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INTERPOLATION IN SPACES OF ENTIRE FUNCTIONS IN $\mathbb{C}^N$

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Let $\{s_m\}$ be a discrete set of points in $\mathbb{C}^N$ and λ_m any sequence of points in $\mathbb{C}$. We shall be interested in finding an entire function $F(z)$ such that $F(s_m)=\lambda_m$. This is of course easy if no restriction is placed on F, but we shall be interested in finding an F which in addition satisfies certain growth conditions.

We shall denote the variable $z=(z_1, \ldots, z_N)$, $z_j=x_j+iy_j$, $\|z\|=(\sum_{j=1}^N |z_j|^2)^{1/2}$. If $f(z)$ is an entire function, we set $M_f(r)=\sup_{\|z\|=r} |f(z)|$, which is an increasing function of r. The function $f(z)$ is of finite exponential type σ if

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M_f(r)}{r} < \infty.$$

If $N=1$ and $\{\lambda_m\} \in \ell^p$, it is known that if $\{s_m\}$ is the set of integers $0, \pm 1, \pm 2, \ldots,$ then there exists an entire function $F(z)$ of exponential type at most π such that $F(s_m)=\lambda_m$ and $F\in L^p(\mathbb{R})$, $\mathbb{R}=\{z{:}y=0\}$. (cf. Boas [1], Chapter 10.6.) When $N>1$, one can obtain the same result by iterating the one variable technique when $\{s_m\}$ is the lattice points all of whose coordinates are integral, but if $\{s_m\}$ only lies near these lattice points, this process will no longer work and different techniques are required. We shall prove the following:

THEOREM. *Let $\{\lambda_m\}\in\ell^2$ and let $\{s_m\}$ be a sequence of points in $\mathbb{C}^N$ such that $|s_m-s_{m'}|\geq 2\delta$ for $m\neq m'$ and $\sup_j |\mathrm{Im}(s_j)_m|\leq C$. Then there exists an entire function $F(z)$ of exponential type at most $A(N, C, \delta)$ with $F\in L^2(\mathbb{R}^N=\mathbb{R}x\cdots x\mathbb{R})$ such that $F(s_m)=\lambda_m$.*

We can take $A(N, C, \delta)=(104N^{5/2}/\delta^2)(1+(C+\delta)^2)^{3/2}$ but this is not necessarily the best possible.

NOTE. Ronkin [4] has shown that if $s_m\in\mathbb{R}^N$ and has a certain density, then if the exponential growth of F is not too large, the solution will be unique.

We begin by stating two lemmas which we shall need in the proof of the theorem.

A function V is plurisubharmonic if it is upper semi-continuous and if $\sum_{j,k} (\partial^2 V/\partial z_j\partial\bar{z}_k)w_j\bar{w}_k$ taken as a distribution defines a positive measure for all $w\in\mathbb{C}^N$. We let $\bar\partial$ be the exterior differential operator $\bar\partial=\sum_{j=1}^N (\partial/\partial\bar{z}_j)\,d\bar{z}_j$. If α is a $(0, 1)$ differential form with functions for coefficients $\alpha=\sum_{j=1}^N C_j\,d\bar{z}_j$, then $|\alpha|=\sum_{j=1}^N |C_j|$.

LEMMA 1. *Let V be a plurisubharmonic function and $C(z)$ a continuous function such that $\sum_{j,k}(\partial^2 V/\partial z_j \partial \bar{z}_k) w_j \bar{w}_k - C(z)\,\|w\|^2$ defines a positive measure for all $w \in \mathbb{C}^N$. Then for every* $(0, 1)$ *form α such that $\bar{\partial}\alpha = 0$, there exists a function β such that $\int |\beta|^2 \exp(-V)\, d\lambda \le \int (|\alpha|^2/C(z)) \exp(-V)\, d\lambda$ when the right hand side is finite.*

Proof. The proof is exactly the same as that of Lemma 4.4.1 of [2] as extended in Theorem 4.4.2.

LEMMA 2. *Let $F(z)$ be an entire function of finite exponential type τ. Then if $F(x+iy) \in L^2(\mathbb{R}^N)$ for some $y=(y_1, \ldots, y_N)$, $F(x+iy) \in L^2(\mathbb{R}^N)$ for all y and*

$$\int_{\mathbb{R}^N} |F(x+iy)|^2\, dx_1 \cdots dx_N \le \exp 2\tau \left(\sum_{j=1}^{N} |y_j| \right) \int_{\mathbb{R}^N} |F(x)|^2\, dx_1 \cdots dx_N.$$

Proof. This follows by iteration of the one variable analogue (cf. Boas [1], p. 98).

Proof of Theorem. For $w = u + iv \in \mathbb{C}$, we define the function

$$V(w) = \sum_{n=1}^{\infty} \log\left\{ \left(\frac{(1+|w-n|^2)}{n^2} \right) \left(\frac{(1+|w+n|^2)}{n^2} \right) \right\} + \log(1+|w|^2).$$

We note that this defines a locally convergent series since

$$(1)\qquad (1+|w-n|^2)(1+|w+n|^2) = (1+u^2+v^2+n^2)^2 - 4u^2 n^2$$

and $\log(1+c/n^2) \le c/n^2$ for all $c \ge 0$. Furthermore, $V(w)$ is periodic with period 1 so V is uniformly bounded for $|v| < B$.

Since V is periodic, we may assume $|u| < 1$. Then for $|v|$ sufficiently large, by (1)

$$\log\left\{ \frac{(1+|w-n|^2)}{n^2} \frac{(1+|w+n|^2)}{n^2} \right\} \le \log\left\{ \frac{((|v|+1)^2 + n^2)^2}{n^4} \right\}$$
$$\le 2 \log\left\{ 1 + \frac{(|v|+1)^2}{n^2} \right\}$$

By comparing V with $\log|\sin \pi w| = \log|w| + \sum_{n=1}^{\infty} \log|1-(w^2/n^2)|$ for $w = iv$, we see that there exists $k > 0$ such that

$$(2)\qquad V(w) \le 2\pi|v| + k$$

We also note that $(\partial^2 V/\partial w \partial \bar{w})(w) = \sum_{n=-\infty}^{\infty} (1/(1+|w-n|^2)^2)$ which converges for all w and is periodic with period 1. We can assume without loss of generality that $0 \le u \le \frac{1}{2}$. Then

$$\sum_{n=1}^{\infty} \frac{1}{(1+B^2+(n-u)^2)^2} > \int_1^{\infty} \frac{ds}{(1+B^2+(s-u)^2)^2}$$
$$= \frac{(s-u)}{2(1+B^2)(1+B^2+(s-u)^2)} \Bigg]_1^{\infty} + \frac{1}{2(1+B^2)} \int_1^{\infty} \frac{ds}{1+B^2+(s-u)^2}$$
$$= \frac{-(1-u)}{2(1+B^2)(1+B^2+(1-u)^2)} + \frac{1}{2(1+B^2)^{3/2}} \left[\frac{\pi}{2} - \tan^{-1} \frac{(1-u)}{(1+B^2)^{1/2}} \right]$$

Similarly

$$\sum_{n=-\infty}^{0} \frac{1}{(1+B^2+(n-u)^2)^2} > \int_1^\infty \frac{ds}{(1+B^2+(s+u)^2)^2}$$

$$= \frac{-u}{2(1+B^2)(1+B^2+u^2)} + \frac{1}{2(1+B^2)^{3/2}}\left[\frac{\pi}{2} - \tan^{-1}\frac{u}{(1+B^2)^{1/2}}\right]$$

So

$$\frac{\partial^2 V(w)}{\partial w\,\partial \bar{w}} > \frac{\pi}{2(1+B^2)^{3/2}} \tag{3}$$

Let $\alpha(r)$ be any $\mathscr{C}^2$ function with $0 \le \alpha \le 1$, $\alpha \equiv 1$ for $r \le \frac{1}{2}$, $\alpha \equiv 0$ for $r \ge 1$. In particular, we will choose

$$\alpha = 1 - \frac{\int_{1/2}^{r} (1-x)^2(x-\frac{1}{2})^2\,dx}{\int_{1/2}^{1} (1-x)^2(x-\frac{1}{2})^2\,dx} \qquad \tfrac{1}{2} < r < 1$$

(The constant A will depend on the choice of α.) An easy computation shows that $\int_{1/2}^{1} (1-x)^2(x-\frac{1}{2})^2\,dx = \frac{1}{960}$. For $\frac{1}{2} < r < 1$, $\alpha'(r) = -960(1-r)^2(r-\frac{1}{2})^2$ and $\sup_r |\alpha'(r)| = \frac{15}{4}$; $\alpha''(r) = 960 \cdot 2(1-r)(r-\frac{1}{2})(2r-\frac{3}{2})$ which takes its maximum when

$$r = \frac{3}{4} + \frac{1}{4\sqrt{3}} \quad \text{and} \quad \sup_r |\alpha''(r)| = \frac{40}{\sqrt{3}}.$$

Let $S(z) = \sum_{m=1}^{\infty} \alpha(z - s_m/\delta)\log(\|z-s_m\|/\delta)$. Since $\log\|z-s_m\|$ is plurisubharmonic we find

$$\sum_{j,k} \frac{\partial^2 S(z)}{\partial z_j\,\partial \bar{z}_k} w_j \bar{w}_k \ge \sum_{j,k} \left\{\frac{-40}{\sqrt{3}}\frac{\log 2}{\delta^2} - \frac{15}{2\delta^2}\right\} |w_j|\,|w_k| \tag{4}$$

$$\ge -\frac{26}{\delta^2} \sum_{j,k} |w_j|\,|w_k|$$

$$\ge -\frac{26}{\delta^2} N \|w\|^2$$

Let $V_1(z) = \sum_{j=1}^{N} V(z_j)$. From (3),

$$\sum_{j,k} \frac{\partial^2 V_1(z)}{\partial z_j\,\partial \bar{z}_k} w_j \bar{w}_k > \frac{\pi \|w\|^2}{2(1+(C+\delta)^2)^{3/2}}$$

for $z \in A = \{z : (\delta/2) \le \|z-s_m\| \le \delta\}$ and $S(z)$ is plurisubharmonic outside of A. Thus by (4)

$$V^*(z) = 2N\left\{\frac{52(1+(C+\delta)^2)^{3/2}}{\pi\delta^2} N V_1(z) + S(z)\right\}$$

is plurisubharmonic and there exists a constant c such that

$$\sum_{j,k} \frac{\partial^2 V^*}{\partial z_j \, \partial \bar{z}_k} w_j \bar{w}_k \geq c \, \|w\|^2 \quad \text{for} \quad \sup_j |y_j| \leq C+\delta.$$

Let $s(z)=\sum_{m=1}^{\infty} \lambda_m \alpha(z-s_m)$. Then if $\alpha=\bar{\partial} s$, $\bar{\partial}\alpha=0$. Let $c(z)$ be any continuous function such that $(c/2)\leq c(z)\leq c$ for $\sup_j |y_j|\leq C+\delta$ and $c(z)\leq \inf \sum_j \sum_{n=-\infty}^{+\infty} \times \{1/(1+|z_j-n|^2)^2\}$. Then $\int |\alpha|^2 \exp(-V^*)/c(z)) \, d\lambda<\infty$ since $\lambda_m \in \ell^2$ so by Lemma 1, there exists a function β such that $\bar{\partial}\beta=\alpha$. Since $\alpha \in \mathscr{C}^\infty$ so is β (cf. Hormander [2], Chapter 4). Notice that $\exp(-V^*)$ has a non-integrable singularity at each point s_m so that $\beta(s_m)=0$. Then $F(z)=s(z)-\beta(z)$ is a holomorphic function, hence entire, and $F(s_m)=\lambda_m$. Furthermore,

$$\int |F|^2 \exp(-V^*) \, d\lambda < \infty.$$

This implies (cf. Lelong [3], Theorem 5.4.3) that F has exponential type at most $\sigma=N^{5/2}(104/\delta^2)(1+(C+\delta)^2)^{3/2}$ since $V_1(z)\leq \sum 2\pi \, |y_i|+k\leq 2\pi \sqrt{N}+k$ by (2). Furthermore, since V^* is bounded above for $\sup_j |y_j|\leq C$, $\int_{|y_j|\leq C} |F|^2 \, d\lambda<\infty$ and hence by Lemma 2, $F\in L^2(\mathbb{R}^N)$. Q.E.D.

References

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