y * a \leq x \leq y \leq (y * a) * a$. Thus $y * a = (y * a)[(y * a) * a] = a$.

REMARK. We can interpret this result as follows. Let $a \in \mathcal{I}(L)$ and let $x \geq a$. If there exists a prime in $[a, x]$, then either $x \in D_a$ or the prime is x, that is, if $x \notin D_a$, then the only possible prime in $[a, x]$ is x.

COROLLARY 3.10. *Let L be a pseudocomplemented semilattice and let p_1, p_2 be distinct non-dense primes of L. Then p_1 and p_2 are incomparable.*

Proof. By Proposition 3.9 and the remark above.

To prove the next few results we need a few properties of $*$ that seem to be new.

LEMMA 3.11. *Let $a, c \in \mathcal{I}(L)$. Then for each $b \in L$ we have $a(b * c) = a[(ab) * (ac)]$.*

Proof. Since $a, c \in \mathcal{I}(L)$ we have $ac \in \mathcal{I}(L)$ since $\mathcal{I}(L)$ is a subsemilattice of

L. Now, $(ab) * (ac) = [(ab) * a][(ab) * c] = (ab) * c = a * (b * c)$. Hence $a[(ab) * (ac)] = a[a * (b * c)] = a(b * c)$.

LEMMA 3.12. *Let $b, c \in \mathcal{I}(L)$ with $b \geq c$. Then for each $a \in L$ we have $(a * b) * c = [(a * c) * c][b * c]$.*

Proof. We have $b \leq a * b$. Hence $(a * b) * c \leq b * c$. Also $c \leq b$ gives $a * c \leq a * b$, and hence $(a * b) * c \leq (a * c) * c$. Thus, $(a * b) * c \leq [(a * c) * c][b * c]$. We now apply Lemma 3.11 and obtain $[(a * c) * c] \times [b * c][a * b] = [(a * c) * c][b * c][\{(b * c)a\} * \{(b * c)b\}]$. But $b(b * c) = bc = c = c(b * c)$ since $c \leq b$, and $c \leq b * c$. Thus $[(a * c) * c][b * c][a * b] = [(a * c) * c][b * c][\{(b * c)a\} * \{(b * c)c\}] = [(a * c) * c][b * c][a * c]$ (by Lemma 3.11) $= [(a * c) * c][a * c][b * c] = (a * c)c(b * c) = c$. This means that $[(a * c) * c][b * c] \leq (a * b) * c$ and completes the proof of the Lemma.

PROPOSITION 3.13. *Let $a, b \in \mathcal{I}(L)$ with $a \leq b$, and suppose that x is an a-cover. Then either $x * a = x * b$ or $b * a \leq x * a$.*

Proof. Suppose that $x * a \neq x * b$. Since $a \leq b$, we have $x * a \leq x * b$. This means that $x * a < x * b$. Since $x * a$ is prime by Proposition 3.8, it follows by Proposition 3.9 that $x * b \in D_a$. Thus $(x * b) * a = a$. But since $b \geq a$, it follows by Lemma 3.12 that $(x * b) * a = [(x * a) * a][b * a]$. Thus $[(x * a) * a][b * a] = a$, and hence $1 = a * a = \{[(x * a) * a][b * a]\} * a = (b * a) * (x * a)$. This means that $b * a \leq x * a$.

LEMMA 3.14. *Let $a \in \mathcal{I}(L)$ and let $x \geq a$ be prime. Then either $x \in D_a$ or $x \in C_a$.*

Proof. Suppose that $x \notin D_a$. Since $a \leq x \leq (x * a) * a$, by Proposition 3.9, we have that either $x = (x * a) * a$ or $(x * a) * a \in D_a$. But $(x * a) * a \in D_a$ would mean that $x * a = a$, that is, $x \in D_a$. Thus $(x * a) * a = x$, that is, $x \in C_a$.

REMARK. It is easily checked that if $a \in \mathcal{I}(L)$, then $C_a \cap D_a = \{1\}$. Also, it is easily checked that if $a \in L$ is prime, then $a \in \mathcal{I}(L)$. In fact, we then have $x * a = a$ if $x \not\leq a$, and $x * a = 1$ if $x \leq a$, for all $x \in L$.

PROPOSITION 3.15. *Suppose that L is a modular semilattice and let $a \in \mathcal{I}(L)$. Suppose that x is an a-cover. If $p \in C_a$ is prime, then either $x \leq p$ or $p = x * a$.*

Proof. Suppose that $x \not\leq p$. Then we must have $p \leq x * a$. For, if $p \not\leq x * a$, then $p * (x * a) = x * a$ since $x * a$ is prime by Proposition 3.8. Thus $(px) * a = x * a$. But $x > a$, and $p \in C_a$ also means that $p \geq a$. Then, by Lemma 2.9 of [6], we have that $pxd = xd$ for some $d \in D_a$, that is, $xd \leq p$. Since p is prime and $x \not\leq p$, we have $d \leq p$, and hence $p \in D_a$ since D_a is a filter. Thus $p \in C_a \cap D_a = \{1\}$. This gives $x \leq p$, a contradiction. Thus $p \leq x * a$. But this means that $a \leq p \leq x * a$, and hence, by Proposition 3.9, either $p = x * a$ or $x * a \in D_a$.

But if $x * a \in D_\alpha$, we have $x * a \in D_a \cap C_a = \{1\}$, and hence that $x \leq a$, a contradiction. Thus $p = x * a$.

§4. **Implicative lattices**

THEOREM 4.1. *Let L be a complete semilattice and let $a \in \mathcal{I}(L)$. If x is an a-cover and if $y \geq a$, then $x * y$ is defined.*

Proof. For each $s \in L$, let $P(s) = \{\text{primes } p \mid s \leq p\}$. We claim that $x * y =$ greatest lower bound of the set $[P(y) - P(x)] = \{\text{primes } p \mid y \leq p,\ x \nleq p\}$. Let $d = \Lambda[P(y) - P(x)]$, which exists since L is complete. We need to show that $z \leq d$ if and only if $zx \leq y$. Suppose that $zx \leq y$. If $p \in \mathbb{P}(y) - P(x)$, we have $zx \leq y \leq p$, and hence $z \leq p$ since p is prime and $x \nleq p$. Then $z \leq d$. Conversely, suppose that $z \leq d$. We first observe that if $p \in P(y)$, then $zx \leq p$. In fact, if $x \leq p$, then clearly $zx \leq x \leq p$. If $x \nleq p$, then $p \in P(y) - P(x)$; hence $z \leq d \leq p$. Thus, we have $zx \leq z \leq p$. We now use the fact that x is an a-cover and hence that $x * a$. Since $x * a$ is prime. If $x \leq y$, then clearly $zx \leq x \leq y$. Suppose that $x \nleq y$. Since $y \geq a$, we have $y \in (L - [x)) \cap [a)$. Since x is an a-cover, it follows by Theorem 3.6 that $\langle x, y \rangle \cap [a) = [a, x * a]$. Clearly $y \in \langle x, y \rangle \cap [a)$. Thus $a \leq y \leq x * a$, and $x * a \in P(y)$. Hence, by our observation above, we have that $zx \leq x * a$. Since $x * a$ is prime, we must have $z \leq x * a$ or $x \leq x * a$. But if $x \leq x * a$, then $x = x(x * a) = xa \leq a$, which contradicts the fact that $x > a$. Thus $z \leq x * a$, and hence $zx \leq a \leq y$.

THEOREM 4.2. *Let L be a complete atomistic lattice. Then the following are equivalent:*

(i) *L is implicative*

(ii) *L is $(2, \infty)$ meet distributive*

(iii) *Each element of L is a meet of primes.*

Proof. The fact that (i) and (ii) are equivalent is well known (for example, see Theorem 15, page 147 of [3]). However, it might be instructive to re-prove that (ii) implies (i) in another way. Suppose that L is $(2, \infty)$ meet distributive. Then, by Theorem 1, page 111 of [5], L is pseudocomplemented, that is, $0 \in \mathcal{I}(L)$. Then for each $x, y \in L$, we have $y \geq 0$, and we can write $x = \Sigma a$, a join of atoms. By Theorem 4.1, $a * y$ is defined for each a. We claim that $x * y = \Lambda(a * y)$, the greatest lower bound of the elements $a * y$. We need to show that $z \leq \Lambda(a * y)$ if and only if $zx \leq y$, that is, $z(\Sigma a) \leq y$. But $z \leq \Lambda(a * y) \Leftrightarrow z \leq a * y$ for each a, $\Leftrightarrow az \leq y$ for each a, $\Leftrightarrow \Sigma(az) \leq y \Leftrightarrow z(\Sigma a) \leq y$, using the property of $(2, \infty)$ meet distributivity. Thus (ii) implies (i). We now prove that (iii) implies (ii). Suppose that every element of L is a meet of primes. Since L is complete, so is $\mathcal{I}(L)$. In fact, let $\{a_i \mid i \in I\} \subset \mathcal{I}(L)$. We have to show that $\Lambda a_i \in \mathcal{I}(L)$. We have that $\Lambda a_i \in L$. Let $x \in L$. Then $x * (\Lambda a_i) = \Lambda(x * a_i)$. Thus $\mathcal{I}(L)$ is complete. Since each element of L is a meet of primes, it follows that each element of L is in $\mathscr{I}(L)$, that is, $L=\mathscr{I}(L)$, and hence L is implicative. Thus (iii) implies (ii). Finally, suppose that L is $(2, \infty)$ meet distributive. Let $y \in L$. We have to show that y is a meet of primes. We may assume that $y \neq 1$. Since L is atomistic, we can write $1=\Sigma a$, a join of atoms, and $y=1 * y=\Lambda(a * y)$. In this expression, we need only keep those atoms a for which $a \not\leq y$. We claim that there will be such atoms. For if not, then each atom a in $1=\Sigma a$ satisfies $a \leq y$. Then $1=\Sigma a \leq y$, contradicting the hypothesis that $y \neq 1$. Thus we have $y=\Lambda(a * y)$ where a are atoms for which $a \not\leq y$. Since a is an atom, we have that a^* is prime. Also $a^* \leq a * y$. Thus, we have a prime $a^* \in [0, a * y]$. Clearly $a * y \notin D_y$. For if $a * y \in D_y$, then $(a * y) * y=y$. Hence $a * y=1$, that is, $a \leq y$, a contradiction. Thus, by the remark following Proposition 3.9, it follows that $a * y=a^*$, a prime. Thus $y=\Lambda(a * y)=\Lambda a^*$, a meet of primes, and hence (ii) implies (iii).

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