ON THE TRANSCENDENCE OF SOME INFINITE SERIES

JAROSLAV HANČL* AND JAN ŠTĚPNIČKA

Department of Mathematics and Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic
e-mail: hancl@osu.cz, janstepnicka@centrum.cz

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Abstract. The paper deals with a criterion for the sum of a special series to be a transcendental number. The result does not make use of divisibility properties or any kind of equation and depends only on the random oscillation of convergence.

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1. Introduction. Erdős [2] proved that if \( \{a_n\}_{n=1}^{\infty} \) is an increasing sequence of positive integers such that \( \lim_{n \to \infty} a_n^{1/n} = \infty \) then the number \( S = \sum_{n=1}^{\infty} \frac{1}{a_n} \) is irrational. In [4] it is shown that if \( \lim_{n \to \infty} \log_3 \log_2 a_n > 1 \) and \( a_n \in \mathbb{N} \) for all \( n \in \mathbb{N} \) then \( S \) is a transcendental number. Many other criteria for \( S \) to be transcendental can be found in [1], [5], [6] or [8] but divisibility properties or fulfilling special equations are necessary. It seems to be the case that in general it is not easy to decide when \( S \) is a transcendental number if \( \limsup_{n \to \infty} \log_3 \log_2 a_n < 1 \) holds and divisibility properties or fulfilling certain equations are not required. In this paper we give conditions on sequences \( \{a_n\}_{n=1}^{\infty} \) with \( \limsup_{n \to \infty} \log_3 \log_2 a_n < 1 \) such that \( S \) is transcendental. We prove the following.

**Theorem 1.1.** Let \( \{a_n\}_{n=1}^{\infty} \) be an eventually non-decreasing sequence of positive integers such that \( a_n > 2^n \) for every sufficiently large \( n \). Suppose that \( a_n < 2^{3^{n/2}} \) and that \( a_{2n} > 2^{3^{n/2} - 4^n} \) for infinitely many \( n \). Then the number \( \sum_{n=1}^{\infty} \frac{1}{a_n} \) is transcendental.

**Example 1.1.** Let \( a_1 = a_2 = 1 \). For every \( s = 0, 1, 2, 3, \ldots \) set

\[
a_n = \begin{cases} 
2^{3^{n/2} + 1} + 3 & \text{if } 2^3 < n \leq 2.2^{3^n} \\
2^{3^{n/2} + 1} + 2^{3^{n/2} + 1} + 3 & \text{if } 2.2^{3^n} < n \leq 2^{3^n + 1}.
\end{cases}
\]

Then the number \( \sum_{n=1}^{\infty} \frac{1}{a_n} \) is transcendental.

It is unclear to the authors if there exists a sequence \( \{a_n\}_{n=1}^{\infty} \) of positive integers such that \( \sum_{n=1}^{\infty} \frac{1}{a_n} \) is an algebraic number and \( a_n > 2^{(\frac{3}{2})^n} \) for all \( n \in \mathbb{N} \).

2. Main results. In the sequel, for a real number \( x \) we use \( \lfloor x \rfloor \) to denote the greatest integer less than or equal to \( x \). Theorem 1.1 is an immediate consequence of the following theorem.

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THEOREM 2.1. Let $\alpha$, $\beta$, $\gamma$ and $\nu$ be real numbers with $0 < \beta < \alpha < \log_2 3$, $0 \leq \nu < 1$ and $\gamma > 0$. Assume that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are sequences of positive integers with $\{a_n\}_{n=1}^\infty$ eventually non-decreasing such that for every sufficiently large $n$

\[ b_n < a_n^\nu \log_2^\gamma a_n \quad (1) \]

and

\[ a_n > 2^n . \quad (2) \]

Suppose that there exists a positive real number $k$ with

\[ k < \frac{(\alpha - \beta)}{\log_2(\frac{2}{1-\nu} + 1) - \alpha} \quad (3) \]

such that for infinitely many $n$

\[ a_n < 2^{3^{jn}} \quad (4) \]

and

\[ a_{n+[k.n]} > 2^{2^{a_n+\lfloor k.a \rfloor}} . \quad (5) \]

Then the number $\sum_{n=1}^\infty \frac{b_n}{a_n}$ is transcendental.

As an immediate consequence of Theorem 2.1 we obtain the following corollary.

COROLLARY 2.1. Let $\alpha$, $\beta$ and $\gamma$ be real numbers with $0 < \beta < \alpha < 1$ and $\gamma > 0$. Assume that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are sequences of positive integers with $\{a_n\}_{n=1}^\infty$ eventually non-decreasing such that for every sufficiently large $n$

\[ b_n < \log_2^\gamma a_n \]

and

\[ a_n > 2^n . \]

Suppose that there exists positive real number $k$ with

\[ k < \frac{(\alpha - \beta)}{1 - \alpha} \]

such that for infinitely many $n$

\[ a_n < 2^{3jn} \]

and

\[ a_{n+[k.n]} > 2^{2^{a_n+\lfloor k.a \rfloor}} . \]

Then the number $\sum_{n=1}^\infty \frac{b_n}{a_n}$ is transcendental.

REMARK 2.1. Let the sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ satisfy all conditions (1)--(5). Then Theorem 2.1 implies that the number $\sum_{n=1}^\infty \frac{b_n}{a_n}$ is transcendental. If in addition
there exists a fixed \( \epsilon > 0 \) such that

\[
a_n < 2^{(2-\epsilon)n}
\]

holds for all sufficiently large \( n \) then there exists a sequence \( \{c_n\}_{n=1}^{\infty} \) of positive integers such that \( \sum_{n=1}^{\infty} \frac{h_n}{c_n a_n} \) is a rational number. For more information see [3].

3. Proof. Proof: (of Theorem 2.1) Let \( N \) be a sufficiently large positive integer satisfying (4) and (5). Assume that \( \delta \) is a sufficiently small positive real number.

Let us define the finite sequence \( \{c_t\}_{t=N}^{T} \) by

\[
c_t = \begin{cases} 
  d_t, & \text{if } t = N \\
  \left(\frac{1}{\sum_{i=N}^{T} \alpha_i^{1+\delta}}\right)^{1-N}, & \text{if } t = N + 1, N + 2, \ldots, N + [k.N].
\end{cases}
\]

Set

\[
c_T = \max_{t=N,N+1,\ldots,N+[k.N]} c_t. \tag{6}
\]

If \( c_T = c_N \) then from (4) and (5) we obtain

\[
2N^{2\alpha N} > d_N^N = c_N \geq c_{N+[k.N]} = d_{N+[k.N]} > 2^{\frac{\varphi(N+[k.N])} {\sum_{i=N}^{T} \alpha_i^{1+\delta}}} = 2^{(N+[k.N]) \log \left( \frac{2^{1+\delta}}{1-\nu} \right).}
\]

Applying \( \log_2 \) twice to the above inequality we get

\[
\log_2 N + \beta N > \alpha (N + [k.N]) - [k.N] \log_2 \left( \frac{2^{1+\delta}}{1-\nu} + 1 + \delta \right)
\]

and this is a contradiction with (3).

Therefore \( c_T \neq c_N \) and thus

\[
c_T \geq \max_{j=N,N+1,\ldots,T-1} c_j. \tag{7}
\]

From this and from the fact that the sequence \( \{a_n\}_{n=1}^{\infty} \) is eventually non-decreasing we obtain that

\[
a_T \geq \left( \max_{j=N,N+1,\ldots,T-1} c_j \right)^{(\frac{2}{1-\nu} + 1+\delta)^{T-N}} \prod_{i=N}^{T-1} \left( \max_{j=N,N+1,\ldots,T-1} c_j \right)^{(\frac{2}{1-\nu} + 1+\delta)^{i-N}}. \tag{7}
\]

Here the second inequality comes from the fact that

\[
\frac{\left( \frac{2}{1-\nu} + 1 + \delta \right)^{T-N}} {\left( \frac{2}{1-\nu} + 1 + \delta \right) - 1} > \frac{\left( \frac{2}{1-\nu} + 1 + \delta \right)^{T-N} - 1} {\left( \frac{2}{1-\nu} + 1 + \delta \right) - 1} = \left( \frac{2}{1-\nu} + 1 + \delta \right)^{T-N-1} + \left( \frac{2}{1-\nu} + 1 + \delta \right)^{T-N-2} + \cdots + 1.
\]
positive integers we obtain that
\[ a_n = \left( \prod_{i=N}^{T-1} \left( \max_{j=N,N+1,...,T-1} c_j \right)^{\left( \frac{2}{\nu} + 1 + \delta \right)^{i-N}} \right)^{\frac{1}{2^{T-1} + \delta}} \geq \left( \prod_{i=N}^{T-1} c_i^{\left( \frac{2}{\nu} + 1 + \delta \right)^{i-N}} \right)^{\frac{1}{2^{T-1} + \delta}} = \left( \prod_{i=N}^{T-1} a_i \right)^{\frac{1}{2^{T-1} + \delta}}. \]

This implies that
\[ a_{1-\nu}^T = \left( \frac{\sum_{i=N}^{\infty} a_i}{a_{N+1}^{\frac{1}{2^{T-1} + \delta}}} \right) \geq \left( \prod_{i=1}^{T-1} a_i \right)^{\frac{1}{2^{T-1} + \delta}} > a_{T^{\frac{1}{2^{T-1} + \delta}}} \cdot \left( \prod_{i=1}^{T-1} a_i \right)^{\frac{1}{2^{T-1} + \delta}}. \]

Now we will prove that for every sufficiently large \( N \)
\[ \sum_{n=N+1}^{\infty} \frac{b_n}{a_n} < \frac{2 \log_2 a_{N+1}}{a_{N+1}^{1-\nu}}. \]

From (1), (2) and the fact that \( \{a_n\}_{n=1}^{\infty} \) is an eventually non-decreasing sequence of positive integers we obtain that
\[ \sum_{n=N+1}^{\infty} \frac{b_n}{a_n} < \sum_{n=N+1}^{\infty} \log_2 a_n = \sum_{n=N+1}^{\infty} \log_2 a_n^{1-\nu} + \sum_{n=N+1}^{\infty} \log_2 a_n^{1-\nu} - \log_2 a_{N+1}^{1-\nu} + \sum_{n=0}^{\infty} \frac{n}{2^{n+1-(1-\nu)}} \leq 2 \log_2 a_{N+1}^{1-\nu} \cdot a_{N+1}^{1-\nu}. \]

Let \( T \) satisfies (6). Inequalities (8) and (9) imply that for every sufficiently large \( T \)
\[ \left| \sum_{n=1}^{\infty} \frac{b_n}{a_n} - \sum_{n=1}^{T-1} \frac{b_n}{a_n} \right| = \left| \sum_{n=1}^{\infty} b_n - \prod_{n=1}^{T-1} a_n \sum_{n=1}^{T-1} b_n \prod_{n=1}^{T-1} a_n \right| = \left| \sum_{n=T}^{\infty} b_n \right| \leq \frac{2 \log_2 a_T}{a_T^{1-\nu}} \cdot a_T^{1-\nu} \cdot \left( \prod_{i=1}^{T-1} a_i \right) \leq \frac{2 \log_2 a_T}{a_T^{1-\nu}} \cdot a_T^{1-\nu} \cdot \left( \prod_{i=1}^{T-1} a_i \right) \leq \frac{2 \log_2 a_T}{a_T^{1-\nu}} \cdot a_T^{1-\nu} \cdot \left( \prod_{i=1}^{T-1} a_i \right)^{1-\nu}. \]
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Set \( q_T = \prod_{n=1}^{T-1} a_n, \quad p_T = \prod_{n=1}^{T-1} d_n \sum_{n=1}^{T-1} \frac{b_n}{a_n} \) and \( \epsilon = \frac{\delta(1-\nu)}{1+\frac{\nu}{2}(1-\nu)}. \) Because

\[
\frac{2 \log_2 a_T}{\frac{\delta}{(1-\nu)^3}} \quad \text{tends to zero when} \quad T \quad \text{tends to infinity}
\]

we obtain the inequality

\[
\left| \sum_{n=1}^{\infty} \frac{b_n}{a_n} - \frac{p_T}{q_T} \right| < \frac{1}{q_T^{2+\epsilon}} \quad (10)
\]

which holds true for all sufficiently large \( T. \)

The fact that we can find infinitely many pairs \((p_T, q_T)\) satisfying (10) and the Roth’s Theorem [7] imply that the number \( \sum_{n=1}^{\infty} \frac{b_n}{a_n} \) is transcendental.

\[\Box\]

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