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A NOTE ON ROOT DECISION PROBLEMS IN GROUPS

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1. Introduction. Consider a positive integer $r > 1$. We say that the rth root problem is solvable for a group G if we can decide for any $W \in G$ whether or not W has an rth root, i.e. whether or not there exists $V \in G$ such that $W = V^r$.

Baumslag, Boone and Neumann [**1**] proved that there exists a finitely presented group with all root problems unsolvable. Here we are concerned with the relationship between the different root problems. We prove:

THEOREM 1. *Let r and s be positive integers such that neither divides the other. Then the corresponding root problems are independent.*

THEOREM 2. *Let $r_1, r_2, \ldots, r_n$ be positive integers such that each has a prime divisor which does not divide any of the others. Then the corresponding root problems are independent.*

Recall that a group G with generators $x_1, x_2, \ldots$ and defining relations $R_1, R_2, \ldots$ is said to be *recursively presented* if there exists an effective process which lists the words R_n. We say that the decision problems $D_1, D_2, \ldots, D_n$ for groups are *independent* if for any subset $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$ there exists a recursively presented group with $D_{i_1}, \ldots, D_{i_m}$ solvable, but the remaining D_i's unsolvable.

Our proofs use a one-to-one recursive function $\phi : \mathbf{N} \rightarrow \mathbf{N}$ whose image, Im ϕ, is non-recursive, i.e. given a positive integer k, we can compute $\phi(k)$ but we cannot decide if k belongs to Im ϕ. Such functions are known to exist; c.f. Britton [**2**, Lemma 2.31]. We will also use the following theorem which follows from elementary properties of free products.

THEOREM A. *Let G be the free product of groups A_i. Then the* rth *root problem is solvable for G if the word problem and* rth *root problem are solvable for the factors A_i.*

2. Proof of Theorem 2. Let p be a prime and let $\phi : \mathbf{N} \rightarrow \mathbf{N}$ be a one-to-one recursive function with a non-recursive image (see introduction). Let G_p be the group with generators

$$x_1, x_2, x_3, \ldots \text{ and } y_1, y_2, y_3, \ldots$$

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and defining relations

$$x_{\phi(1)} = y_1^p, \qquad x_{\phi(2)} = y_2^p, \qquad x_{\phi(3)} = y_3^p, \ldots$$

Clearly G_p is recursively presented since ϕ is recursive. We claim:

LEMMA. *The word problem is solvable for G_p. The* rth *root problem is unsolvable for G_p if p divides r, and solvable if p does not divide r.*

Proof of Lemma. Observe that G_p is the free product of the infinite cyclic groups generated by the y_i and the infinite cyclic groups generated by the x_j for $j \notin \operatorname{Im} \phi$. Since ϕ is recursive, we can always write $W \in G_p$ as a word in the x's and y's so that y_i does not appear next to x_j whenever $\phi(i) = j$. This gives the syllable (free product) length of W. Thus we can solve the word problem for G_p.

Suppose p divides r, say $r = pt$. Then x_i^t has an rth root if and only if $i \in \operatorname{Im} \phi$; if $\phi(j) = i$ then y_j is an rth root of x_i^t. But $\operatorname{Im} \phi$ is non-recursive; hence the rth root problem is unsolvable for G_p if p divides r.

On the other hand, suppose p does not divide r. Since the word problem is solvable for G_p and since G_p is a free product, it suffices to solve the rth root problem for a factor H of G_p. Let $V \in H$. Then $V = y_i^m$ or $V = x_i^m$. We claim that V has an rth root if and only if r divides m. This is clearly true in the case that $V = y_i^m$, or $V = x_i^m$ for $i \notin \operatorname{Im} \phi$. Suppose $\phi(j) = i$; then $V = x_i^m = y_j^{mp}$. But p does not divide r; hence V has an rth root if and only if r divides m. Thus the rth root problem is solvable for H, and hence it is solvable for G_p. Accordingly, the lemma is proved.

We now prove Theorem 2. Let $\{i_1, \ldots, i_m\}$ be a subset of $\{1, 2, \ldots, n\}$ and let G be the direct product

$$G = G_{z_1} \times G_{z_2} \times \ldots \times G_{z_m}, \qquad z_j = p_{i_j}$$

By the above lemma, the r_ith root problem is solvable for G if and only if $i \notin \{i_1, \ldots, i_m\}$. Thus, Theorem 2 is proved.

3. Proof of Theorem 1. Let $d = \gcd(r, s)$; say $r = da$ and $s = db$. Then $\gcd(a, b) = 1$; also $a \neq 1$ and $b \neq 1$ since neither r nor s divides the other. Let G be the group with generators

$$x_1, x_2, x_3, \ldots \quad \text{and} \quad y_1, y_2, y_3, \ldots$$

and defining relations

$$x_{\phi(1)}^d = y_1^r, \qquad x_{\phi(2)}^d = y_2^r, \qquad x_{\phi(3)}^d = y_3^r, \ldots$$

Clearly G is recursively presented since ϕ is recursive. We claim that the sth root problem is solvable for G, but the rth root problem is not. This will prove our theorem since we can interchange r and s to obtain a group for which the rth root problem is solvable but the sth root problem is not.

Note that G is the free product of the groups

$$H_i = \langle x_{\phi(i)}, y_i; x_{\phi(i)}^d = y_i^r \rangle, \text{ and } K_j = \langle x_j \rangle, j \notin \operatorname{Im} \phi.$$

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Observe that x_k^d has an rth root if and only if $k \in \operatorname{Im} \phi$; hence the rth root problem is unsolvable for G because $\operatorname{Im} \phi$ is non-recursive. It remains to show that the sth root problem is solvable for G. We claim, first of all, that the word problem is solvable for G. Let $W \in G$, i.e. let W be a word in the x's and y's. Now if a y_i appears next to an x_j in W, then we can decide whether or not they belong to the same factor of G because we can decide whether or not $\phi(i) = j$. Moreover, we can solve the word problem for the factors of G. Thus we can determine the syllable length of W, and hence solve the word problem for G. Accordingly, it suffices to solve the sth problem for the factors of G. That is, given an element V in a factor of G, we have to decide whether or not V has an sth root. There are two cases.

Case I. V is a power of y_i, or a word in Y_i and x_j with $\phi(i) = j$. Then V belongs to H_i. How H_i is a free product of two infinite cyclic groups with a cyclic amalgamation. The sth root problem is solvable for such a group; cf. [**3**]. Thus we can decide whether or not V has an sth root.

Case II. V is a power of x_i; say $V = x_i^n$. We claim that V has an sth root if and only if s divides n. If $i \notin \operatorname{Im} \phi$ then V would belong to the infinite cyclic group K_i generated by x_i, and the claim clearly holds. On the other hand, suppose $i \in \operatorname{Im} \phi$; say $\phi(k) = i$. Then V belongs to the group

$$H_k = \langle x_i, y_k; x_i^d = y_k^r \rangle.$$

We view H_k as a free product with an amalgamation. If s divides n then clearly V has an sth root. Suppose, however, that V has an sth root. We claim that one such sth root U has syllable length one. If d does not divide n, then U must have syllable length one. If d does divide n, then V belongs to the center of H_k. It follows that V has an sth root U which is cyclically reduced. This sth root U has syllable length one.

We now have that $V = U^s$ where U has syllable length one. There are two possibilities:

Case A. $U = x_i^c$. Then $x_i^{cs} = U^s = V = x_i^n$, whence s divides n.

Case B. $U = y_k^e$. Then $y_k^{es} = U^s = V = x_i^n$. Then V lies in the amalgamated subgroup, whence y_k^{es} is a power of y_k^r; say

$$es = rf.$$

Recall $r = ad$ and $s = bd$ where $\gcd(a, b) = 1$. It follows that b divides f; say $f = bg$. Using the relation $x_i^d = y_k^r$, we have

$$V = U^s = y_k^{es} = y_k^{rf} = x_i^{df} = x_i^{dbg} = x_i^{sg}.$$

But $V = x_i^n$; hence s divides n.

We have shown in Case II that $V = x_i^n$ has an sth root if and only if s divides n. Thus we can decide whether or not V has an sth root.

Accordingly, the sth root problem is solvable for G, and therefore the theorem is proved.

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