OSCILLATIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Sufficient conditions are established for the oscillation of solutions of hyperbolic equations of neutral type of the form

\[
\frac{\partial^2}{\partial t^2} \left[ u(x,t) + p(t)u(x, t - \tau) \right] = a(t)\Delta u(x,t) + q(t)f \left( u(x, \sigma(t)) \right)
\]

\[(x,t) \in \Omega \times R_+ \equiv G,
\]

where \( R_+ = (0, \infty) \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a piecewise smooth boundary \( \partial \Omega \).

Recently there has been much interest in studying the oscillatory behavior of solutions of partial differential equations with deviating arguments. We refer the reader to the papers by Georgiou and Kreith [1], Mishev and Bainov [3, 4], and Yoshida [6].

The purpose of this paper is to obtain the sufficient conditions for the oscillation of the solutions of hyperbolic equations of neutral type of the form

\[
\frac{\partial^2}{\partial t^2} \left[ u(x,t) + p(t)u(x, t - \tau) \right] = a(t)\Delta u(x,t) + q(t)f \left( u(x, \sigma(t)) \right)
\]

\[(x,t) \in \Omega \times R_+ \equiv G,
\]

where \( R_+ = [0, \infty) \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a piecewise smooth bounded \( \partial \Omega \).

Suppose that the following conditions (A) hold:

(A1) \( \tau = \text{const} > 0, \sigma(t) \) is a continuous function in \( R_+ \) such that \( \lim_{t \to \infty} \sigma(t) = \infty, \sigma(t) \leq t \);

(A2) \( a(t) \) is a nonnegative continuous function on \( R_+ \), \( f(u) \in C(R, R) \) are convex in \( (0, \infty) \) and \( uf(u) > 0 \) for \( u \neq 0 \); and

(A3) \( q(t) \in C \left( [0, \infty) \right) \) and \( p(t) \in C \left( [0, \infty), [0, 1] \right) \).
Consider first the boundary condition

\[ \frac{\partial u}{\partial n} + \mu u = 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}^+, \]

where \( \mu \) is a continuous, nonnegative function on \( \partial \Omega \times \mathbb{R}^+ \), and \( n \) denotes the unit exterior normal vector to \( \partial \Omega \).

**Definition:** The solution \( u(u, t) \) of the problem (1), (2) is called oscillatory in \( G \) if \( u(x, t) \) has zero in \( \Omega \times [t_0, \infty) \) for each \( t_0 > 0 \).

**Theorem 1.** Let the conditions (A) hold and suppose there exists a positive constant \( \alpha \) such that

\[ \frac{f(u)}{u} > \alpha \quad \text{for} \quad u \neq 0, \quad \sigma'(t) \geq 0 \quad \text{for every} \quad t \geq 0. \]

If

\[ \int_0^\infty q(s) \left[ 1 - p(\sigma(s)) \right] ds = \infty, \]

then every solution \( u(x, t) \) of the problem (1), (2) is oscillatory.

**Proof:** Suppose to the contrary that there is a nonoscillatory solution \( u(x, t) \) of the problem (1), (2) which has no zero in \( \Omega \times [t_0, \infty) \). Without loss of generality we may assume that \( u(x, t) > 0 \) in \( \Omega \times [t_0, \infty) \). From condition \( (A_1) \) there exists a \( t_1 > t_0 \) such that \( u(x, t) > 0, \ u(x, \sigma(t)) > 0 \) and \( u(x, t - \tau) > 0 \) in \( \Omega \times [t_1, \infty) \). We integrate (1) with respect to \( x \) over the domain \( \Omega \) and obtain for \( t \geq t_1 \)

\[ \frac{d^2}{dt^2} \left[ \int_\Omega u(x, t) dx + p(t) \int_\Omega u(x, t - \tau) dx \right] = a(t) \int_\Omega \Delta u(x, t) dx - q(t) \int_\Omega f(u(x, \sigma(t))) dx. \]

Green’s formula yields

\[ \int_\Omega \Delta u dx = \int_{\partial \Omega} \frac{\partial u}{\partial n} ds = - \int_{\partial \Omega} \mu u ds \leq 0. \]

Moreover, from \( (A_2) \) and using Jensen’s inequality it follows that

\[ \int_\Omega f(u(x, \sigma(t))) dx \geq |\Omega| \left( \frac{\int_\Omega u(x, \sigma(t)) dx}{|\Omega|} \right), \quad \text{for} \quad t \geq t_1, \ |\Omega| = \int_\Omega dx. \]

Then from (5), (6) and (7) it follows that for \( t \geq t_1 \)

\[ \frac{d^2}{dt^2} [V(t) + p(t)V(t - \tau)] + q(t)f(V(\sigma(t))) \leq 0, \]
where \( V(t) = \int_\Omega u(x,t)dx, \quad t \geq t_0. \) Thus \( V(t) \) is a positive solution of the inequality (8). Set \( Z(t) = V(t) + p(t)V(t - \tau) \). Obviously \( Z(t) > 0 \) for \( t \geq t_1 \) and
\[
Z''(t) \leq 0 \quad \text{for} \quad t \geq t_1.
\]
Hence \( Z'(t) \) is a decreasing function. We claim that \( Z'(t) > 0 \) for \( t \geq t_1 \). If \( Z'(t) < 0 \) for \( t \geq t_1 \), then there exists \( t_2 \geq t_1 \) such that \( Z'(t_2) < 0 \). Then
\[
Z(t) - Z(t_2) \leq \int_{t_2}^t Z'(t)ds = Z'(t_2)(t - t_2) \quad \text{for} \quad t \geq t_2
\]
and \( \lim_{t \to \infty} Z(t) = -\infty \), which contradicts the fact that \( Z(t) > 0 \). In view of (3), we have
\[
Z''(t) + \alpha q(t)V(\sigma(t)) \leq 0, \quad \text{for} \quad t \geq t_1,
\]
or
\[
Z''(t) + \alpha q(t)[Z(\sigma(t)) - p(\sigma(t))V(\sigma(t) - \tau)] \leq 0, \quad t \geq t_1.
\]
Since \( Z(t) \geq V(t) \) and \( Z(t) \) is nondecreasing, it follows that
\[
Z''(t) + \alpha q(t)[1 - p(\sigma(t))]Z(\sigma(t)) \leq 0, \quad t \geq t_1.
\]
Let
\[
W(t) = t^\beta \frac{Z'(t)}{Z^{\sigma(t)}}, \quad t \geq t_1, \quad \beta > 0.
\]
We obtain for \( t \geq t_1 \)
\[
W''(t) = \beta t^{\beta-1} \frac{Z'(t)}{Z(\sigma(t))} + t^\beta \frac{Z''(t)Z(\sigma(t)) - Z'(t)Z'(\sigma(t))\sigma'(t)}{[Z(\sigma(t))]^2}
\]
\[
= \beta t^{\beta-1} \frac{Z'(t)}{Z(\sigma(t))} - t^\beta \alpha q(t)[1 - p(\sigma(t))] - \frac{Z'(t)Z'(\sigma(t))\sigma'(t)}{[Z(\sigma(t))]^2}
\]
\[
\leq \beta t^{-1}W(t) - t^\beta \alpha q(t)[1 - p(\sigma(t))].
\]
From this it follows that
\[
\left( \frac{W(t)}{t^\beta} \right)' \leq -\alpha q(t)[1 - p(\sigma(t))] \quad \text{for} \quad t \geq t_1,
\]
or
\[
0 \leq \frac{W(t)}{t^\beta} \leq \frac{W(t_1)}{t_1^\beta} - \int_{t_1}^t \alpha q(s)[1 - p(\sigma(s))]ds, \quad t \geq t_1.
\]
Thus we have
\[
\lim_{t \to \infty} \frac{W(t)}{t^\beta} = -\infty,
\]
which leads to a contradiction.

If \( u(x,t) \leq 0 \) for \( (x,t) \in \Omega \times [t_0, \infty) \), then the proof follows from the fact that \(-u(x,t)\) is a positive solution of the problem (1), (2). This completes the proof of the theorem. \( \Box \)
REMARK. If \( u(x,t) \equiv u(t) \), \( f(u) \equiv u \) and \( \sigma = t - \sigma \), then Theorem 1 and the Theorem in [2] are the same.

We consider now the boundary condition

\[
(11) \quad u = 0, \quad \text{on} \quad \partial \Omega \times [0,\infty),
\]
and we consider the following Dirichlet problem in the domain \( \Omega \)

\[
\Delta v + \alpha v = 0, \quad \text{in} \quad \Omega
\]
\[
v|_{\partial \Omega} = 0,
\]
where \( \alpha \) is a constant. It is well-known that the smallest eigenvalue \( \alpha_0 \) of the problem (12) is positive and the corresponding eigenfunction \( \psi(x) \geq 0 \) for \( x \in \Omega \).

With a solution \( u(x,t) \) of the problem (1), (11) we associate the function

\[
(13) \quad H(t) = \frac{\int_{\Omega} u(x,t)\psi(x)dx}{\int_{\Omega} \psi(x)dx}, \quad t \geq 0.
\]

**THEOREM 2.** If all conditions of Theorem 1 hold, then every solution of the problem (1), (11) is oscillatory in \( \Omega \).

**PROOF:** Let \( u(x,t) \) be a positive solution of the problem (1), (11) in \( \Omega \times [t_0,\infty) \) for some \( t_0 \). By condition \( (A_1) \) there exists a \( t_1 \geq t_0 \) such that \( u(x,\sigma(t)) \geq 0 \) and \( u(x,t - \tau) > 0 \) in \( \Omega \times [t_1,\infty) \). Multiply both sides of equation (1) by the eigenfunction \( \psi(x) \geq 0 \), and integrate with respect to \( x \) over the domain \( \Omega \); then we have

\[
(14) \quad \frac{d^2}{dt^2} \left[ \int_{\Omega} u(x,t)\psi(x)dx + p(t) \int_{\Omega} u(x,t - \tau)\psi(x)dx \right]
\]
\[
= a(t) \int_{\Omega} \Delta u \psi(x)dx - q(t) \int_{\Omega} f \left( u(x,\sigma(t)) \right) \psi(x)dx, \quad t > t_1.
\]

From the divergence theorem it follows that

\[
(15) \quad \int_{\Omega} \Delta u \psi(x)dx = -\alpha_0 \int_{\Omega} u \psi(x)dx, \quad t \geq t_1,
\]
where \( \alpha_0 \) is the smallest eigenvalue of the problem (12).

Using condition \( (A_2) \) and Jensen's inequality it follows that

\[
(16) \quad \int_{\Omega} f \left( u(x,\sigma(t)) \right) \psi(x)dx \geq \int_{\Omega} \psi(x)dx f \left( \frac{1}{\int_{\Omega} \psi(x)dx} \int_{\Omega} u(x,\sigma(t))\psi(x)dx \right), \quad t \geq t_1.
\]
Using (13), (15) and (16), we obtain

\begin{equation}
\frac{d^2}{dt^2} \left[ \dot{H}(t) + p(t)H(t - \tau) \right] \leq -\alpha_0 a(t)H(t) - q(t)f\left(H(\sigma(t))\right), \quad t \geq t_1.
\end{equation}

Since, for \( t \geq t_1 \), \( H(t) \geq 0 \) and \( H(\sigma(t)) \geq 0 \), then by (17)

\begin{equation}
\frac{d^2}{dt^2} \left[ \dot{H}(t) + p(t)H(t - \tau) \right] + q(t)f\left(H(\sigma(t))\right) \leq 0, \quad t \geq t_1.
\end{equation}

The remainder of the proof is similar to that of Theorem 1; we omit it.

We need the following lemma [5].

**Lemma.** Consider the differential inequality

\begin{equation}
x'(t) + b(t)x(g(t)) \leq 0, \quad t \geq t_0,
\end{equation}

where \( b(t) \in C\left(R, [0, \infty)\right) \), \( g(t) \in C(R, R) \), \( g(t) \leq t \) and \( g(t) \) is a nondecreasing function with \( \lim_{t \to \infty} g(t) = \infty \). If

\[
\liminf_{t \to \infty} \int_{g(t)}^{t} b(s)ds \geq \frac{1}{e},
\]

then the inequality (18) has no ultimately positive solutions.

**Theorem 3.** Let conditions (A) and (3) hold. If

\begin{equation}
\liminf_{t \to \infty} \int_{\sigma(t)}^{t} Q(s)ds > \frac{1}{e},
\end{equation}

where \( Q(t) = \alpha_0 \varepsilon_0 q(t)\sigma(t)\left[1 - p(\sigma(t))\right] \exp\left[\int_{T}^{t} \alpha_0 \varepsilon_0 a(s)\sigma(1 - p(s))ds\right] \), for all large \( T, \varepsilon_0 \in (0, 1) \) is a constant, then every solution \( u(x, t) \) of the problem (1), (11) is oscillatory in \( G \).

**Proof:** Let \( u(x, t) \) be a positive solution of the problem (1)–(11). As in the proof of Theorem 2 we get (17). From (17) and (3) it follows that

\begin{equation}
\frac{d^2}{dt^2} \left[ \dot{H}(t) + p(t)H(t - \tau) \right] + \alpha_0 a(t)H(t) + \alpha q(t)H(\sigma(t)) \leq 0, \quad t \geq t_1.
\end{equation}

Moreover,

\[
Z''(t) + \alpha_0 a(t)\left[Z(t) - p(t)H(t - \tau)\right] + \alpha q(t)\left[Z(\sigma(t)) - p(\sigma(t))H(\sigma(t) - \tau)\right] \leq 0, \quad t \geq t_1.
\]
Since $Z(t) \geq H(t)$ and $Z(t)$ is nondecreasing, we have

\begin{equation}
Z''(t) + \alpha_0 a(t) [1 - p(t)] Z(t) + \alpha q(t) [1 - p(\sigma(t))] Z(\sigma(t)) \leq 0, \quad t \geq t_1.
\end{equation}

Obviously, $Z(t) \geq 0$, $Z''(t) \leq 0$ and $Z'(t) \geq 0$ for $t \geq t_1$ (as in the proof of Theorem 1).

By Lemma 1 in [5] there exists a $T \geq t_1$ such that

\begin{equation}
Z(t) \geq \varepsilon_0 t Z'(t) \quad \text{and} \quad Z(\sigma(t)) \geq \varepsilon(\sigma) Z'(\sigma(t)) \quad \text{for} \quad t \geq T.
\end{equation}

Hence, from (20) we have

\begin{equation}
Z''(t) + \alpha_0 \varepsilon_0 \omega(t) [1 - p(t)] Z(t) + \alpha_0 \varepsilon q(t) \sigma(t) [1 - p(\sigma(t))] Z'(t) \leq 0, \quad t \geq T.
\end{equation}

Set

$Y(t) = Z'(t) \exp \left[ \int_{t_2}^{t} \alpha_0 \varepsilon_0 \sigma(s) [1 - p(s)] ds \right], \quad t \geq T$

and using (21) we obtain

\begin{equation}
Y'(t) + Q(t) Y(\sigma(t)) \leq 0, \quad t \geq T,
\end{equation}

where

$Q(t) = \alpha_0 \varepsilon_0 q(t) \sigma(t) [1 - p(\sigma(t))] \exp \left[ \int_{T}^{t} \alpha_0 \varepsilon_0 \sigma(s) [1 - p(s)] ds \right].$

Thus (22) has a positive solution $Y(t)$, which contradicts the conclusion of the lemma.

THEOREM 4. Let conditions (A) and (3) hold. If

\begin{equation}
\limsup_{t \to \infty} \int_{\sigma(t)}^{t} \alpha_0 q(\sigma(t)) [1 - p(\sigma(t))] d\sigma(t) \geq 1,
\end{equation}

then every solution $u(x, t)$ of the problem (1), (11) is oscillatory.

PROOF: As in the proof of Theorem 3, we obtain (21). Thus

\begin{equation}
Y'(t) + \alpha_0 \varepsilon_0 q(t) \sigma(t) [1 - p(\sigma(t))] Y(\sigma(t)) \leq 0, \quad t \geq T,
\end{equation}

where $Y(t) = Z'(t)$, $\varepsilon_0 \in (0, 1)$, such that

\begin{equation}
\limsup_{t \to \infty} \int_{\sigma(t)}^{t} \alpha_0 \varepsilon_0 q(s) \sigma(s) [1 - p(\sigma(s))] ds \geq 1.
\end{equation}

Integrating (23) from $\sigma(t)$ to $t$, we have

\begin{equation}
Y(t) - Y(\sigma(t)) + \alpha_0 \varepsilon_0 \int_{\sigma(t)}^{t} q(s) \sigma(s) [1 - p(\sigma(s))] Y(\sigma(s)) ds \leq 0, \quad t \geq T.
\end{equation}
Since $Y(t)$ is decreasing and $\sigma'(t) \geq 0$, it follows that

$$Y(t) - Y(\sigma(t)) + \alpha_0 \varepsilon_0 Y(\sigma(t)) \int_{\sigma(t)}^t q(s)\sigma(s)\left[1 - p(\sigma(s))\right] ds \leq 0, \quad t \geq T,$$

or

$$\alpha_0 \varepsilon_0 \int_{\sigma(t)}^t q(s)\sigma(s)\left[1 - p(\sigma(s))\right] ds \leq 1 - \frac{Y(t)}{Y(\sigma(t))} \leq 1, \quad t \geq T.$$

Hence

$$\limsup_{t \to \infty} \int \alpha_0 \varepsilon_0 q(s)\sigma(s)\left[1 - p(\sigma(s))\right] ds \leq 1,$$

which contradicts (24). Thus the proof of Theorem 4 is complete.

**EXAMPLE 1.** Consider the equation

$$\frac{\partial^2}{\partial t^2} \left[ u(x,t) + \frac{1}{2} u \left( x, t - \frac{\pi}{2} \right) \right] + \alpha_0 \varepsilon_0 Y(\sigma(t)) \int_{\sigma(t)}^t q(s)\sigma(s)\left[1 - p(\sigma(s))\right] ds \leq 0, \quad t \geq T,$$

(25)

$$= u_{xx}(x,t) - \frac{1}{2} u \left( x, t - \frac{\pi}{2} \right), \quad (x,t) \in (0,\pi) \times [0,\infty),$$

with a boundary condition

$$u_z(0,t) = u_z(\pi,t) = 0, \quad t \geq 0. \quad (26)$$

It is easy to check the functions: $f(u) = u$, $g(t) = 1/2 = p(t)$, $a(t) = 1$, $\tau = \pi/2$ and $\alpha = 1$. Moreover, $u \equiv 0$ on $\partial \Omega \times [0,\infty)$ where $\Omega = (0,\pi)$, and

$$\int_{\sigma(t)}^\infty q(s)\left[1 - p(\sigma(s))\right] ds = \int_{\sigma(t)}^\infty \frac{1}{2} \left[1 - \frac{1}{2}\right] ds = \infty.$$

Hence the conditions of Theorem 1 are satisfied. Thus all solutions of the problem (25), (26) oscillate in $G = (0,\pi) \times [0,\infty)$. For instance, the function $u(x,t) = \sin t \cos x$ is such a solution.

**EXAMPLE 2.** Consider the equation

$$\frac{\partial^2}{\partial t^2} \left[ u(x,t) + e^{-\pi}u(x,t - \pi) \right] = u_{xx}(x,t) - e^{-\pi}u(x,t - \pi),$$

(27)

$$\left( x,t \right) \in (0,\pi) \times [0,\infty),$$

with a boundary condition

$$u(0,t) = u(\pi,t) = 0, \quad t \geq 0. \quad (28)$$

It is easily verified that the functions

$$p(t) = e^{-\pi}, \quad g(t) = e^{-\pi}, \quad a(t) = 1, \quad \sigma(t) = t - \pi \quad \text{and} \quad f(u) = u$$

satisfy the conditions of Theorem 4. Hence all solutions of the problem (27), (28) oscillate in $G = (0,\pi) \times [0,\infty)$. For instance, the function $u(x,t) = \sin t \sin x$ is such a solution.
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