ON PROJECTIVE CHARACTERS OF THE SAME DEGREE

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0. Introduction. All groups G considered in this paper are finite and all representations of G are defined over the field of complex numbers. The reader unfamiliar with projective representations is referred to [**9**] for basic definitions and elementary results.

Let $\mathrm{Proj}(G, \alpha)$ denote the set of irreducible projective characters of a group G with cocyle α. In previous papers (for exampe [**2**], [**4**], and [**6**]) numerous authors have considered the situation when $|\mathrm{Proj}(G, \alpha)| = 1$ or 2; such groups are said to be of α-central type or of 2α-central type, respectively. In particular in [**4**, Theorem A] the author showed that if $\mathrm{Proj}(G, \alpha) = \{\xi_1, \xi_2\}$, then $\xi_1(1) = \xi_2(1)$. This result has recently been independently confirmed in [**8**, Corollary C].

The aim of this short paper is to provide some positive evidence about the following conjecture, of which the result mentioned above is just a special case.

CONJECTURE. *Let G be a group and α be a cocycle of G. Then either G is of α-central type or* $\mathrm{Proj}(G, \alpha)$ *contains at least two elements of the same degree.*

The reader will discover that groups of α-central type play an important part in our investigation of the conjecture, which we are able to verify in a number of cases; most notably when G is supersoluble or has odd order.

1. Characters of the smallest degree. We start by considering the situation when α is trivial.

LEMMA 1.1. *Let G be a non-trivial group. Then* $\mathrm{Irr}(G)$ *do not all have different degrees.*

Proof. Let G be a counterexample of minimal order. Suppose N is a proper normal subgroup of G. Then $\mathrm{Irr}(G/N)$ contains two elements of the same degree, which lift irreducibly to G. So G must be a non-abelian simple group, and moreover all of its irreducible charcaters must be rational valued. Thus $G \cong Sp_6(2)$ or $O_8^+(2)'$ from [**3**, Corollary B.1], but from [**1**] both these groups do possess irreducible characters of the same degree. □

As a consequence of Lemma 1.1, we can assume henceforward where necessary that $o([\alpha]) > 1$ in $\mathrm{M}(G)$, the Schur multiplier of G. We now proceed to verify the conjecture in a number of easy cases, these cases have in common the fact that we need only to consider irreducible projective characters of the smallest degree. To avoid repetition α will always denote a cocycle of the group G under consideration in the following results.

LEMMA 1.2. *Let G be a p-group. Then either G is of α-central type or* $\mathrm{Proj}(G, \alpha)$ *contains n elements of the smallest degree where $n \equiv 0 \pmod p$.*

Proof. Let $\mathrm{Proj}(G, \alpha) = \{\xi_1, \ldots, \xi_t\}$, with ξ_1 being an element of the smallest degree. Then,

$$|G|/(\xi_1(1))^2 = \sum_{i=1}^{t} (\xi_i(1)/\xi_1(1))^2.$$

Now G is of α-central type if and only if $t = 1$. If $t > 1$ the left hand side of the above equation is congruent to 0 modulo p, so there must be n elements $\xi_i \in \mathrm{Proj}(G, \alpha)$ with $n \equiv 0 \pmod{p}$ such that $\xi_i(1) = \xi_1(1)$. □

COROLLARY 1.3. *Let G be a nilpotent group and $\{p_i : 1 \leq i \leq r\}$ be the distinct prime divisors of $|G|$. Then either G is of α-central type or* $\mathrm{Proj}(G, \alpha)$ *contains n elements of the smallest degree where $n \equiv 0 \pmod{p_i}$ for some i with $1 \leq i \leq r$.*

Proof. Let S_i be the Sylow p_i-subgroup of G. Then it follows from either Corollary 5.1.3 or Theorem 7.1.13 of **[9]** that there exist cocycles α_i of S_i such that $\mathrm{Proj}(G, \alpha) = \{\lambda(\xi_1 \times \cdots \times \xi_r) : \xi_i \in \mathrm{Proj}(S_i, \alpha_i)\}$, where λ is a function from G into the non-zero complex numbers with $\lambda(1) = 1$. The result is now immediate from Lemma 1.2. □

Our next result covers the case of a metacyclic group.

LEMMA 1.4. *Let G be a group, and suppose G contains a normal abelian subgroup N such that $[\alpha_N] = [1]$ and G/N is cyclic. Then either G is of α-central type or* $\mathrm{Proj}(G, \alpha)$ *contains at least two elements of the smallest degree.*

Proof. Let $\xi \in \mathrm{Proj}(G, \alpha)$, then $\xi(1)$ divides $[G : N]$ by **[11**, Theorem 2]. Now assume ξ is of the smallest degree, let λ be an irreducible constituent of ξ_N, and I denote the inertia subgroup $I_G(\lambda)$. Then $\lambda(1) = 1$, since N is abelian and $[\alpha_N]$ is trivial. Also since G/N is cyclic, the elements of $\mathrm{Proj}(I/N, \beta)$ all have degree one for any cocycle β of I/N. It follows from the bijections of Clifford's theorem (described in the proof of Theorem 2.1 below), that the $[I : N]$ distinct elements of $\mathrm{Proj}(G, \alpha)$ which are the constituents of λ^G all have degree $\xi(1)$. We have thus constructed at least two elements of the smallest degree unless $I = N$, and $\lambda^G = \xi$. In this case $\xi(1) = [G : N]$, and consequently every element of $\mathrm{Proj}(G, \alpha)$ has this degree. Once again we have at least two elements of the only degree unless ξ is unique and $[G : N] = |N|$. □

Our final result in this section has an almost identical proof to Lemma 1.4, and so the proof is omitted.

COROLLARY 1.5. *Let G be a group, and suppose G contains a normal subgroup N with $\zeta \in$* $\mathrm{Proj}(N, \alpha_N)$ *such that $I_G(\zeta)/N$ is a non-trivial cyclic group. Then there are at least two elements of* $\mathrm{Proj}(G, \alpha)$ *of the same degree which are constituents of ζ^G.*

2. Supersoluble groups and groups of odd order.

THEOREM 2.1. *Let G be a supersoluble group. Then either G is of α-central type or* $\mathrm{Proj}(G, \alpha)$ *contains at least two elements of the same degree.*

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Proof. Let G be a counterexample of minimal order. Let N be a non-trivial normal subgroup of G, and $\zeta \in \operatorname{Proj}(N, \alpha_N)$. Let $I = I_G(\zeta)$, and $\operatorname{Proj}(I|\zeta, \alpha_I)$ denote the set of irreducible constituents of ζ^I. Then by [**9**, Theorem 7.8.10] there exists a cocycle β of I/N and bijections $\operatorname{Proj}(I/N, \beta) \to \operatorname{Proj}(I|\zeta, \alpha_I) \to \operatorname{Proj}(G|\zeta, \alpha)$ defined by $\gamma \mapsto \gamma\kappa \mapsto (\gamma\kappa)^G$, where $\kappa_N = \zeta$ and $\kappa \in \operatorname{Proj}\left(I, \beta^{-1}\alpha_I\right)$. The cocycle β^{-1} is called an obstruction cocycle, since it obstructs the extension of ζ to an element of $\operatorname{Proj}(I, \alpha_I)$. Now since $|I/N| < |G|$, either $\operatorname{Proj}(I/N, \beta)$ contains at least two elements of the same degree or I/N is of β-central type. In the former case the bijections above yield at least two elements of $\operatorname{Proj}(G|\zeta, \alpha)$ of the same degree, contrary to the assumption that G is a counterexample. So we must assume I/N is of β-central type. Consequently $\xi(x) = 0$ for all $\xi \in \operatorname{Proj}(G, \alpha)$, and all $x \notin N$. Since G is not of α-central type, it must contain a unique minimal normal subgroup $K = \langle x : x \text{ is } \alpha\text{-regular}\rangle$.

Since $|K| = p$ for some prime p, K consists of the α-regular elements of G. Let S be a Sylow p-subgroup of G. Then $K \leq Z(S)$, so that $S \leq I_G(\lambda)$ for all $\lambda \in \operatorname{Proj}(K, \alpha_K)$. Let H be a Hall p'-subgroup of G. Then $H \leq I_G(\lambda)$ for some $\lambda \in \operatorname{Proj}(K, \alpha_K)$ by [**5**, Proposition 1.5 and Corollary 2.4]. It follows from the bijections above that exactly one element δ of $\operatorname{Proj}(K, \alpha_K)$ is G-invariant, and there is a unique $\xi \in \operatorname{Proj}(G|\delta, \alpha)$ with $\xi_K = e\delta$ and $e^2 = [G : K]$. Now $\operatorname{Proj}(K, \alpha_K) = \left\{\delta\nu : \nu \in \operatorname{Irr}(K)\right\}$. Let ν be a non-trivial element of $\operatorname{Irr}(K)$, so that ν is faithful. Then $I_G(\delta\nu) = I_G(\nu) = C_G(K) \lhd G$. Thus the G-orbits on $\left\{\delta\nu : \nu \neq 1\right\}$ all have the same length, and for each such orbit we obtain from the bijections above $\xi \in \operatorname{Proj}(G, \alpha)$ with $\xi(1)^2 = [G : K][G : C_G(K)]$. Thus there must be a unique such orbit. This implies that G is of 2α-central type, contrary to [**4**, Theorem A]. □

Theorem 2.2. *Let G be a group of odd order. Then either G is of α-central type or* $\operatorname{Proj}(G, \alpha)$ *contains at least two elements of the same degree.*

Proof. Let G be a counterexample of minimal order. Then the results of the first paragraph of the proof of Theorem 2.1 still hold, and in particular G must contain a unique minimal normal subgroup $K = \langle x : x \text{ is } \alpha\text{-regular}\rangle$. Moreover K is abelian since G has odd order. Now if $K \leq Z(G)$, then $\operatorname{Proj}(G, \alpha)$ consists of $|K|$ elements of degree $[G : K]^{1/2}$, a contradiction. It follows from [**7**, Theorem 2.7(b)] that either K is of α_K-central type or $[\alpha_K] = [1]$. In the former case we obtain that G is of α-central type, a contradiction. So $[\alpha_K] = [1]$.

Our argument now follows that of the proof of Theorem A of [**4**]. Let $C = C_G(K)$, and $V = \operatorname{Irr}(K)$. Let $\bar{R} = R/C$ be a chief factor of G. Then $\bar{R}$ acts faithfully on V and $C_V(\bar{R})$ is trivial, so that $\bar{R}$ has order coprime to p. Thus we may use the arguments in the proofs of Lemmas 2.4 and 2.5 of [**10**] to show that some $\delta \in \operatorname{Proj}(G, \alpha)$ is G-invariant. Let ν be a non-trivial element of V, then $I_G(\delta\nu) = I_G(\nu) = I_G\left(\nu^{-1}\right) = I_G\left(\delta\nu^{-1}\right)$. However since G has odd order ν and ν^{-1} are not conjugate, and so $\delta\nu$ and $\delta\nu^{-1}$ lie in two different orbits of the same length. It follows from the bijections in the proof of Theorem 2.1 that if ξ_1 is an irreducible constituent of $(\delta\nu)^G$ and ξ_2 is an irreducible constituent of $\left(\delta\nu^{-1}\right)^G$, then $\xi_1(1) = \xi_2(1)$, a contradiction. □

If the conjecture is true in general then it has the following immediate application to ordinary character theory.

Proposition 2.3 (Modulo Conjecture). *Let G be a group, N be a normal sub group of G, and $\vartheta \in \operatorname{Irr}(N)$. Then either ϑ^G has at least two irreducible constituents of the same degree, or each irreducible constituent of ϑ^G vanishes on $G - N$.*

Proof. Let $I = I_G(\vartheta)$ and β^{-1} denote the cocycle of I/N which obstructs the extension of ϑ to an element of $\mathrm{Irr}(I)$. Then assuming the conjecture holds either $\mathrm{Proj}(I/N, \beta)$ contains at least two elements of the same degree or I/N is of β-central type. In the former case the bijections in the proof of Theorem 2.1 yield at least two elements of $\mathrm{Irr}(G|\vartheta)$ of the same degree. In the latter case using the notation of Theorem 2.1, $\mathrm{Irr}(G|\vartheta) = \{(\gamma\kappa)^G\}$, where γ is the unique element of $\mathrm{Proj}(I/N, \beta)$. Consequently $(\gamma\kappa)^G(x) = 0$ for all $x \notin N$. □

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