J. Austral. Math. Soc. (*Series A*) 30 (1981), 438–452

HARMONIC ANALYSIS ON THE FOURIER ALGEBRAS $A_{1,p}(G)$

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(Received 21 April 1980)

Communicated by G. Brown

Abstract

Let G be a locally compact group G (which may be non-abelian) and $A_p(G)$ the p-Fourier algebra of Herz (1971). This paper is concerned with the Fourier algebra $A_{1,p}(G) = A_p(G) \cap L_1(G)$ and various relations that exist between $A_{1,p}(G)$, $A_p(G)$ and G.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 43 A 15, 43 A 20, 46 H 10, 46 H 20, 46 J 20.

Keywords: Banach algebra, Fourier algebra, amenable group, factorization property, weak factorization property, multiplier.

1. Introduction

Let G be a locally compact group with left Haar measure λ. In this paper, we will discuss some properties of the Fourier algebra $A_{1,p}(G) = A_p(G) \cap L_1(G)$, $1 < p < \infty$, where $A_p(G)$ is the p-Fourier algebra in the sense of C. Herz (1971). The space $A_p(G)$, $1 \leqslant p < \infty$, was introduced in 1964 by Figà-Talamanca who studied the case where G is a locally compact abelian group. Eymard in the same year studied $A_2(G)$ where G is non-abelian.

For $1 < p,\ q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, the space $A_p(G)$ is defined by the set of all functions u on G satisfying the following conditions:

$$u = \sum_{i=1}^{\infty} f_i * \check{g}_i, \qquad f_i \in L_p(G), g_i \in L_q(G)$$

The first author was partially supported by the National Science Council, Taiwan, Republic of China.

such that $\sum_{i=1}^{\infty}\|f_i\|_p\|g_i\|_q < \infty$ where $\check{g}_i(x) = g_i(x^{-1})$ for $x \in G$. $A_p(G)$ is equipped with the norm

$$\|u\|_{A_p} = \inf\left\{\sum_{i=1}^{\infty}\|f_i\|_p\|g_i\|_q\right\}$$

where the infimum is taken over all possible representations for u. This norm $\|\ \|_{A_p}$ is stronger than the uniform norm $\|\ \|_\infty$.

Originally concerned with the multipliers of $L_p(G)$, Figà-Talamanca (1964) introduced this space $A_p(G)$ and proved that the multiplier space of $L_p(G)$ is isometrically isomorphic to the dual space $A_p^*(G)$ of $A_p(G)$ provided G is abelian. For nonabelian locally compact group G, Eymard (1964) studied the Fourier algebra $A_2(G) = A(G)$, and for general p, $1 \leqslant p < \infty$, Herz (1971) proved that $A_p(G)$ is a Banach algebra under pointwise multiplication. In a later paper, Herz (1973) also proved that $A_p(G)$ is a regular tauberian algebra of the functions on G.

In this paper, we introduce the algebra $A_{1,p}(G) = A_p(G) \cap L_1(G)$, $1 < p < \infty$, by using the sum norm given by

$$\|u\|_{1,p} = \|u\|_{A_p} + \|u\|_1, \qquad u \in A_{1,p}(G).$$

Evidently, $A_{1,p}(G)$ is nonempty since $C_c * C_c(G) \subset A_{1,p}(G)$, where $C_c(G)$ is the space of continuous functions with compact support in G, and one can verify easily that $(A_{1,p}(G), \|\ \|_{1,p})$ is a Banach algebra under pointwise multiplication.

Note that $A_p(G)$ is a Fourier algebra under pointwise multiplication but is not an algebra under convolution. However, the algebra $A_{1,p}(G)$ with convolution product is a Segal algebra of $L_1(G)$. We are concerned with the structure theory of $A_{1,p}(G)$ in connection with the known properties of $A_p(G)$ and $L_1(G)$. In particular, we will show that the following statements are equivalent:

(1) $A_{1,p}(G) \subset L_r(G)$ for each $0 < r \leqslant \infty$.
(2) $A_{1,p}(G) = A_p(G)$.
(3) $A_{1,p}(G)$ has a bounded approximate identity.
(4) $A_{1,p}(G)$ has the factorization property.
(5) $A_{1,p}(G)$ has the weak factorization property.
(6) G is compact.
(7) Every maximal ideal in $A_{1,p}(G)$ is prime.
(8) Every maximal ideal in $A_{1,p}(G)$ is regular.
(9) Every maximal ideal in $A_{1,p}(G)$ is closed.
(10) Every positive functional on $A_{1,p}(G)$ is continuous.
(11) $A_{1,p}(G)$ has an identity.
(11)′ $A_p(G)$ has an identity.
(12) $A_{1,p}(G) = B_{1,p}(G)$.
(12)′ $A_p(G) = B_p(G)$.

Here

$$B_p(G) = \{\varphi \in C_b(G) | \varphi u \in A_p(G), \forall u \in A_p(G)\},$$
$$B_{1,p}(G) = \{\varphi \in C_b(G) | \varphi u \in A_{1,p}(G), \forall u \in A_{1,p}(G)\}$$

and $C_b(G)$ denotes the set of all bounded continuous functions on G.

2. Structure theorems and approximate identity for $A_{1,p}(G)$

We first consider the connection between $A_{1,p}(G)$ and the p-Fourier algebra $A_p(G)$

THEOREM 2.1. *Let G be a locally compact (Hausdorff) group. Then*

(1) *$A_{1,p}(G)$ is a commutative semisimple Banach algebra under pointwise multiplication and $A_{1,p}(G)$ is a dense ideal in $A_p(G)$.*

(2) *$A_{1,p}(G)$ is a proper ideal of $A_p(G)$ if and only if G is noncompact.*

PROOF. (1) It is easy to see that $A_{1,p}(G)$ is a commutative Banach algebra under pointwise multiplication.

In fact, if $\{u_n\}$ is any Cauchy sequence in $A_{1,p}(G)$, then it is also a Cauchy sequence in $A_p(G)$ and $L_1(G)$. It follows that there exist $v \in A_p(G)$, $v' \in L_1(G)$ such that $\|u_n - v\|_{A_p} \to 0$ and $\|u_n - v'\|_1 \to 0$ as $n \to \infty$. Since $\|u_n - v'\|_1 \to 0$, there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $u_{n(k)}(x) \to v'(x)$ a.e. On the other hand, since $|u_n(x) - v(x)| \leqslant \|u_n - v\|_{A_p} \to 0$, we have $u_{n(k)}(x) \to v(x)$ and hence $v = v'$ a.e. This shows the completeness of $A_{1,p}(G)$. In addition, for any $u, v \in A_{1,p}(G)$, we have

$$\begin{aligned}\|uv\|_{1,p} &\leqslant \|u\|_{A_p}\|v\|_{A_p} + \|u\|_{A_p}\|v\|_1 \\ &\leqslant \|u\|_{1,p}\|v\|_{1,p}.\end{aligned}$$

Next we show that $A_{1,p}(G)$ is semisimple. Note first that the regular maximal ideal space of $A_p(G)$ is G (see Herz (1973, p. 102)). Since $A_p(G) \cap C_c(G) \subset A_{1,p}(G) \subset A_p(G)$ and since $A_p(G) \cap C_c(G)$ is dense in $A_p(G)$, it follows that $A_{1,p}(G)$ is a dense ideal in $A_p(G)$. Hence the regular maximal ideal space of $A_{1,p}(G)$ is also G (see Burnham (1972) or Lai (1975)), and so $A_{1,p}(G)$ is semisimple.

(2) We show that $A_{1,p}(G)$ is a proper ideal if and only if G is noncompact. Evidently, if G is compact, then $A_{1,p}(G) = A_p(G)$. Now suppose G is noncompact. Then there exists a sequence $\{\gamma_n\}$ in G and a compact symmetric neighborhood U of the identity e of G, such that $\gamma_i U^2 \cap \gamma_j U^2 = \varnothing$ provided

$i \neq j$. Let

$$f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{\gamma_k U^2}, \qquad g = \chi_U$$

where $\chi_{\gamma_k U^2} \chi_U$ denote the characteristic functions of $\gamma_k U^2$ and U respectively. For $1 < p,\ q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$, we have $f \in L_p(G)$, $\check{g} \in L_q(G)$ and $f * g \in A_p(G)$. However, $f * g \notin L_1(G)$, because

$$\int_G |f * g(x)|\, dx = \left(\sum_{k=1}^{\infty} \frac{1}{k} \right) \lambda(U)^2$$

diverges. This shows $f * g \notin A_{1,p}(G)$, and hence $A_{1,p}(G)$ is a proper ideal of $A_p(G)$. The proof is complete.

A locally compact group is said to be *amenable* if for any compact set K of G and any $\varepsilon > 0$, there exists a compact set V with positive measure m such that $m(KV) \leqslant (1 + \varepsilon) m(V)$. This definition is equivalent to the existence of an invariant mean on $L^\infty(G)$. A locally compact group G is amenable if and only if $A_p(G)$ has a bounded approximate identity (see Eymard (1971) p. 62, Theorem 3 or Herz (1973) p. 120, Theorem 6). We will show that the existence of an approximate identity of $A_{1,p}(G)$ follows from the existence of a bounded approximate identity of $A_p(G)$. However, any approximate identity of $A_{1,p}(G)$ is unbounded, unless G is compact.

THEOREM 2.2. *To each bounded approximate identity of $A_p(G)$ (if it exists) there corresponds an approximate identity of $A_{1,p}(G)$. All approximate identities in $A_{1,p}(G)$ are unbounded unless G is compact.*

PROOF. Let $\{e_\alpha\}$ be the usual bounded approximate identity of $A_p(G)$ with $\|e_\alpha\|_{A_p} \leqslant c$. For any $u \in A_{1,p}(G) \subset L_1(G)$ and any $\varepsilon > 0$, there exists a compact subset K of G depending on u and ε such that

$$\int_{G-K} |u(x)|\, dx < \frac{\varepsilon}{8(c+1)}$$

and there exists α_1 such that

$$\|e_\alpha u - u\|_{A_p} < \varepsilon/4 \quad \text{whenever } \alpha > \alpha_1.$$

Since $A_{1,p}(G)$ is an ideal of $A_p(G)$, we have $e_\alpha u \in A_{1,p}$ and so $e_\alpha u - u \in L_1(G)$.

Thus for any α,

$$\begin{aligned}
&\int_{G-K} |e_\alpha(x)u(x) - u(x)|\, dx \\
&\quad \leqslant (\|e_\alpha\|_\infty + 1)\int_{G-K} |u(x)|\, dx \quad (\text{since } A_p(G) \subset L_\infty(G)) \\
&\quad \leqslant (\|e_\alpha\|_{A_p} + 1)\varepsilon/8(c+1) \\
&\quad < \frac{1}{8}\varepsilon.
\end{aligned}$$

It follows from the regularity of the Banach algebra $A_p(G)$ (Herz (1973)), that there exists a function $h \in A_p(G)$ such that $h = 1$ on K. Thus for every $x \in K$,

$$|e_\alpha(x) - 1| = |e_\alpha(x)h(x) - h(x)| \leqslant \|e_\alpha h - h\|_{A_p}$$

and since $\{e_\alpha\}$ is an approximate identity for $A_p(G)$, we can choose α_2 such that

$$\begin{aligned}
|e_\alpha(x) - 1| &\leqslant \|e_\alpha h - h\|_{A_p} \\
&\leqslant \varepsilon\left[8\lambda(k)(\|u\|_\infty + 1)\right]^{-1} \quad \text{whenever } \alpha > \alpha_2.
\end{aligned}$$

Hence

$$\int_K |e_\alpha(x)u(x) - u(x)|\, dx = \int_K |e_\alpha(x) - 1|\, |u(x)|\, dx < \varepsilon/8$$

whenever $\alpha > \alpha_2$. Now letting $\alpha_0 = \max\{\alpha_1, \alpha_2\}$, we have

$$\begin{aligned}
\|e_\alpha u - u\|_{1,p} &= \|e_\alpha u - u\|_{A_p} + \|e_\alpha u - u\|_1 \\
&< \frac{1}{4}\varepsilon + \int_G |e_\alpha(x)u(x) - u(x)|\, dx \\
&= \frac{1}{4}\varepsilon + \int_K |e_\alpha(x)u(x) - u(x)|\, dx \\
&\quad + \int_{G-K} |e_\alpha(x)u(x) - u(x)|\, dx \\
&< \frac{1}{4}\varepsilon + \frac{1}{8}\varepsilon + \frac{1}{8}\varepsilon \\
&= \frac{1}{2}\varepsilon
\end{aligned}$$

whenever $\alpha > \alpha_0$.

Since $A_{1,p}(G)$ is a dense ideal in $A_p(G)$, for any $e_\beta \in A_p(G)$ and $\delta > 0$ there corresponds an element $f_{(\beta,\delta)} \in A_{1,p}(G)$ such that

$$\|f_{(\beta,\delta)} - e_\beta\|_{A_p} < \delta.$$

Here we order the set of pairs $\{(\alpha, \varepsilon)\}$ as follows:

$$(\alpha, \varepsilon) < (\beta, \delta) \Leftrightarrow \alpha < \beta \quad \text{and} \quad \varepsilon > \delta.$$

https://doi.org/10.1017/S1446788700017912 Published online by Cambridge University Press

Then $\{(\alpha, \varepsilon)\}$ is a directed set with this ordering, and we ssert that the set $\{f_{(\alpha,\varepsilon)}\}$ forms an approximate identity for $A_{1,p}(G)$. In fact, letting

$$\gamma_0 = \left(\alpha_0, \varepsilon/2(1 + \|u\|_{1,p})\right)$$

and for any $\gamma = (\beta, \delta) > \gamma_0$, we have

$$(*) \qquad \begin{aligned} \|f_\gamma u - e_\beta u\|_{1,p} &\leqslant \|f_\gamma - e_\beta\|_{A_p}\|u\|_{1,p} \\ &\leqslant \delta\|u\|_{1,p} < \frac{1}{2}\varepsilon \end{aligned}$$

and hence

$$\begin{aligned} \|f_\gamma u - u\|_{1,p} &\leqslant \|f_\gamma u - e_\beta u\|_{1,p} + \|e_\beta u - u\|_{1,p} \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

The inequality $(*)$ follows from the following computation:

$$\begin{aligned} \|f_\gamma u - e_\beta u\|_{1,p} &= \|f_\gamma u - e_\beta u\|_{A_p} + \|f_\gamma u - e_\beta u\|_1 \\ &\leqslant \|f_\gamma - e_\beta\|_{A_p}\|u\|_{A_p} + \|f_\gamma - e_\beta\|_\infty\|u\|_1 \\ &\leqslant \|f_\gamma - e_\beta\|_{A_p}\left(\|u\|_{A_p} + \|u\|_1\right) \quad \left(\because \| \ \|_\infty \leqslant \| \ \|_{A_p}\right) \\ &= \|f_\gamma - e_\beta\|_{A_p}\|u\|_{1,p}. \end{aligned}$$

All the approximate identity in $A_{1,p}(G)$ are unbounded unless G is compact (see Lai (1969) p. 574 or Burnham (1972), Theorem 1.2).

If G is amenable, then $A_{1,p}(G)$ and $A_p(G)$ have the same structure theory. The following proposition follows immediately from Lai (1969), Theorem 2, or Burnham (1972), Theorem 1.1.

PROPOSITION 2.3. *Suppose the locally compact group G is amenable, then the following two statements hold*:

(1) *If I is a closed ideal in $A_p(G)$ then $J = I \cap A_{1,p}(G)$ is a closed ideal in $A_{1,p}(G)$ and the closure of J in $A_p(G)$ is I; and*

(2) *If J i s a closed ideal in $A_{1,p}(G)$ and I is the closure of J in $A_p(G)$, then I is a closed ideal in $A_p(G)$ and $J = I \cap A_{1,p}(G)$.*

3. Some equivalence relations

DEFINITION. Let $(A, \| \ \|_A)$ be a Banach algebra. A is said to have the *factorization property* if for every $u \in A$, there exists $V, W \in A$ such that $u = VW$. A is said to have the *weak factorization property* (in symbols $A^2 = A$) if

https://doi.org/10.1017/S1446788700017912 Published online by Cambridge University Press

for every $u \in A$, there exist $V_1, V_2, \ldots, V_n, W_1, W_2, \ldots, W_n$ in A such that $u = \sum_{i=1}^n V_i W_i$.

In order to show that the compactness of G and the factorization properties of $A_{1,p}(G)$ are connected, we need the following lemmas.

LEMMA 3.1. *Let G be a noncompact locally compact group. Then there exists a sequence $\{U_n\}$ of subsets in G such that for each n, U_n contains a compact subset K_n, and there is a sequence $\{V_n\}$ in $A_{1,p}(G)$ with $\|V_n\|_{1,p} \leqslant c_n$, where $c_n \geqslant 1$ for all n, such that*

(1) *$U_i \cap U_j = \varnothing$ if $i \neq j$, $K_n \subset U_n^\circ$ and $0 < \lambda(U_n) = \alpha < \infty$, $0 < \lambda(K_n) = \beta < \infty$ $(n = 1, 2, \ldots)$ where U_n° is the interior of U_n and α, β are real constants;*

(2) *for all n, $V_n(K_n) = 1$, $\operatorname{supp} V_n \subset U_n$ and $0 \leqslant V_n \leqslant 1$; and*

(3) *for all n, $\|V_n\|_{1,p} \leqslant c_n$ and the series $\sum_{n=1}^\infty 1/c_n^a$ converges if $a > 1$, and diverges if $a \leqslant 1$.*

PROOF. Since G is noncompact, there exists a sequence $\{\gamma_n\}$ in G and a compact symmetric neighborhood U of the identity e in G such that $\gamma_i U \cap \gamma_j U = \varnothing$ if $i \neq j$. Take $\gamma_1 = e$ and a compact symmetric neighborhood K of e in U°. Since $A_p(G)$ is regular, there exists a $V \in A_p(G)$ such that $V(K) = 1$, $\operatorname{supp} V \subset U$ and $0 \leqslant V \leqslant 1$ (see Herz (1973), p. 101). V is an element of $A_{1,p}(G)$ since V has compact support. Now let

$$U_n = \gamma_n U, \qquad K_n = \gamma_n K, \qquad V_n = V_{\gamma_n^{-1}} \quad (n = 1, 2, \ldots)$$

where $V_{\gamma_n^{-1}}(x) = V(\gamma_n^{-1}x)$ for any $x \in G$. Take $d \geqslant 1$ such that $\|V\|_{1,p} \leqslant d$, and let $c_n = nd$, we have

(1) $U_i \cap U_j = \varnothing$ if $i \neq j$ and

$$K_n = \gamma_n K \subset \gamma_n U^\circ \subset (\gamma_n U)^\circ = U_n^\circ$$
$$0 < \lambda(U_n) = \lambda(U) = \alpha < \infty$$
$$0 < \lambda(K_n) = \lambda(K) = \beta < \infty.$$

(2) $0 \leqslant V_n \leqslant 1$, $\operatorname{supp} V_n \subset \gamma_n U = U_n$ and $V_n(K_n) = V_{\gamma_n^{-1}}(\gamma_n K) = V(K) = 1$.

(3) Since

$$\|V_n\|_{1,p} = \|V_{\gamma_n^{-1}}\|_1 + \|V_{\gamma_n^{-1}}\|_{A_p} = \|V\|_1 + \|V\|_{A_p} = \|V\|_{1,p} \leqslant d \leqslant c_n,$$

$$\sum_{n=1}^\infty \frac{1}{c_n^a} = \left(\sum_{n=1}^\infty \frac{1}{n^a}\right)\frac{1}{d^a}\begin{cases}\text{converges if } a > 1,\\ \text{diverges if } a \leqslant 1.\end{cases}$$

LEMMA 3.2. (1) $A_{1,p}(G) \subset L_\gamma(G)$ *for* $1 \leqslant \gamma \leqslant \infty$.
(2) *If* $A_{1,p}(G)$ *has the weak factorization property, then*
(i) $A_{1,p}(G) \subset L_{1/2^n}(G)$ *for any integer* $n \geqslant 1$; *and*
(ii) $A_{1,p}(G) \subset L_{1/3}(G)$.

PROOF. (1) This is clear since $A_{1,p}(G) \subset C_0(G) \cap L_1(G) \subset L_\gamma(G)$ *for* $1 \leqslant \gamma \leqslant \infty$.

(2) Since $A_{1,p}(G)$ has the weak factorizatioin property, it follows that if $u \in A_{1,p}(G) = A^2_{1,p}(G)$, there exist $V_1, V_2, \ldots, V_m$ and $W_1, W_2, \ldots, W_m \in A_{1,p}(G)$ such that

$$u = \sum_{i=1}^{m} V_i W_i.$$

Since

$$\int_G |V_i(x) W_i(x)|^{1/2}\, dx \leqslant \left(\int |V_i(x)|\, dx\right)^{1/2} \left(\int |W_i(x)|\, dx\right)^{1/2} < \infty,$$

we have $V_i W_i \in L_{1/2}(G)$ for $i = 1, 2, \ldots, m$ and so $u \in L_{1/2}(G)$. Continuing this process, we obtain by induction that $A_{1,p}(G) \subset L_{1/2}(G)$ for any integer $n \geqslant 1$.

To see that (ii) holds, we take a function $u \in A_{1,p}(G)$ which we may assume to be the above mentioned u. By Hölder's inequality, we have (since $\frac{1}{3} + \frac{1}{3/2} = 1$)

$$\int_G |V_i(x) W_i(x)|^{l/3}\, dx \leqslant \left(\int |V_i(x)|\, dx\right)^{1/3} \left(\int |W_i(x)|^{1/2}\, dx\right)^{2/3} < \infty,$$

that is, $V_i W_i \in L_{1/3}(G)$. Consequently $u \in L_{1/3}(G)$ as required.

THEOREM 3.3. *The following statements are equivalent.*
(1) *G is compact.*
(2) $A_{1,p}(G) = A_p(G)$.
(3) $A_{1,p}(G)$ *has a bounded approximate identity.*
(4) $A_{1,p}(G)$ *has the factorization property.*
(5) $A_{1,p}(G)$ *has the weak factorization property.*
(6) $A_{1,p}(G) \subset L_r(G)$ *for all* r, $0 < r \leqslant \infty$.

PROOF. (1) $\Leftrightarrow$ (2) This follows from Theorem 2.1.

(2) $\Rightarrow$ (3) (2) implies (1) so that G is amenable, and therefore $A_{1,p}(G) = A_p(G)$ has a bounded approximate identity (see Eymard (1971) or Herz (1973)).

(3) $\Rightarrow$ (4) This follows from Cohen's factorization theorem (Cohen (1959), Theorem 1).

(4) $\Rightarrow$ (5) Trivial.

(5) $\Rightarrow$ (1) Suppose to the contrary that G is noncompact. If $A_{1,p}(G)$ has the weak factorization property, we have $A_{1,p}(G) \subset L_{1/3}(G)$ by Lemma 3.2. According to Lemma 3.1, if we let

$$v = \sum_{n=1}^{\infty} \frac{v_n}{c_n^3},$$

where v_n stands for V_n, then

$$\|v\|_{1,p} \leqslant \sum_{n=1}^{\infty} \frac{\|v_n\|_{1,p}}{c_n^3} \leqslant \sum_{n=1}^{\infty} \frac{1}{c_n^2} < \infty, \qquad v \in A_{1,p}(G)$$

and so $v \in L_{1/3}(G)$. On the other hand, supp $v_n \subset U_n$ and $U_i \cap U_j = \varnothing$ if $i \neq j$, so

$$|v(x)|^{1/3} = \sum_{n=1}^{\infty} \frac{|v_n(x)|^{1/3}}{c_n} \quad \text{if } x \in G.$$

Thus

$$\begin{aligned}\int_G |v(x)|^{1/3}\,dx &= \int_G \sum_{n=1}^{\infty} \frac{|v_n(x)|^{1/3}}{c_n}\,dx \\ &= \sum_{n=1}^{\infty} \int_G \frac{|v_n(x)|^{1/3}}{c_n}\,dx \\ &\geqslant \sum_{n=1}^{\infty} \int_{K_n} \frac{1}{c_n}\,dx \quad (\text{since } v_n(K_n) = 1) \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{c_n}\right)\beta\end{aligned}$$

diverges. This implies $v \notin L_{1/3}(G)$ and therefore by Lemma 3.2, $A_{1,p}(G)$ does not have the weak factorization property. But this contradicts our hypothesis and shows that G must be compact.

(6) $\Leftrightarrow$ (1) It suffices to consider the case where $0 < r < 1$ since $A_{1,p}(G) \subset L_r(G)$ holds for all $1 \leqslant r \leqslant \infty$ (see Lemma 3.2, (1)). If G is compact, then $A_{1,p}(G) = A_{1,p}^2(G)$ (see (1) $\Rightarrow$ (5)) and by Lemma 3.2 (2), we have $A_{1,p}(G) \subset L_{1/2^n}(G)$, $n \geqslant 1$. Since $0 < r < 1$, there is a positive integer m such that $1 < 1/2^m \leqslant r < 1$. Hence $A_{1,p}(G) \subset L_r(G)$ for $0 < r < 1$ and (1) $\Rightarrow$ (6) is proved.

If G is noncompact, we take an element $v = \sum_{n=1}^{\infty} v_n/c_n^3$ as described in the proof of (5) $\Rightarrow$ (1), then $v \in A_{1,p}(G)$ but $v \notin L_{1/3}(G)$. This contradicts the fact that $A_{1,p}(G) \subset L_r(G)$ for all r, $0 < r \leqslant \infty$. Hence (6) $\Rightarrow$ (1) is proved.

In order to prove Theorem 3.5, we shall need the following known result of Porcelli (1966), p. 88.

THEOREM A. *Let A be a commutative Banach algebra such that $A^2 = \{0\}$, then*

(i) *A contains a nonprime maximal ideal if and only if $A^2 \neq A$ and in this case each nonprime maximal ideal is a maximal subspaces of A containing A^2.*

(ii) *If A has no identity, then a maximal ideal is regular if and only if it is prime.*

The proof of the following lemma is similar to that given for $A_p(G)$ (see Herz (1973), p. 101).

LEMMA 3.4. *If W is an open set in G which contains a compact set K, then there exists $v \in A_{1,p}(G)$ such that $v = 1$ on K, $v = 0$ outside W and $0 \leqslant v(x) \leqslant 1$ for all $x \in G$. Namely, $A_{1,p}(G)$ is regular.*

THEOREM 3.5. *If G is a locally compact group, then the following conditions are equivalent:*

(1) *G is compact.*

(2) *$A_{1,p}(G)$ has an identity.*

(3) *$A_p(G)$ has an identity.*

(4) *Every maximal ideal in $A_{1,p}(G)$ is prime.*

(5) *Every maximal ideal in $A_{1,p}(G)$ is regular.*

(6) *Every maximal ideal in $A_{1,p}(G)$ is closed.*

PROOF. (1) $\Leftrightarrow$ (2) Since $A_{1,p}(G)$ is a regular semi-simple Banach algebra, the regular maximal ideal space G of $A_{1,p}(G)$ is compact if and only if $A_{1,p}(G)$ has an identity. (See Loomis (1953), p. 52, Theorem and p. 83 Corollary).

(1) $\Leftrightarrow$ (3) The reason is the same as (1) $\Leftrightarrow$ (2).

(1) $\Leftrightarrow$ (4) This follows immediately from Theorem 3.3, (5) and Theorem A(i).

(2) $\Rightarrow$ (5) Trivial.

(5) $\Rightarrow$ (1) This result follows from Theorem 3.3 and Theorem A.

(6) $\Rightarrow$ (1) If G is noncompact, then Theorem 3.3 and Theorem A, there is a maximal ideal I which is neither prime nor regular. This I is also a maximal subspace of $A_{1,p}(G)$ containing $A^2_{1,p}(G)$ by Theorem A(i). Furthermore, I is not closed: otherwise

$$A^2_{1,p}(G) \subset \bar{I} \subset I \subset A_{1,p}(G),$$

$A^2_{1,p}(G)$ is dense in $A_{1,p}(G)$ (see the following remark) which would imply $I = A_{1,p}(G)$. This is a contradiction since I is a maximal subspace of $A_{1,p}(G)$.

(1) $\Rightarrow$ (6) Trivial.

REMARK. $A^2_{1,p}(G)$ is a dense subset of $A_{1,p}(G)$. In fact, $A^2_{1,p}(G)$ is dense in $C_0(G)$ by the Stone–Weierstrass Theorem, and $A_{1,p}(G)$ is dense in $C_0(G)$ if and only if $A^2_{1,p}(G)$ is dense in $A_{1,p}(G)$ (Hahn–Banach Theorem).

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COROLLARY 1. *A locally compact group G is noncompact if and only if $A_{1,p}(G)$ contains a maximal ideal which is neither closed, prime, nor regular.*

Varopoulos (1964) proved that every positive functional on a *-Banach algebra A with a bounded approximate identity is continuous. We know that $A_{1,p}(G)$ has a bounded approximate identity if and only if G is compact (Theorem 3.3). Now if we define $u^*(x) = \overline{u(x)}$, $x \in G$ for each $u \in A_{1,p}(G)$, then $A_{1,p}(G)$ is a *-Banach algebra. Thus if G is compact, then every positive functional on $A_{1,p}(G)$ is continuous, we assert that if every positive functional on $A_{1,p}(G)$ is continuous, then G is compact. Indeed this follows from the fact that if a *-Banach algebra A such that A^2 is a dense proper subset of A, then there exists a discontinuous positive functional on A (see Wang (1972), Theorem 5.1). Hence if every positive functional on $A_{1,p}(G)$ is continuous, the weak factorization property must hold in $A_{1,p}(G)$, so it follows from Theorem 3.3 that G is compact. We summarize this in the following

THEOREM 3.6. *Every positive functional on $A_{1,p}(G)$ is continuous if and only if G is compact.*

4. Regular Tauberian algebra and local properties of $A_{1,p}(G)$

Herz (1973) proved that the Fourier algebras $A_p(G)$, $1 < p < \infty$, are regular Tauberian algebras. Similar result holds for $A_{1,p}(G)$. Recall that a Banach algebra A is said to be a *regular Tauberian algebra* of functions on G if the following conditions hold:

(R) given a compact subset $K \subset G$ and a closed subset F disjoint from K, there exists $u \in A$ such that $u = 1$ on K and $u = 0$ on F;

(T) the elements of compact support are dense in A; and

(G) if T is a continuous multiplicative linear functional on A whose support is a single point $\{x_0\} \subset G$, then $T = \delta_{x_0}$, that is, $\langle u, T\rangle = u(x_0)$ for all $u \in A$.

We state the following theorem for $A_{1,p}(G)$ without proof since it is similar to that given for $A_p(G)$ (Herz (1973)).

THEOREM 4.1. *The Banach algebra $A_{1,p}(G)$ $(1 < p < \infty)$ is a regular Tauberian algebra*

We say that a commutative Banach algebra A satisfies the Ditkin condition if, for any $x \in M \in \mathfrak{M}$ there exists a sequence $x_n \in A$ such that $\hat{x}_n = 0$ in a

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neighborhood V_n of M, the maximal ideal, and $xx_n \to x$. Here $\mathfrak{M}$ is the maximal ideal space of A and $\hat{x}$ is the Gelfand transform of the algebra A.

The following results are analogous to those for $L_1(G)$ in case G is an abelian group (see Loomis (1953) p. 151 and p. 86, 25F Theorem, see also Lai (1970), p. 62 or Yap (1970) for other Banach algebras).

THEOREM 4.2 (see Loomis (1953), p. 151). *Let G be a locally compact amenable group and let I be a closed ideal of* $A_{1,p}(G)$. *Then*

$I \supset \{u \in k(h(I)) \mid$ (*Boundary of* hull(x)) $\cap\, h(I)$ *includes no non-zero perfect set*$\}$.

That is, if $u \in k(h(I))$ *such that the intersection of the boundary of* hull(x) *with* hull(I) *includes no non-zero perfect set, then* $u \in I$. *Here* $k(h(I))$ *denotes kernel* (hull(I)).

The above result can be proved easily by similar arguments, mutatis mutandis, as that in Lai (1970) for $A^p(G)$ (the space of all $f \in L^1(G)$ such that the Fourier transform $\hat{f} \in L^p(\hat{G})$, where $\hat{G}$ is the dual group of the locally compact abelian group G). Therefore, we shall only sketch the proof here.

Note first that $A_{1,p}(G)$ is a commutative semisimple Banach algebra by Theorem 2.1. Furthermore, $A_{1,p}(G)$ satisfies the Ditkin condition (see Loomis (1953), p. 86), that is, if $u \in A_{1,p}(G)$, $x_0 \in G$, $u(x_0) = 0$, then there exists a net $\{u_\alpha\} \subset A_{1,p}(G)$ with $u_\alpha = 0$ in a neighborhood of x_0 in G such that

$$\lim_{\alpha} \|uu_\alpha - u\|_{1,p} = 0.$$

Indeed, this holds in view of the following lemma, the proof of which is elementary.

LEMMA 4.3 (see Lai (1970), Theorem 4). *Suppose that* $u \in A_{1,p}(G)$, $x_0 \in G$, $u(x_0) = 0$. *Furthermore, suppose that there is a neighbohood system* $\{W_\beta\}$ *of* x_0 *in G with Haar measure less than or equal to* 1. *Then there is a net* $\{V_\beta\}$ *in* $A_{1,p}(G)$ *such that*

(1) $\|v_\beta\|_{1,p} < 5$;
(2) $v_\beta = 1$ *on some neighborhood of* x_0 *in* W_β *and* $v_\beta = 0$ *outside* W_β;
(3) $\lim_\beta \|uv_\beta\|_1 = 0$; *and*
(4) $\lim_\beta \|uv_\beta\|_{1,p} = 0$ *if G is amenable.*

Note that $A_{1,p}(G)$ also satisfies the Ditkin's condition at the point of infinity since it has an approximate identity with compact support (Loomis (1953), p. 149 Lemma). It follows that Theorem 4.2 is valid.

We list several easy consequences of Theorem 4.2.

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COROLLARY 1. *If G is amenable, then every closed ideal I in $A_{1,p}(G)$ is included in a regular maximal ideal, that is if I is a closed ideal in $A_{1,p}(G)$ and if* $\text{hull}(I) = \varnothing$, *then $I = A_{1,p}(G)$ provided G is amenable.*

COROLLARY 2. *Suppose that G is an amenable, locally compact group and I is an ideal in $A_{1,p}(G)$ such that I is contained in exactly one regular maximal ideal M. Then $\bar{I} = M$.*

COROLLARY 3. *If G is amenable, then every closed primary ideal in $A_{1,p}(G)$ (or $A_p(G)$) is maximal.*

We remark that Theorem 4.2 and its corollaries still hold if $A_{1,p}(G)$ is replaced by $A_p(G)$.

5. Multipliers for $A_{1,p}(G)$

If A is a Banach algebra, a mapping $T\colon A \to A$ is called a *multiplier* of A if $u(Tv) = (Tu)v$ for all $u, u \in A$. We denote the set of all multipliers of A by $M(A)$. If A is commutative Banach algebra without order and $T \in M(A)$, then T is continuous and $T(uv) = u(Tv) = (Tu)v$.

Now, let $B_p(G) = \{\varphi \in C_b(G)\colon \varphi u \in A_p(G),\ \forall u \in A_p(G)\}$, and define the norm for $B_p(G)$ by

$$\|\varphi\|_{B_p} = \sup_{u \neq 0} \frac{\|\varphi u\|_{A_p}}{\|u\|_{A_p}},$$

then $B_p(G)$ is a Banach algebra under pointwise multiplication (see Eymard (1971)).

Now we define

$$B_{1,p}(G) = \left\{\varphi \in C_b(G)\colon \varphi u \in A_{1,p}(G), \forall u \in A_{1,p}(G)\right\}.$$

Then

$$\begin{aligned}\|\varphi u\|_{1,p} &= \|\varphi u\|_{A_p} + \|\varphi u\|_1 \leqslant \|\varphi\|_{B_p}\|u\|_{A_p} + \|\varphi\|_\infty\|u\|_1 \\ &\leqslant \left(\|\varphi\|_{B_p} + \|\varphi\|_\infty\right)\|u\|_{1,p}\end{aligned}$$

and we write

$$\|\varphi\|_{B_{1,p}} = \sup_{u \neq 0} \frac{\|\varphi u\|_{1,p}}{\|u\|_{1,p}}.$$

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It is immediately clear that $B_{1,p}(G)$ is also a commutative Banach algebra provided the multiplication is the pointwise product.

In order to find a multiplier algebra of $A_{1,p}(G)$, we shall need the following known result of Wang (1961).

THEOREM B. *Let A be a commutative Banach algebra without order and let $T \in M(A)$. Then there exists a unique bounded continuous function φ on $\mathfrak{M}(A)$ (the maximal ideal space of A) such that*

(1) $(Tu)\hat{} = \varphi\hat{u}$ for all $u \in A$.

(2) $\|\varphi\|_\infty \leqslant \|T\|$.

LEMMA 5.1. *Let $u \in A_{1,p}(G)$ and $uv = 0$ for all $v \in A_{1,p}(G)$. Then $u = 0$. That is, $A_{1,p}(G)$ is without order.*

PROOF. If $u \neq 0$, then there is a $x_0 \in G$ such that $u(x_0) \neq 0$. But $A_{1,p}(G)$ is regular (by Lemma 3.4), there exists a $v_0 \in A_{1,p}(G)$ such that $v_0(x_0) = 1$, thus $u(x_0)v_0(x_0) \neq 0$. This contradicts to $uv = 0$. Hence $u = 0$.

REMARK. $A_p(G)$ is also without order.

The following theorem holds for $A_{1,p}(G)$ and $A_p(G)$.

THEOREM 5.2. *The following statements are equivalent:*

(1) $T \in M(A_{1,p}(G))$ (respectively $M(A_p(G))$).

(2) There exists a unique function $\varphi \in B_{1,p}(G)$ (respectively $B_p(G)$) such that $Tu = \varphi u$ for each $u \in A_{1,p}(G)$ (respectively $A_p(G)$). Moreover the correspondence between T and φ defines an isometric linear isomorphism from $M(A_{1,p}(G))$ (respectively $M(A_p(G))$) onto $B_{1,p}(G)$ (respectively $B_p(G)$). That is

$$M(A_{1,p}(G)) \simeq B_{1,p}(G) \quad \big(\text{respectively } M(A_p(G)) \simeq B_p(G)\big).$$

PROOF. (1) $\Rightarrow$ (2) It follows immediately from Lemma 4.1, Theorem B and the property that the Gelfand transform on $A_{1,p}(G)$ (respectively $A_p(G)$) is an identity mapping.

(2) $\Rightarrow$ (1) If $\varphi \in B_{1,p}(G)$ (respectively $B_p(G)$) and we define $T_\varphi u = \varphi u$ for each $u \in A_{1,p}(G)$ (respectively $A_p(G)$), then $T_\varphi \in M(A_{1,p}(G))$ (respectively, $M(A_p(G))$).

Moreover, $M(A_{1,p}(G))$ (respectively, $M(A_p(G))$) and $B_{1,p}(G)$ (respectively $M(B_p(G))$) are isometric isomorphic onto.

From this theorem, we have

COROLLARY 1. *The following statements are equivalent*:
(1) $A_{1,p}(G) = B_{1,p}(G)$.
(2) $A_{1,p}(G)$ *has an identity*.
(3) $A_p(G) = B_p(G)$.

COROLLARY 2. *If G is compact, then*

$$B_{1,p}(G) = A_{1,p}(G) = A_p(G) = B_p(G).$$

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