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QUANTIZATIONS OF THE MODULE OF TENSOR FIELDS OVER THE WITT ALGEBRA

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ABSTRACT After introducing the q analogue of the enveloping algebra of the Witt algebra we construct q analogues of the module of tensor fields over the Witt algebra and prove a partial q analogue of Kaplansky s Theorem concerning this module of tensor fields

0 **Introduction.** The representation $V_{\alpha\beta}$ of the Witt algebra on the space of "the tensor fields" of the form $P(z)z^{\alpha}(dz)^{\beta}$ is usually called the module of tensor fields over the Witt algebra Here α and β are complex numbers and $P(z)$ is an arbitrary polynomial in z and z^{-1} The module $V_{\alpha\beta}$ over the Witt algebra plays a very important role in the representation theory of the Virasoro algebra In 1982, I Kaplansky proved in [3] that if $V = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}v_n$ is a $\mathbb{Z}$-graded module of the Witt algebra $W = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}d_n$ and $d_{\pm 1}$ are injective operators on V, then V is isomorphic to the module $V_{\alpha\beta}$ of tensor fields for some $\alpha, \beta \in \mathbb{C}$ We call this result Kaplansky's Theorem The main purpose of this paper is to prove a partial q-analogue of Kaplansky's Theorem

Throughout the paper, we assume that

- All vector spaces are the vector spaces over complex number field $\mathbb{C}$,
- $\mathbb{C}^* = \{x \in \mathbb{C} \mid x \neq 0\}$,
- q is a complex number satisfying $q^2 \neq 0, 1$,
- $\ln(z)$ is the principal value of the function $\ln(z)$,
- $q^{\alpha} = e^{\alpha \ell n(q)}$ for $\alpha \in \mathbb{C}$,
- $[\alpha] = \frac{q^{\alpha}-q^{-\alpha}}{q-q^{-1}}$ for $\alpha \in \mathbb{C}$

In Section 1, after defining q-analogue $U(W_q)$ of the enveloping algebra of the Witt algebra, we will construct two kinds of $U(W_q)$-modules $A(\lambda, \alpha, \beta)$ and $B(\lambda, \alpha, \beta)$ by using a version of the operations over $\mathbb{Z}$-graded modules of the Witt algebra introduced by B L Feigin and D B Fuchs [1], where $(\lambda, \alpha, \beta) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$ Both $A(1, \alpha, \beta)$ and $B(1, \alpha, \beta)$ become the module of tensor fields over the Witt algebra when $q \longrightarrow 1$ In Section 2, we will find the necessary and sufficient conditions for $X(\lambda, \alpha, \beta) \simeq Y(\lambda', \alpha', \beta')$ (where $X, Y \in \{A, B\}$) and study the reducibility and unitarity of $X(\lambda, \alpha, \beta)$ In Section 3, we will prove a partial q-analogue of Kaplansky's Theorem

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1. **The constructions of $U(W_q)$-modules $X(\lambda, \alpha, \beta)$.** Based on Proposition 1.1 in [5], we introduce the following definition:

DEFINITION 1.1. The q-analogue $U(W_q)$ of the enveloping algebra of the Witt algebra is defined as the associative algebra with generators $\{J^{\pm 1}, d_m \mid m \in \mathbb{Z}\}$ and the following relations:

$$JJ^{-1} = J^{-1}J = 1, \quad Jd_mJ^{-1} = q^m d_m, \tag{1.1}$$

$$q^m d_m d_n J - q^n d_n d_m J = [m-n] d_{m+n}, \tag{1.2}$$

where $m, n \in \mathbb{Z}$.

DEFINITION 1.2. A $U(W_q)$-module V is called a *$\mathbb{Z}$-graded module* if $V = \bigoplus_{n\in\mathbb{Z}} V_n$ and $d_m(v_n) \in V_{m+n}$ for $m, n \in \mathbb{Z}$.

For every $\lambda \in \mathbb{C}^*$, we define an algebra isomorphism $\varphi(\lambda)$ of $U(W_q)$ as follows:

$$\varphi(\lambda)\colon J^{\pm 1} \longmapsto \lambda^{\pm 1} J^{\pm 1}, \quad d_m \longmapsto \lambda^{-1} d_m \text{ for } m \in \mathbb{Z}.$$

If $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is a $\mathbb{Z}$-graded module of $U(W_q)$-module with $J(v_n) = q^n v_n$ for all $n \in \mathbb{Z}$ and $v_n \in V_n$, then we can construct three more modules from V: *contragradient module* $\bar{V} := \bigoplus_{n\in\mathbb{Z}} (\bar{V})_n$, *adjoint module* $V^* := \bigoplus_{n\in\mathbb{Z}} (V^*)_n$ and *inverted module* $V^\circ := \bigoplus_{n\in\mathbb{Z}} (V^\circ)_n$, where

$$(\bar{V})_n := \mathrm{Hom}(V_n, \mathbb{C}), \quad J|(\bar{V})_n := q^n \cdot \mathrm{id};$$
$$(V^*)_n := \mathrm{Hom}(V_{-n}, \mathbb{C}), \quad J|(V^*)_n := q^{-n} \cdot \mathrm{id};$$
$$(V^\circ)_n := V_{-n}, \quad J|(V^\circ)_n := q^{-n} \cdot \mathrm{id}$$

and the definitions of the operators d_m on $\bar{V}$, V^* and V° are the same as in [1].

It is easy to check that $\bar{V}$ is a $\mathbb{Z}$-graded $U(W_q)$-module and V^*, as well as V°, is a $\mathbb{Z}$-graded $U(W_{q^{-1}})$-module. As $U(W_q)$-modules, $(V^*)^\circ \simeq \bar{V}$.

In particular, if $V = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k$ is a $\mathbb{Z}$-graded $U(W_q)$-module with the natural $\mathbb{Z}$-grading and the following module action on V:

$$J(v_k) := q^k v_k, \quad d_n(v_k) := a(q, n, k) v_{n+k}, \tag{1.3}$$

where $n, k \in \mathbb{Z}$ and $a(q, n, k) \in \mathbb{C}$, then we can describe the contragradient module $\bar{V}$, the adjoint module V^* and the inverted module V° as follows:

$$\begin{gathered} \bar{V} = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k, \\ J(v_k) = q^k v_k, \quad d_n(v_k) = a(q, -n, n+k) v_{n+k}; \end{gathered} \tag{1.4}$$

$$\begin{gathered} V^* = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k, \\ J(v_k) = q^{-k} v_k, \quad d_n(v_k) = -a(q, n, -n-k) v_{n+k}; \end{gathered} \tag{1.5}$$

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$$
(1.6)\qquad \begin{aligned} &V^\circ = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k,\\ &J(v_k) = q^{-k}v_k, \quad d_n(v_k) = -a(q,-n,-k)v_{n+k}. \end{aligned}
$$

For $\alpha, \beta \in \mathbb{C}$, set

$$
(1.7)\qquad a(q,n,k) := -([k+\alpha]q^\alpha + [n+1][\beta]q^{n+k}),
$$

where $n, k \in \mathbb{Z}$. Then (1.3) and (1.7) define a $U(W_q)$-module action on $V(\alpha,\beta) := \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k$.

Let us check (1.2), *i.e.*

$$
(1.8)\qquad q^m d_m d_n(v_k) - q^n d_n d_m(v_k) = [m-n]d_{m+n}J^{-1}(v_k) \text{ for } m,n,k \in \mathbb{Z}.
$$

Let $q^m d_m d_n(v_k) = A_{m,n,k}v_{m+n+k}$, then $q^n d_n d_m(v_k) = A_{n,m,k,}v_{m+n+k}$. By (1.7), we have

$$
\begin{aligned} A_{m,n,k} = {} & q^{m+2\alpha}[k+\alpha][n+k+\alpha] + q^{2m+n+k+\alpha}[k+\alpha][m+1][\beta]\\ & + q^{m+n+k+\alpha}[n+1][\beta][n+k+\alpha]\\ & + q^{2m+2n+2k}[n+1][m+1][\beta]^2. \end{aligned}
$$

It follows that

$$
\begin{aligned} q^m d_m d_n(v_k) - q^n d_n d_m(v_k) & \\ = {} & (A_{m,n,k} - A_{n,m,k})v_{m+n+k}\\ = {} & \Big([k+\alpha](q^{m+2\alpha}[n+k+\alpha] - q^{n+2\alpha}[m+k+\alpha])\\ & + q^{m+n+k}[\beta]\Big((q^{m+\alpha}[k+\alpha][m+1] + q^\alpha[n+1][n+k+\alpha])\\ & - (q^{n+\alpha}[k+\alpha][n+1] + q^\alpha[m+1][m+k+\alpha])\Big)\Big)v_{m+n+k}\\ = {} & \Big([k+\alpha](-q^{-k}q^\alpha[m-n])\\ & + q^{m+n+k}[\beta](-q^{-k}[m-n][m+n+1])\Big)v_{m+n+k}\\ = {} & -[m-n]([k+\alpha]q^\alpha + [m+n+1][\beta]q^{m+n+k})q^{-k}v_{m+n+k}\\ = {} & [m-n]d_{m+n}J^{-1}(v_k), \end{aligned}
$$

so (1.8) is true.

By the discussion above, (1.4) defines a $\mathbb{Z}$-graded $U(W_q)$-module $\bar{V}(\alpha,\beta)$. If we replace q by q^{-1} in (1.5), then (1.5) also defines a $\mathbb{Z}$-graded $U(W_q)$-module $V(\alpha,\beta)^{(1)}$ as follows:

$$
\begin{aligned} &V(\alpha,\beta)^{(1)} := \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k,\\ &J(v_k) := q^k v_k, \quad d_n(v_k) := -a(q^{-1}, n, -n-k)v_{n+k}. \end{aligned}
$$

After replacing q by q^{-1} in (1.6), we get the following $\mathbb{Z}$-graded $U(W_q)$-module $V(\alpha,\beta)^{(2)}$:

$$
(1.9)\qquad \begin{aligned} &V(\alpha,\beta)^{(2)} := \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k,\\ &J(v_k) := q^k v_k \quad d_n(v_k) := -a(q^{-1}, -n, -k)v_{n+k}. \end{aligned}
$$

By (1.7), we know that

$$a(q^{-1},-n,-k) = [k-\alpha]q^{-\alpha} + [n-1][\beta]q^{n+k}$$
$$= \frac{q^k}{q-q^{-1}}\big(-q^{-2k} + q^{-1}[\beta]q^{2n} + (q^{-2\alpha} - q[\beta])\big).$$

Choose $\alpha', \beta' \in \mathbb{C}$ such that

$$q[\beta'] = q^{-2\alpha} - q[\beta] \text{ and } q^{2\alpha'} - q^{-1}[\beta'] = q^{-1}[\beta].$$

Then we get

$$a(q^{-1},-n,-k) = \frac{q^k}{q-q^{-1}}\big(-q^{-2k} + (q^{2\alpha'} - q^{-1}[\beta'])q^{2n} + q[\beta']\big)$$
$$= [n+k+\alpha']a^{n+\alpha'} + [1-n][\beta']q^{n+k}.$$

A direct computation shows that

$$\bar{V}(\alpha,\beta) \simeq V(\alpha',\beta')^{(2)} \text{ and } V(\alpha,\beta)^{(1)} \simeq V(\alpha'',\beta'')$$

for some $\alpha', \beta', \alpha'', \beta'' \in \mathbb{C}$. Therefore, the construction which produces the modules $\bar{V}(\alpha,\beta)$ (resp. $V(\alpha,\beta)^{(1)}$) does not take us out of the class of the $\mathbb{Z}$-graded $U(W_q)$-module $V(\alpha,\beta)^{(2)}$ (resp. $V(\alpha,\beta)$).

Hence, for any $(\lambda,\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$, we can construct two kinds of $\mathbb{Z}$-graded $U(W_q)$-module $A(\lambda,\alpha,\beta)$ and $B(\lambda,\alpha,\beta)$ by using (1.3), (1.7), (1.9), (1.10) and $\varphi(\lambda)$ as follows (where $n, k \in \mathbb{Z}$):

(1.11)
$$A(\lambda,\alpha,\beta) := \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k,$$
$$J(v_k) := \lambda q^k v_k,$$
$$d_n(v_k) := -\lambda^{-1}([k+\alpha]q^{\alpha} + [1+n][\beta]q^{n+k})v_{n+k}$$
$$= -\frac{\lambda^{-1}q^k}{q-q^{-1}}\big(-q^{-2k} + q[\beta]q^{2n} + (q^{2\alpha} - q^{-1}[\beta])\big)v_{n+k}$$

and

(1.12)
$$B(\lambda,\alpha,\beta) := \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k,$$
$$J(v_k) := \lambda q^k v_k,$$
$$d_n(v_k) := -\lambda^{-1}([n+k+\alpha]q^{n+\alpha} + [1-n][\beta]q^{n+k})v_{n+k}$$
$$= -\frac{\lambda^{-1}q^k}{q-q^{-1}}\big(-q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])q^{2n} + q[\beta]\big)v_{n+k}.$$

Let X be A or B, we define

$$c\ell(X) := \{X(\lambda,\alpha,\beta) \mid (\lambda,\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}\}.$$

A $U(W_q)$-module V is said to be in $c\ell(X)$ if $V \simeq X(\lambda,\alpha,\beta)$ (as $U(W_q)$-modules) for some $X(\lambda,\alpha,\beta) \in c\ell(X)$.

REMARK. For any fixed $h \in \mathbb{Z}$, $X(\lambda,\alpha,\beta) \simeq X(\lambda q^h, \alpha+h, \beta')$ as $U(W_q)$-modules, where $\beta' \in \mathbb{C}$ with $[\beta'] = [\beta]q^{2h}$.

2. **The properties of $U(W_q)$-modules $X(\lambda,\alpha,\beta)$.** In this section, we assume that q is not a root of unity.

If q is in the real number field $\mathbb{R}$, then $U(W_q)$ has an antilinear anti-involution θ such that $\theta(J^{\pm 1}) := J^{\pm 1}$ and $\theta(d_n) := d_{-n}$ for all $n \in \mathbb{Z}$.

DEFINITION 2.1. Let $q \in \mathbb{R}$. A $U(W_q)$-module V is *unitary* with respect to θ if there is an Hermitian form $\langle \cdot | \cdot \rangle$ on V such that

$$\langle v|v\rangle > 0 \text{ for } v \in V \text{ and } v \neq 0,$$

$$(*) \qquad \langle x(u)|v\rangle = \langle u|\theta(x)v\rangle \text{ for } u, v \in V \text{ and } x \in U(W_q).$$

An Hermitian form $\langle \cdot|\cdot \rangle$ satisfying $(*)$ is called a *contravariant form*.

Let $X(\alpha,\beta) := X(1,\alpha,\beta)$; then the following proposition is clear:

PROPOSITION 2.1. *For $(\lambda,\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$, we have*

(1) $X(\lambda,\alpha,\beta)$ is reducible if and only if $X(\alpha,\beta)$ is reducible.

(2) If $q \in \mathbb{R}$, then $X(\lambda,\alpha,\beta)$ is unitary with respect to θ if and only if $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $X(\alpha,\beta)$ is unitary with respect to θ. ∎

Now we prove

PROPOSITION 2.2. *Let $X, Y \in \{A, B\}$. Then*

(1) $X(\lambda,\alpha,\beta) \simeq Y(\lambda_1,\alpha_1,\beta_1) \Longleftrightarrow$ there exist some $h \in \mathbb{Z}$ such that $\lambda = \lambda_1 q^h$ and some $\mathbb{Z}$-grading preserving isomorphism φ such that $\varphi : X(\alpha,\beta) \simeq Y(\alpha',\beta')$, where $\alpha' = \alpha_1 + h$ and $[\beta'] = [\beta_1]q^{2h}$.

(2) Every submodule of $X(\lambda,\alpha,\beta)$ respects the $\mathbb{Z}$-grading of $X(\lambda,\alpha,\beta)$.

PROOF. (1) $\Longrightarrow$: Let

$$X(\lambda,\alpha,\beta) = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k, \quad Y(\lambda_1,\alpha_1,\beta_1) = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}u_k$$

$$\psi : X(\lambda,\alpha,\beta) \simeq Y(\lambda_1,\alpha_1,\beta_1), \quad \psi(v_k) = a_{j_1}u_{j_1} + \cdots + a_{j_r}u_{j_r}$$

where $a_{j_s} \in \mathbb{C}^*$ and $j_s \neq j_t$ if $s \neq t$. That $\psi J(v_k) = J\psi(v_k)$ gives that $\lambda = \lambda_1 q^{j_s - k}$ for all $1 \leq s \leq r$. Because q is not a root of unity, $r = 1$. It follows that

$$\psi(v_k) = a_{f(k)}u_{f(k)}, \text{ where } f(k) \in \mathbb{Z}.$$

Since $q^{f(k)-k} = \frac{\lambda}{\lambda_1}$, $f(k) - k = f(k') - k$ for all $k, k' \in \mathbb{Z}$. Let $h := f(k) - k$ for $k \in \mathbb{Z}$. Then

$$\psi(v_k) = a_{k+h}u_{k+h} \text{ for } k \in \mathbb{Z}.$$

By the remark in Section 1, $\eta : Y(\lambda_1,\alpha_1,\beta_1) \simeq Y(\lambda_1 q^h, \alpha', \beta')$, where $\alpha' = \alpha_1 + h$ and $[\beta'] = [\beta_1]q^{2h}$. Let $\varphi := \eta\psi$; then φ preserves the $\mathbb{Z}$-grading and $\varphi : X(\lambda,\alpha,\beta) \simeq Y(\lambda,\alpha',\beta')$. Using the automorphism $\varphi(\lambda)$ of $U(W_q)$, we get that $\varphi : X(\alpha,\beta) \simeq Y(\alpha',\beta')$.

$\Longleftarrow$: If $X(\alpha,\beta) \simeq Y(\alpha',\beta')$, then

$$X(\lambda,\alpha,\beta) \simeq Y(\lambda,\alpha',\beta') = Y(\lambda_1 q^h, \alpha_1 + h, \beta') \simeq Y(\lambda_1,\alpha_1,\beta_1).$$

(2) follows from the application of the operator J. ∎

The proposition above tells us that if q is not a root of unity, then in order to study the properties of the $U(W_q)$-module $X(\lambda,\alpha,\beta)$, it suffices to study the properties of the $U(W_q)$-module $X(\alpha,\beta)$.

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PROPOSITION 2.3. *Let* $(\alpha,\beta) \in \mathbb{C}\times\mathbb{C}$. *Then* $B(\alpha,\beta) \in c\ell(A) \iff q^{2\alpha+1} + (eq^4-1)[\beta] \neq 0$.

PROOF. $\Longleftarrow$: Since $q^{2\alpha+1} + (q^4-1)[\beta] \neq 0$, we can find $(\alpha'\beta') \in \mathbb{C}\times\mathbb{C}$ such that

$$q^{2\alpha'} - q^{-1}[\beta'] = q[\beta] \text{ and } q[\beta'] = q^{2\alpha} - q^{-1}[\beta].$$

Hence, $B(\alpha,\beta) = A(\alpha',\beta') \in c\ell(A)$ by (1.11) and (1.12).

$\Longrightarrow$: If $B(\alpha,\beta) \in c\ell(A)$, then, by Proposition 2.2, there exists a $\mathbb{Z}$-grading preserving isomorphism φ such that

$$\varphi : A(\alpha',\beta') \simeq B(\alpha,\beta) \text{ for some } (\alpha',\beta') \in \mathbb{C}\times\mathbb{C}.$$

Set

$$A(\alpha',\beta') = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v'_k, \quad B(\alpha,\beta) = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}v_k,$$
$$\varphi(v'_k) = a_k v_k, \text{ where } a_k \in \mathbb{C}$$

and

(2.1) $$a_{\alpha\beta} := q^{2\alpha}, \quad b_{\alpha,\beta} := a_{\alpha,\beta} - 1, \quad c_\beta := q[\beta] - 1.$$

Using (1.11), (1.12) and $\varphi d_n(v'_k) = d_n\varphi(v'_k)$, we have

(2.2) $$(-x + q[\beta']y + a_{\alpha'\beta'})a_{n+k} = (-x + a_{\alpha\beta}y + q[\beta])a_k,$$

where

$$x := q^{-2k} \text{ and } y := q^{2n}.$$

It follows from (2.2) that

(2.3) $$(b_{\alpha'\beta'}x + q[\beta']y)a_{n+k} = (c_\beta x + a_{\alpha\beta}y)a_0,$$

(2.4) $$(b_{\alpha'\beta'}x + q[\beta'])a_k = (c_\beta x + a_{\alpha\beta})a_0.$$

Multiplying both sides of (2.2) by

$$(b_{\alpha'\beta'}x + q[\beta']y)(b_{\alpha'\beta'}x + q[\beta']),$$

we get by using (2.3) and (2.4)

$$(-x + q[\beta']y + a_{\alpha'\beta'})(c_\beta x + a_{\alpha\beta}y)(b_{\alpha'\beta'}x + q[\beta'])$$
$$= (-x + a_{\alpha\beta}y + q[\beta])(b_{\alpha'\beta'}x + q[\beta']y)(c_\rho x + a_{\alpha\beta}).$$

Comparing the coefficients of x^2y and xy^2 gives us the following identities:

(2.5) $$-a_{\alpha\beta}b_{\alpha'\beta'} + q[\beta']c_\beta b_{\alpha'\beta'} = -q[\beta']c_\beta + a_{\alpha\beta}b_{\alpha'\beta'}c_\beta;$$

$$[\beta']a_{\alpha\beta}b_{\alpha'\beta'} = [\beta']c_\beta a_{\alpha\beta}. \tag{2.6}$$

Let $n = 0$ in (2.2); we have

$$q[\beta'] + a_{\alpha'\beta'} = q[\beta] + a_{\alpha\beta}. \tag{2.7}$$

- If $a_{\alpha\beta} = 0$, then $[\beta] = q^{2\alpha+1}$ by (2.1). Hence,

$$q^{2\alpha+1} + (q^4 - 1)[\beta] = q^{2\alpha+5} \neq 0.$$

- If $a_{\alpha\beta} \neq 0$ and $[\beta'] \neq 0$, then, by (2.1) and (2.6), we get

$$q^{2\alpha'} - q^{-1}[\beta'] = q[\beta]. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$q^{2\alpha} - q^{-1}[\beta] = q[\beta']. \tag{2.9}$$

Using (2.8) and (2.9), we have that

$$\begin{aligned} 0 \neq q^{2\alpha'} &= q^{-1}[\beta'] + q[\beta] \\ &= q^{-2}(q^{2\alpha} - q^{-1}[\beta] + q[\beta]) \\ &= q^{-3}\big(q^{2\alpha+1} + (q^4 - 1)[\beta]\big), \end{aligned}$$

so $q^{2\alpha+1} + (q^4 - 1)[\beta] \neq 0$.

- If $a_{\alpha\beta} \neq 0$ and $[\beta'] = 0$, then $b_{\alpha',\beta'} \neq 0$ by (2.4). It follows from (2.5) that $c_\beta = -1$, *i.e.* $[\beta] = 0$. Hence,

$$q^{2\alpha+1} + (q^4 - 1)[\beta] = q^{2\alpha+1} \neq 0.$$ ■

A similar argument can prove the following proposition:

PROPOSITION 2.4. *Let φ be a $\mathbb{Z}$-grading preserving linear map and $\alpha, \beta, \alpha'\beta' \in \mathbb{C}$. We have*

(1) $\varphi : A(\alpha,\beta) \simeq A(\alpha',\beta')$ if and only if one of the following conditions holds:
- $(q^{2\alpha} - 1)(q^{2\alpha'} - 1) \neq 0$ *and* $[\beta] = [\beta'] = 0$;
- $q^{2\alpha} = q^{2\alpha'}$ *and* $[\beta] = [\beta']$;
- $q^{2\alpha+1} = [\beta] = q^{2\alpha'-1}$, $[\beta'] = 0$ *and* $q^{2\alpha'} \notin \{q^{2k} \mid k \in \mathbb{Z}\}$;
- $q^{2\alpha'+1} = [\beta'] = q^{2\alpha-1}$, $[\beta] = 0$ *and* $q^{2\alpha} \notin \{q^{2k} \mid k \in \mathbb{Z}\}$.

(2) $\varphi : B(\alpha,\beta) \simeq B(\alpha',\beta')$ if and only if one of the following conditions holds:
- $q^{2\alpha} = q^{-1}[\beta]$, $q^{2\alpha'} = q^{-1}[\beta']$ *and* $(q[\beta] - 1)(q[\beta'] - 1) \neq 0$;
- $q^{2\alpha} = q^{2\alpha'}$ *and* $[\beta] = [\beta']$;
- $q^{2\alpha+1} = [\beta] = q^{2\alpha'-1}$, $[\beta'] = 0$ *and* $q^{2\alpha'} \notin \{q^{2k} \mid k \in \mathbb{Z}\}$;
- $q^{2\alpha'+1} = [\beta] = q^{2\alpha-1}$, $[\beta] = 0$ *and* $q^{2\alpha} \notin \{q^{2k} \mid k \in \mathbb{Z}\}$. ■

PROPOSITION 2.5. *For $(\alpha,\beta) \in \mathbb{C} \times \mathbb{C}$, we have*

(1) $A(\alpha,\beta)$ is reducible if and only if either $q^{2\alpha} - q^{2t} = [\beta] = 0$ or $q^{2\alpha+1} = [\beta] = q^{-2t-1}$ for some $t \in \mathbb{Z}$.

(2) If $B(\alpha,\beta) \notin c\ell(A)$, then $B(\alpha,\beta)$ is irreducible.

PROOF. The arguments of proving (1) and (2) are similar. Let us explain them by proving 2.

If $B(\alpha,\beta) \notin c\ell(A)$, then, by Proposition 2.3,

$$q^{2\alpha+1} + (q^4 - 1)[\beta] = 0. \tag{2.10}$$

Assume that $N \neq 0$ is a submodule of $B(\alpha,\beta)$; it follows from Proposition 2.2 that $N = \oplus_{k\in S} \mathbb{C}v_k$ for some non-empty subset S of $\mathbb{Z}$.

• If there exists $v_m \notin N$ and $v_n \notin N$ with $m \neq n$, then by (1.12),

$$N \ni d_{m-k}(v_k) = -\frac{q^k}{q-q^{-1}}\big(-q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])q^{2(m-k)} + q[\beta]\big)v_m$$
$$N \ni d_{n-k}(v_k) = -\frac{q^k}{q-q^{-1}}\big(-q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])q^{2(n-k)} + q[\beta]\big)v_n$$

where $v_k \in N$. So the coefficients of v_m and v_n have to be zero. This implies that $q^{2\alpha} - q^{-1}[\beta] = 0$. Going back to (2.10), we get that $[\beta] = 0$, which contradicts to (2.10).

• If $\mathbb{Z}\backslash S = \{s\}$, then we can choose $v_m \in N$ and $v_n \in N$ with $m \neq n$. As above, it follows from $d_{s-m}(v_m) \in N$ and $d_{s-n}(v_n) \in N$ that $[\beta] = 0$, which is impossible. ∎

For $t \in \mathbb{Z}$, we define

$$A_t(\alpha,\beta) = \begin{cases} \frac{A(\alpha,\beta)}{\mathbb{C}v_t}, & \text{if } q^{2\alpha} - q^{-2t} = [\beta] = 0; \\ \bigoplus_{\substack{k\in\mathbb{Z}\\ k\neq t}} \mathbb{C}v_k, & \text{if } q^{2\alpha+1} = [\beta] = q^{-2t-1}. \end{cases}$$

Then, $A_t(\alpha,\beta)$ is an irreducible $U(W_q)$-module.

PROPOSITION 2.6. *Let $q \in \mathbb{R}$, then with respect to the antilinear anti-involution θ of $U(W_q)$,*

(1) $A(\alpha,\beta)$ is unitary $\Longleftrightarrow q > 0$ and $q^{2\alpha} = q^{-1}[\beta] + q\overline{[\beta]}$.

(2) $A_t(\alpha,\beta)$ and $B(\alpha,\beta)$ are not unitary, where $B(\alpha,\beta) \notin c\ell(A)$.

PROOF. (1) $\Longrightarrow$: Assume that $A(\alpha,\beta) = \oplus_{k\in\mathbb{Z}} \mathbb{C}v_k$ is unitary and $\langle\cdot|\cdot\rangle$ the contravariant form on V. So

$$\langle d_m(v_k)|v_\ell\rangle = \langle v_k|d_{-m}(v_\ell)\rangle \text{ for } m,k,\ell \in \mathbb{Z}.$$

Let $n := \ell := m + k$, then we have by (1.11)

$$\begin{aligned} &q^k\big(-q^{-2k} + q[\beta]q^{2n-2k} + (q^{2\alpha} - q^{-1}[\beta])\big)\langle v_n|v_n\rangle \\ &\qquad = q^n\big(-q^{-2n} + q\overline{[\beta]}q^{-2n+2k} + (\overline{q^{2\alpha}} - q^{-1}\overline{[\beta]})\big)\langle v_k|v_k\rangle, \end{aligned} \tag{2.11}$$

where $n,k \in \mathbb{Z}$. Let $k = 0$ in (2.11), we get

$$(q[\beta]q^{2n} + b_{\alpha\beta})\langle v_n|v_n\rangle = q^n(\overline{c_\beta}q^{-2n} + \overline{a_{\alpha\beta}})\langle v_0|v_0\rangle, \tag{2.12}$$

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where $a_{\alpha,\beta}$, $b_{\alpha,\beta}$ and c_β are defined by (2.1).

It follows from (2.11) and (2.12) that

$$(2.13)\qquad \begin{aligned}&(-x+q[\beta]xy+a_{\alpha\beta})(b_{\alpha\beta}x+q[\beta])(\overline{a_{\alpha\beta}}y+\overline{c_\beta})\\&\quad=(-x+\overline{a_{\alpha\beta}}xy+q\overline{[\beta]})(\overline{c_\beta}x+\overline{a_{\alpha\beta}})(q[\beta]y+b_{\alpha\beta}),\end{aligned}$$

where $x := q^{-2k}$ and $y := q^{2n}$. Comparing the coefficients of x^2y, xy and y, we get

$$(2.14)\qquad -\overline{a_{\alpha\beta}}b_{\alpha\beta}+q[\beta]\overline{c_\beta}b_{\alpha\beta}=-q[\beta]\overline{c_\beta}+\overline{c_\beta}\overline{a_{\alpha\beta}}b_{\alpha\beta};$$

$$(2.15)\qquad q^2[\beta]^2\overline{c_\beta}+a_{\alpha\beta}\overline{a_{\alpha\beta}}b_{\alpha\beta}=\overline{a_{\alpha\beta}}^2b_{\alpha\beta}+q^2[\beta]\overline{[\beta]}\overline{c_\beta};$$

$$(2.16)\qquad [\beta]a_{\alpha\beta}\overline{a_{\alpha\beta}}=q[\beta]\overline{[\beta]}\overline{a_{\alpha\beta}}.$$

Suppose that $[\beta]=0$, then $a_{\alpha\beta}=q^{2\alpha}\neq 0$, $c_\beta=-1$ and $b_{\alpha\beta}\neq 0$ by (2.1) and (2.12). It follows from (2.15) that $a_{\alpha\beta}=\overline{a_{\alpha\beta}}$. So (2.12) becomes that

$$(q^{2\alpha}-1)\langle v_n|v_n\rangle = q^n(q^{2\alpha}-q^{-2n})\langle v_0|v_0\rangle \text{ for } n\in\mathbb{Z}.$$

This implies that $f(n) := \frac{q^{2\alpha}-q^{4n}}{q^{2\alpha}-1} > 0$ for all $n\in\mathbb{Z}$, which is impossible because $f(n)f(-n)<0$ for large $n>0$. Therefore, we have proved that $[\beta]\neq 0$.

Similarly, we can prove that $a_{\alpha\beta}\neq 0$ by using (2.14).

Going back to (2.16), we have $a_{\alpha\beta}=q\overline{[\beta]}$, *i.e.* $q^{2\alpha}=q^{-1}[\beta]+q\overline{[\beta]}$.

Finally, choose an odd $n_0\in\mathbb{Z}$ such that $q[\beta]q^{2n_0}+b_{\alpha\beta}\neq 0$, then (2.12) gives that

$$\langle v_{n_0}|v_{n_0}\rangle = q^{-n_0}\langle v_0|v_0\rangle,$$

which implies that $q>0$.

$\Longleftarrow$: Define an Hermitian form $\langle\cdot|\cdot\rangle$ on V by

$$\langle v_n|v_m\rangle := \delta_{nm}q^{-n} \text{ for all } n,m\in\mathbb{Z}.$$

It is easy to check that $\langle\cdot|\cdot\rangle$ is a contravariant form.

(2) Use the same argument as above. ∎

3. A partial q-analogue of Kaplansky's Theorem.

LEMMA 3.1. *Let q be not a root of unity, then for all integers n and all positive integers s, we have in $U(W_q)$*

$$d_nd_{-n}^s = q^{-2ns}d_{-n}^sd_n + q^{-sn}\frac{[2n][sn]}{[n]}d_{-n}^{s-1}d_0J^{-1} + [sn][(s-1)n]d_{-n}^{s-1}J^{-2}.$$

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PROOF. We use induction on s. It is clear that the Lemma is true for $s = 1$. Now we assume that the Lemma is true for s, then

$$\begin{aligned}
d_n d_{-n}^{s+1} &= (d_n d_{-n}^s) d_{-n} \\
&= \Big(q^{-2ns} d_{-n}^s d_n + q^{-sn} \frac{[2n][sn]}{[n]} d_{-n}^{s-1} d_0 J^{-1} \\
&\quad + [sn][(s-1)n] d_{-n}^{s-1} J^{-2} \Big) d_{-n} \\
&= q^{-2ns} d_{-n}^s (q^{-2n} d_{-n} d_n + q^{-n}[2n] d_0 J^{-1}) \\
&\quad + [sn][(s-1)n] q^{2n} d_{-n}^s J^{-2} \\
&\quad + q^{-sn} \frac{[2n][sn]}{[n]} d_{-n}^{s-1} q^n (q^{-n} d_{-n} d_0 + [n] d_{-n} J^{-1}) J^{-1} \\
&= q^{-2n(s+1)} d_{-n}^{s+1} d_n + q^{-(s+1)n} (q^{-sn}[n] + q[sn]) \frac{[2n]}{[n]} d_{-n}^s d_0 J^{-1} \\
&\quad + \big(q^{-(s-1)n}[2n] + q^{2n}[(s-1)n] \big) [sn] d_{-n}^s J^{-2} \\
&= q^{-2n(s+1)} d_{-n}^{s+1} d_n + q^{-(s+1)n} \frac{[(s+1)n][2n]}{[n]} d_{-n}^s d_0 J^{-1} \\
&\quad + [(s+1)n][sn] d_{-n}^s J^{-2}.
\end{aligned}$$

This proves the Lemma. ∎

Now we begin to prove the following partial q-analogue of Kaplansky's Theorem:

THEOREM 3.2. *Let q be not a root of unity and $V = \oplus_{k\in\mathbb{Z}} \mathbb{C}v_k$ a $\mathbb{Z}$-graded $U(W_q)$-module with $J(v_k) \in \mathbb{C}v_k$ for $k \in \mathbb{Z}$. If d_1 and d_{-1} are injective operators on V and*

$$(Jd_1d_{-1}J - Jd_{-1}d_1J)(v_0) \neq \frac{1}{q-q^{-1}} v_0,$$

then $V \simeq A(\lambda, \alpha, \beta)$ or $V \simeq B(\lambda, \alpha, \beta)$ for some $(\lambda, \alpha, \beta) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$.

PROOF. Since $J(v_k) \in \mathbb{C}v_k$, $d_1(v_k) \neq 0$ and $Jd_1J^{-1} = qd(v_k)$, there exists some $\lambda \in \mathbb{C}^*$ such that $J(v_k) = \lambda q^k v_k$ for all $k \in \mathbb{Z}$. Using the automorphism $\varphi(\lambda^{-1})$, we can assume that $\lambda = 1$, in which case, we will prove that either $V \simeq A(\alpha, \beta)$ or $V \simeq B(\alpha, \beta)$ for some $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$.

Set

$$d_0(v_0) = av_0, \quad d_1d_{-1}(v_j) = x_jv_j, \quad d_{-1}d_1(v_j) = y_jv_j,$$

where $a, x_j, y_j \in \mathbb{C}$ and $j \in \mathbb{Z}$. We consider the system (i) with respect to α and β:

$$-([\alpha]q^\alpha + [\beta]) = a; \tag{3.2}$$

$$[\alpha]q^\alpha([\alpha-1]q^\alpha + [2][\beta]) = x_0 \tag{3.3}$$

and the system (ii) with respect to α and β:

$$-([\alpha]q^\alpha + [\beta]) = a;$$

$$[\alpha+1]q^\alpha([\alpha]q^\alpha + [2][\beta]q) = y_0 \tag{3.4}$$

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First, we assume that there exist α and β such that (i) holds. Using induction on j and (3.2) gives us

$$d_0(v_j) = -([\alpha+j]q^\alpha + [\beta]q^j)v_j \text{ for } j \in \mathbb{Z}. \tag{3.5}$$

Since $(qd_1d_{-1}J - q^{-1}d_{-1}d_1J)(v_j) = [2]d_0v_j$, we have by (3.5)

$$q^{j+1}x_j - q^{j-1}y_j = -[2][\alpha+j]q^\alpha - [2][\beta]q^j \text{ for } j \in \mathbb{Z}. \tag{3.6}$$

Furthermore, computing $d_1d_{-1}d_1(v_{j-1})$ in two ways produces the following relation between x_j and y_j:

$$x_j = y_{j-1} \text{ for } j \in \mathbb{Z}. \tag{3.7}$$

Going back to (3.6), we get

$$q^{j+1}x_j - q^{j-1}x_{j+1} = -[2][\alpha+j]q^\alpha - [2][\beta]q^j \text{ for } j \in \mathbb{Z}. \tag{3.8}$$

Now we claim that

$$x_j = [\alpha+j]q^\alpha([\alpha+j-1]q^\alpha + [2][\beta]q^j) \text{ for } j \in \mathbb{Z}. \tag{3.9}$$

By (3.3), (3.9) is true for $j = 0$. Assume that (3.9) is true for j, then (3.9) is also true for $j \pm 1$. For example, let us prove that (3.9) is true for $j+1$. By (3.8),

$$\begin{aligned}
x_{j+1} &= q^2x_j + [2][\alpha+j]q^{\alpha-j+1} + [2][\beta]q \\
&= q^2[\alpha+j]q^\alpha([\alpha+j-1]q^\alpha + [2][\beta]q^j) + [2][\alpha+j]q^{\alpha-j+1} + [2][\beta]q \\
&= [\alpha+j]q^{2\alpha}(q^2[\alpha+j-1] + [2]q^{-\alpha-j+1}) \\
&\quad + [2][\beta]q^\alpha([\alpha+j]q^{2+j} + q^{1-\alpha}) \\
&= [\alpha+j]q^{2\alpha}[\alpha+j+1] + [2][\beta]q^\alpha q^{j+1}[\alpha+j+1] \\
&= [\alpha+j+1]q^\alpha([\alpha+j]q^\alpha + [2][\beta]q^{j+1}).
\end{aligned}$$

Hence, (3.9) is true for all $j \in \mathbb{Z}$ by induction.

Let $j = 1$ in (3.9), we get (3.4). So we have proved that if α and β satisfy (i), then α and β also satisfy (ii).

Similarly, we can prove that if α and β satisfy (ii), then α and β also satisfy (i).

A direct calculation shows that either (i) has a solution or (ii) has a solution. Therefore there exists $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$ such that (i) holds.

Using (3.5) and (3.9), we can choose a basis of V, say $\{v_k \mid k \in \mathbb{Z}\}$, such that

$$d_n(v_j) = -([\alpha+j]q^\alpha + [n+1][\beta]q^{n+j})v_{n+j}, \tag{3.10}$$

where $n = 0, \pm 1$ and $j \in \mathbb{Z}$.

For $j \in \mathbb{Z}$, set

$$d_2(v_j) := e(j)v_{j+2}, \quad d_{-2}(v_j) := g(j)v_{j-2},$$

(3.12) $$e(j) := f(j) - ([j+\alpha]q^{\alpha} + [3][\beta]q^{j+2}),$$

(3.13) $$g(j) := h(j) - ([j+\alpha]q^{\alpha} - [\beta]q^{j-2}),$$

where $e(j), g(j), f(j), h(j) \in \mathbb{C}$.

Using (3.10) and following identities:

$$(q^2 d_2 d_{-1} J - q^{-1} d_{-1} d_2 J)(v_j) = [3] d_1(v_j),$$
$$(q^{-2} d_{-2} d_1 J - q d_1 d_{-2} J)(v_j) = -[3] d_{-1}(v_j),$$

we get

$$q^{\alpha+j-1}[\alpha+j+2]e(j) - q^{\alpha+j+2}[\alpha+j]e(j-1) = -[3]([\alpha+j]q^{\alpha} + [2][\beta]q^{j+1}),$$

$$q^{j+1}([\alpha+j-2]q^{\alpha} + [2][\beta]q^{j-1})g(j) - q^{j-2}([\alpha+j]q^{\alpha} + [2][\beta]q^{j+1})g(j+1) = [3][\alpha+j]q^{\alpha}.$$

It follows from (3.12) and (3.13) that

$$[\alpha+j+2]f(j) = q^3[\alpha+j]f(j-1),$$
$$([\alpha+j]q^{\alpha} + [2][\beta]q^{j+1})h(j+1) = q^3([\alpha+j-2]q^{\alpha} + [2][\beta]q^{j-1})h(j).$$

These identities imply that

(3.14) $$f(j) = \frac{q^{3j}[\alpha+1][\alpha+2]}{[\alpha+j+1][\alpha+j+2]} f(0),$$

(3.15) $$h(j) = \frac{q^{3j}([\alpha-1]q^{\alpha} + [2][\beta])([\alpha-2]q^{\alpha} + [2][\beta]q^{-1})}{([\alpha+j-2]q^{\alpha} + [2][\beta]q^{j-1})([\alpha+j-1]q^{\alpha} + [2][\beta]q^{j})} h(0),$$

where $j \in \mathbb{Z}$. Note, that denominators in (3.14) and (3.15) are non-zero follows from that (3.10) and $d_{\pm 1}(v_j) \neq 0$ for all $j \in \mathbb{Z}$.

Let $z := q^{-j}[j]$. We can rewrite (3.12)–(3.15) as follows:

(3.16) $$q^{-j}e(j-2) = q^{-j}f(j-2) - (q^2 z + [\alpha-2]q^{\alpha} + [3][\beta]),$$

(3.17) $$q^{-j}g(j) = q^{-j}h(j) - (z + [\alpha]q^{\alpha} - [\beta]q^{-2}),$$

(3.18) $$q^{-j}f(j-2) = \frac{q^{-2}[\alpha+1][\alpha+2]f(0)}{(q^{3-\alpha}z + [\alpha+1] - [2]q^{1-\alpha})(q^{2-\alpha}z + [\alpha+2] - [2]q^{-\alpha})},$$

(3.19) $$q^{-j}h(j) = \frac{([\alpha-1]q^{\alpha} + [2][\beta])([\alpha-2]q^{\alpha} + [2][\beta]q^{-1})h(0)}{(qz + [\alpha-1]q^{\alpha} + [2][\beta])(q^2 z + [\alpha-2]q^{\alpha} + [2][\beta]q^{-1})}.$$

where $j \in \mathbb{Z}$.

By Lemma 3.1 and a direct computation, we can get

(3.20) $$q^{-j}e(j-2) \cdot q^{-j}g(j) = q^2 z^2 + c_1 z + c_2 \text{ for large even } j,$$

where c_1 and c_2 are complex numbers, which are independent of j.

Using (3.16)–(3.19), we have

$$q^{-j}e(j-2) = -q^2\frac{R_1}{R_2}, \quad q^{-j}g(j) = -\frac{T_1}{T_2}, \tag{3.21}$$

where

$$\begin{aligned} R_1 &:= (z+[\alpha-2]q^{\alpha-2}+[3][\beta]q^{-2})(z+[\alpha+1]q^{\alpha-3}-[2]q^{-2}) \\ &\quad\times(z+[\alpha+2]q^{\alpha-2}-[2]q^{-2}) - q^{2\alpha-9}[\alpha+1][\alpha+2]f(0), \\ R_2 &:= (z+[\alpha+1]q^{\alpha-3}-[2]q^{-2})(z+[\alpha+2]q^{\alpha-2}-[2]q^{-2}), \\ T_1 &:= (z+[\alpha]q^{\alpha}-[\beta]q^{-2})(z+[\alpha-1]q^{\alpha-1}+[2][\beta]q^{-1}) \\ &\quad\times(z+[\alpha-2]q^{\alpha-2}+[2][\beta]q^{-3}) \\ &\quad - q^{-3}([\alpha-1]q^{\alpha}+[2][\beta])([\alpha-2]q^{\alpha}+[2][\beta]q^{-1})h(0), \\ T_2 &:= (z+[\alpha-1]q^{\alpha-1}+[2][\beta]q^{-1})(z+[\alpha-2]q^{\alpha-2}+[2][\beta]q^{-3}). \end{aligned}$$

(3.20) implies that as the polynomials with respect to z, we have

$$R_2T_2 \text{ divides } R_1T_1. \tag{3.22}$$

Now we have two cases to discuss:

• CASE 1. $f(0)g(0) = 0$, in which case, either $f(0) = 0$ or $g(0) = 0$. If $f(0) = 0$, then (3.22) becomes

$$T_2 \text{ divides } (z+[\alpha-2]q^{\alpha-2}+[3][\beta]q^{-2})T_1.$$

It follows that

$$-\frac{q^3([\alpha-1]q^{\alpha}+[2][\beta])([\alpha-2]q^{\alpha}+[2][\beta]q^{-1})h(0)}{T_2} \tag{3.23}$$

is a polynomial of z, hence, it is zero. Since $d_1(v_{-1}) \neq 0$ and $d_1(v_{-2}) \neq 0$, the coefficient of $h(0)$ in (3.23) is not zero. So we have to have $h(0) = 0$.

Similarly, if $h(0) = 0$, then we also have $f(0) = 0$.

Therefore, $f(0)g(0) = 0$ implies that $f(0) = g(0) = 0$. By (3.12)–(3.15), (3.10) is also true for $n = \pm 2$ and $j \in \mathbb{Z}$. This proves that $V = A(\alpha, \beta)$ because $U(W_q)$ is generated by $\{J^{\pm 1}, d_0, d_{\pm 1}, d_{\pm 2}\}$.

• CASE 2. $f(0)g(0) \neq 0$. Since $d_{\pm 1}(v_j) \neq 0$ for all $j \in \mathbb{Z}$, the coefficients of $f(0)$ and $g(0)$ in R_1 and T_1 are non-zero. It follows from (3.22) that R_2 divides T_1 and T_2 divides R_1, *i.e.*

$$\begin{aligned} T_1 &= (z+[\alpha+1]q^{\alpha-3}-[2]q^{-2})(z+[\alpha+2]q^{\alpha-2}-[2]q^{-2})(z+G), \\ R_1 &= (z+[\alpha-1]q^{\alpha-1}+[2][\beta]q^{-1})(z+[\alpha-2]a^{\alpha-2}+[2][\beta]q^{-3})(z+H), \end{aligned}$$

where $G, H \in \mathbb{C}$. Comparing the coefficients of z^2, we get

$$G = [\alpha-2]q^{\alpha-2}+[3][\beta]q^{-2}, \quad H = q^{\alpha}[\alpha]-[\beta]q^{-2}.$$

Going back to (3.21), we have

$$q^{-j}e(j) = -\frac{(z+[\alpha+2]q^{\alpha+2}-[\beta]q^2)(z+[\alpha+1]q^{\alpha+1}+[2][\beta]q^3)}{(z+[\alpha+1]q^{\alpha+1})(z+[\alpha+2]q^{\alpha+2})} \times (z+[\alpha]q^\alpha+[2][\beta]q), \tag{3.24}$$

$$q^{-j}g(j) = -\frac{(z+[\alpha+1]q^{\alpha-3}-[2]q^{-2})(z+[\alpha+2]q^{\alpha-2}-[2]q^{-2})}{(z+[\alpha-1]q^{\alpha-1}+[2][\beta]q^{-1})(z+[\alpha-2]q^{\alpha-2}+[2][\beta]q^{-3})} \times (z+[\alpha-2]q^{\alpha-2}+[3][\beta]q^{-2}) \tag{3.25}$$

for large even j. In particular, the rational function $q^{-j}e(j)$ of z and the rational function of the right side of (3.24) take the same values at infinite different points:

$$\{q^{-j}[j] \mid \text{ for large even } j\}.$$

It follows that (3.24) is true for all $j \in \mathbb{Z}$.

Similarly, (3.25) is also true for all $j \in \mathbb{Z}$.

Now we choose $a_0 = 1$ and $a_j \in \mathbb{C}^*$ for $j \in \mathbb{Z}$ such that

$$\frac{a_{j+k}}{a_{j+k+1}} = \frac{z+[\alpha+k+1]q^{\alpha+k+1}}{z+[\alpha+k]q^{\alpha+k}+[2][\beta]q^{1+2\alpha}} \text{ for } j,k \in \mathbb{Z}. \tag{3.26}$$

Set $u_j := a_j v_j$, we get

$$\begin{aligned} d_n(u_j) &= -q^j(z+[\alpha+n]q^{\alpha+n}+[1-n][\beta]q^n)u_{n+j} \\ &= -([\alpha+n+j]q^{\alpha+n}+[1-n][\beta]q^{n+j})u_{n+j} \end{aligned} \tag{3.27}$$

for $n = 0, \pm 1, \pm 2$ and $j \in \mathbb{Z}$.

For example, let us check that (3.27) is true for $n = 2$ and all $j \in \mathbb{Z}$. By (3.26), we have

$$\frac{a_j}{a_{j+2}} = \frac{(z+[\alpha+1]q^{\alpha+1})(z+[\alpha+2]q^{\alpha+2})}{(z+[\alpha]q^\alpha+[2][\beta]q)(z+[\alpha+1]q^{\alpha+1}+[2][\beta]q^3)}, \tag{3.28}$$

(3.24) and (3.28) imply that

$$\begin{aligned} d_2(u_j) &= a_j(v_j) = a_j e(j) v_{j+2} = q^j \frac{a_j}{a_{j+2}} q^{-j} e(j) u_{j+2} \\ &= -q^j \frac{(z+[\alpha+1]q^{\alpha+1})(z+[\alpha+2]q^{\alpha+2})}{(z+[\alpha]q^\alpha+[2][\beta]q)(z+[\alpha+1]q^{\alpha+1}+[2][\beta]q^3)} \\ &\quad \times \frac{(z+[\alpha+2]q^{\alpha+2}-[\beta]q^2)(z+[\alpha+1]q^{\alpha+1}+[2][\beta]q^3)}{(z+[\alpha+1]q^{\alpha+1})(z+[\alpha+2]q^{\alpha+2})} \\ &\quad \times (z+[\alpha]q^\alpha+[2][\beta]q)u_{j+2} \\ &= -q^j(z+[\alpha+2]q^{\alpha+2}-[\beta]q^2)u_{j+2}. \end{aligned}$$

Therefore, $V = \oplus_{n\in\mathbb{Z}} \mathbb{C}u_n = B(\alpha,\beta)$ by (3.27). ∎

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