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UNBOUNDED B-FREDHOLM OPERATORS ON HILBERT SPACES

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Abstract This paper is concerned with the study of a class of closed linear operators densely defined on a Hilbert space H and called B-Fredholm operators. We characterize a B-Fredholm operator as the direct sum of a Fredholm closed operator and a bounded nilpotent operator. The notion of an index of a B-Fredholm operator is introduced and a characterization of B-Fredholm operators with index 0 is given in terms of the sum of a Drazin closed operator and a finite-rank operator. We analyse the properties of the powers T^m of a closed B-Fredholm operator and we establish a spectral mapping theorem.

Keywords: unbounded B-Fredholm operators; index; Drazin inverse

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1. Introduction

By $\mathcal{C}(H)$ we denote the set of all linear closed operators from H to H, where H is a Hilbert space. We write $\mathcal{D}(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the domain, nullspace and range of an operator $T \in \mathcal{C}(H)$. An operator $T \in \mathcal{C}(H)$ is called a *Fredholm* operator if both the nullity of T, $n(T) = \dim \mathcal{N}(T)$, and the defect of T, $d(T) = \operatorname{codim} \mathcal{R}(T)$, are finite. The index $\kappa(T)$ of a Fredholm operator T is defined by $\kappa(T) = n(T) - d(T)$. It is well known that if T is Fredholm, then $\mathcal{R}(T)$ is closed. The class of *quasi-Fredholm* operators on a Hilbert space was first studied by Labrousse [**8**]. Most of the results given therein extend easily to the Banach space case.

Definition 1.1. Let $T \in \mathcal{C}(H)$ and let

$$\Delta(T) = \{n \in \mathbb{N} : \forall m \in \mathbb{N},\ m \geqslant n \Rightarrow \mathcal{R}(T^n) \cap \mathcal{N}(T) \subseteq \mathcal{R}(T^m) \cap \mathcal{N}(T)\}.$$

Then the *degree of stable iteration of* T is defined as $\operatorname{dis}(T) = \inf \Delta(T)$ (with $\operatorname{dis}(T) = \infty$ if $\Delta(T) = \emptyset$).

Definition 1.2. Let $T \in \mathcal{C}(H)$ and let $d \in \mathbb{N}$. Then T is *quasi-Fredholm of degree d* if and only if the following three conditions hold:

(i) $\operatorname{dis}(T) = d$;

(ii) $\mathcal{R}(T^d) \cap \mathcal{N}(T)$ is closed in H;

(iii) $\mathcal{R}(T) + \mathcal{N}(T^d)$ is closed in H.

By q-$\Phi(d)$ we denote the set of all quasi-Fredholm operators of degree d. The class of B-Fredholm bounded linear operators acting on a Banach space was studied first by Berkani in [**1**] (and further in [**2**, **3**]). Given a bounded linear operator T, for each integer n, define T_n as the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$. Then T is called a *B-Fredholm operator* if, for some integer n, the range space $\mathcal{R}(T^n)$ is closed and T_n is a Fredholm operator. The index of a B-Fredholm operator T is defined as the index of the Fredholm operator T_n. In [**1**], it was shown that a bounded operator T is a B-Fredholm operator if and only if there exists an integer $d \in \mathbb{N}$ such that $T \in$ q-$\Phi(d)$ and such that $\mathcal{N}(T) \cap \mathcal{R}(T^d)$ is of finite dimension and $\mathcal{R}(T) + \mathcal{N}(T^d)$ is of finite codimension. Moreover, if T is B-Fredholm then $\operatorname{ind}(T) = \dim \mathcal{N}(T) \cap \mathcal{R}(T^d) - \operatorname{codim} \mathcal{R}(T) + \mathcal{N}(T^d)$. Based on this characterization of B-Fredholm bounded operators, we introduce the class of B-Fredholm closed linear operators acting on a Hilbert space H and study its properties. Mainly, we prove that an operator $T \in \mathcal{C}(H)$ densely defined on H is a B-Fredholm operator if and only if $T = T_0 \oplus T_1$, where T_0 is a closed Fredholm operator and T_1 is a nilpotent operator. Associated with a B-Fredholm operator we introduce the notion of index and we characterize B-Fredholm operators of index 0 in terms of the sum of a Drazin closed operator and a finite-rank operator.

Finally, we prove that the powers T^m of a closed B-Fredholm operator are closed B-Fredholm operators and their indices are related by $\operatorname{ind}(T^m) = m \cdot \operatorname{ind}(T)$ and we establish a spectral mapping theorem. These results are an extension of similar results obtained in [**1**, **2**] for the class of closed Fredholm operators acting on a Hilbert space.

2. Definition and characterizations of the B-Fredholm closed operators

In this section we define the set of B-Fredholm operators and we investigate its properties.

Definition 2.1. Let $T \in \mathcal{C}(H)$ be densely defined on H. Then T is called a *B-Fredholm operator* if there exists an integer $d \in \Delta(T)$ such that the following conditions are satisfied:

(i) $\dim \mathcal{N}(T) \cap \mathcal{R}(T^d) < \infty$;

(ii) $\operatorname{codim} \mathcal{R}(T) + \mathcal{N}(T^d) < \infty$.

In this case, the *index of T* is defined as the number

$$\operatorname{ind}(T) = \dim \mathcal{N}(T) \cap \mathcal{R}(T^d) - \operatorname{codim} \mathcal{R}(T) + \mathcal{N}(T^d).$$

From [**4**, Lemma 2.3] it follows that each B-Fredholm operator is a quasi-Fredholm operator of degree $d_0 = \mathrm{dis}(T)$. We also observe that the definition of the index of a B-Fredholm operator is independent of the integer $d \in \Delta(T)$ chosen. Indeed, if $d_0 = \mathrm{dis}(T)$ and we take any $d \in \Delta(T)$, then $\mathcal{N}(T) \cap \mathcal{R}(T^{d_0}) = \mathcal{N}(T) \cap \mathcal{R}(T^d)$. On the other hand, we have that $d_0 \in \Delta(T)$. Hence, by [**8**, Proposition 3.1.1] it follows that $\mathcal{R}(T) + \mathcal{N}(T^d) \subseteq \mathcal{R}(T) + \mathcal{N}(T^{d_0})$ and, consequently, $\mathcal{R}(T) + \mathcal{N}(T^d) = \mathcal{R}(T) + \mathcal{N}(T^{d_0})$ because $d \geqslant d_0$.

Let $\mathrm{BF}(H)$ be the class of all B-Fredholm closed operators densely defined on the Hilbert space H.

Remark 2.2. Assume that $T \in \mathcal{C}(H)$ is a B-Fredholm operator densely defined on H, and let $d = \mathrm{dis}(T)$. Then $\mathcal{N}(T) \cap \mathcal{R}(T^d)$ is of finite dimension and $\mathcal{R}(T) + \mathcal{N}(T^d)$ is of finite codimension. From [**8**, Corollary 3.3.1], we know that $\mathcal{R}(T^n)$ is closed for all $n \geqslant d$. Consider the restriction of T to $\mathcal{R}(T^d)$, $T_d : \mathcal{R}(T^d) \to \mathcal{R}(T^d)$. We have that $\mathcal{D}(T_d) = \mathcal{D}(T) \cap \mathcal{R}(T^d)$ and

$$\dim \mathcal{N}(T_d) = \dim \mathcal{N}(T) \cap \mathcal{R}(T^d) < \infty.$$

On the other hand, we have $\mathcal{R}(T_d) = \mathcal{R}(T^{d+1})$ and, by [**6**, Lemma 3.2],

$$\frac{\mathcal{R}(T^d)}{\mathcal{R}(T^{d+1})} \cong \frac{\mathcal{D}(T^d)}{(\mathcal{R}(T) + \mathcal{N}(T^d)) \cap \mathcal{D}(T^d)} \cong \frac{\mathcal{D}(T^d) + \mathcal{R}(T) + \mathcal{N}(T^d)}{\mathcal{R}(T) + \mathcal{N}(T^d)}.$$

Hence, we get

$$\dim \frac{\mathcal{R}(T^d)}{\mathcal{R}(T^{d+1})} = \dim \frac{\mathcal{D}(T^d)}{(\mathcal{R}(T) + \mathcal{N}(T^d)) \cap \mathcal{D}(T^d)} \leqslant \dim \frac{H}{\mathcal{R}(T) + \mathcal{N}(T^d)} < \infty.$$

Hence, we have proved that T_d is a Fredholm closed operator. However, the fact that, for some integer n, the range space $\mathcal{R}(T^n)$ is closed and the restriction operator T_n is a Fredholm closed operator does not allow us to conclude that T is a B-Fredholm operator in the sense of Definition 2.1, as in the case of bounded operators [**4**].

We observe that if T is densely defined, then the adjoint operator T^* exists, belongs to $\mathcal{C}(H^*)$ and is densely defined.

Proposition 2.3. *Let $T \in \mathcal{C}(H)$ be densely defined. If T is a B-Fredholm, then T^* is B-Fredholm and* $\mathrm{ind}(T^*) = -\mathrm{ind}(T)$.

Proof. Assume that T is B-Fredholm and let $d = \mathrm{dis}(T)$. Since $T \in \text{q-}\Phi(d)$, from [**8**, Proposition 3.3.5] it follows that $T^* \in \text{q-}\Phi(d)$ and

$$\mathcal{R}(T) + \mathcal{N}(T^d) = (\mathcal{N}(T^*) \cap \mathcal{R}(T^{*d}))^{\perp}, \quad \mathcal{R}(T^*) + \mathcal{N}(T^{*d}) = (\mathcal{N}(T) \cap \mathcal{R}(T^d))^{\perp}.$$

Hence, we get

$$\begin{aligned}\mathrm{ind}(T^*) &= \dim \mathcal{R}(T^{*d}) \cap \mathcal{N}(T^*) - \mathrm{codim}\, \mathcal{R}(T^*) + \mathcal{N}(T^{*d}) \\ &= \mathrm{codim}\, \mathcal{R}(T) + \mathcal{N}(T^d) - \dim \mathcal{R}(T^d) \cap \mathcal{N}(T) = -\mathrm{ind}(T).\end{aligned}$$

□

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Now we give a characterization of B-Fredholm closed operators based on the Kato decomposition of quasi-Fredholm operators defined in [**8**].

Theorem 2.4. *Let $T \in \mathcal{C}(H)$ with $\mathcal{D}(T)$ dense in H. Then T is B-Fredholm with $d = \operatorname{dis}(T)$ if and only if there exists two closed subspaces M and N of H such that*

(a) $H = M \oplus N$, $T(\mathcal{D}(T) \cap M) \subseteq M$, $N \subseteq \mathcal{N}(T^d)$ *and* $N \nsubseteq \mathcal{N}(T^{d-1})$;

(b) *if* $T_0 = T|_M$, *then* T_0 *is a closed Fredholm operator densely defined on* M *to* M;

(c) *if* $T_1 = T|_N$, *then* T_1 *is nilpotent of degree* d.

In this case $\operatorname{ind}(T) = \kappa(T_0)$.

Proof. Assume that T is B-Fredholm, that is, $T \in$ q-$\Phi(d)$, $\dim(\mathcal{N}(T) \cap \mathcal{R}(T^d)) < \infty$ and $\operatorname{codim}(\mathcal{R}(T) + \mathcal{N}(T^d)) < \infty$.

By [**8**, Theorem 3.2.1], there exist two closed subspaces M and N such that conditions (a) and (c) given in Theorem 2.4 are fulfilled and also the following condition is satisfied:

($\tilde{b}$) T_0 is a closed operator on M to M, $\mathcal{R}(T_0)$ is closed in M and $\operatorname{dis}(T_0) = 0$.

We observe that T_0 is densely defined on M because T is densely defined on H. Let us prove that T_0 is Fredholm and $\operatorname{ind}(T) = \operatorname{ind}(T_0)$. From the proof of [**8**, Theorem 3.2.1, Equations (3.2.22) and (3.2.23)] we get

$$\mathcal{R}(T_0) \oplus N = \mathcal{R}(T) + \mathcal{N}(T^d) \quad \text{and} \quad \mathcal{N}(T_0) = \mathcal{N}(T) \cap \mathcal{R}(T^d). \tag{2.1}$$

Hence, we obtain $\dim \mathcal{N}(T_0) = \dim \mathcal{N}(T) \cap \mathcal{R}(T^d) < \infty$.

On the other hand, since $\operatorname{codim} \mathcal{R}(T) + \mathcal{N}(T^d) < \infty$, we have $H = L \oplus \mathcal{R}(T) + \mathcal{N}(T^d)$, where L is a finite-dimensional subspace. Now, let $P_M : X \to M$ be the linear projection onto M along N. Then, using the first relation in (2.1), we find that $M = P_M(L) + \mathcal{R}(T_0)$ and, consequently, $\mathcal{R}(T_0)$ is of finite codimension in M. Thus, we have proved that T_0 is Fredholm. Finally, we can write $M = K \oplus \mathcal{R}(T_0)$, where K is a finite-dimensional subspace of M. Therefore, $H = K \oplus (\mathcal{R}(T_0) \oplus N) = K \oplus (\mathcal{R}(T) + \mathcal{N}(T^d))$ and, hence, we conclude that the codimension of $\mathcal{R}(T) + \mathcal{N}(T^d)$ in H coincides with the codimension of $\mathcal{R}(T_0)$ in M. Therefore, $\operatorname{ind}(T) = \kappa(T_0)$.

Conversely, assume that there exist two closed subspaces M and N satisfying conditions (a)–(c). By [**8**, Theorem 3.2.2], and the proof of this result, we find that $T \in$ q-$\Phi(d)$ and the relations (2.1) given in the present theorem are verified. Hence, we obtain

$$\dim \mathcal{N}(T) \cap \mathcal{R}(T^d) = \dim \mathcal{N}(T_0) < \infty$$

and

$$\dim \frac{H}{\mathcal{R}(T) + \mathcal{N}(T^d)} = \dim \frac{M \oplus N}{\mathcal{R}(T_0) \oplus N} = \dim \frac{M}{\mathcal{R}(T_0)} < \infty.$$

Consequently, T is B-Fredholm and $\operatorname{ind}(T) = \kappa(T_0)$. □

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Next, we consider the perturbation of a closed B-Fredholm operator by a finite-rank operator. Recall that a linear bounded operator F is called a *finite-rank operator* if $\dim \mathcal{R}(F) < \infty$.

Proposition 2.5. *Let $T \in \mathcal{C}(H)$ be a B-Fredholm operator with $\mathcal{D}(T)$ dense in H and let $F \in \mathcal{B}(H)$ be a finite-rank operator. Then $T + F$ is a B-Fredholm operator and $\operatorname{ind}(T + F) = \operatorname{ind}(T)$.*

Proof. Let $d = \operatorname{dis}(T)$. We observe that, by [**9**, Theorem 15],

$$\mathcal{N}(T + F) \cap \mathcal{R}((T + F)^d) = \mathcal{N}(T) \cap \mathcal{R}(T^d) + S$$

and

$$\mathcal{N}(T^* + F^*) \cap \mathcal{R}((T + F)^{*d}) = \mathcal{N}(T^*) \cap \mathcal{R}(T^{*d}) + G,$$

where S and G are finite-dimensional subspaces of H and H^*, respectively. Hence, we conclude that $\dim \mathcal{N}(T + F) \cap \mathcal{R}((T + F)^d) \leqslant \dim \mathcal{N}(T) \cap \mathcal{R}(T^d) + \dim S < \infty$ and

$$\begin{aligned}\operatorname{codim} \mathcal{R}(T) + \mathcal{N}(T^d) &= \dim \mathcal{N}(T^* + F^*) \cap \mathcal{R}((T + F)^{*d}) \\ &\leqslant \dim \mathcal{N}(T^*) \cap \mathcal{R}(T^{*d}) + \dim G < \infty.\end{aligned}$$

Therefore, $T + F$ is B-Fredholm.

By Theorem 2.4, relative to the space decomposition $H = M \oplus N$ we have $T = T_0 \oplus T_1$, where T_0 is a closed Fredholm operator on M and T_1 is a nilpotent operator on N. Moreover, $\operatorname{ind}(T) = \kappa(T_0)$. Then we can write

$$T + \frac{1}{n} I = T_0 + \frac{1}{n} I_M \oplus T_1 + \frac{1}{n} I_N.$$

From the stability result [**7**, Theorem 5.22], it follows that $T_0 + I_M/n$ is Fredholm and $\kappa(T_0 + I_M/n) = \kappa(T_0)$ for all sufficiently large n. On the other hand, $T_1 + I_N/n$ is invertible and, thus, $\kappa(T_1 + I_N/n) = 0$. Hence, $T + I/n$ is Fredholm with $\kappa(T + I/n) = \kappa(T_0) = \operatorname{ind}(T)$. Analogously, we can see that the operator $T + F + I/n$ is Fredholm and $\kappa(T + F + I/n) = \operatorname{ind}(T + F)$ for all sufficiently large n. Now, since $T + I/n$ is Fredholm and F is a finite-rank operator, it follows that $\kappa(T + (I/n) + F) = \kappa(T + I/n)$. Thus, we have proved that $\operatorname{ind}(T + F) = \operatorname{ind}(T)$. □

Recall that the *ascent of* an operator A, denoted by $\alpha(A)$, is the smallest non-negative integer n such that $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1})$; the *descent of* A, denoted by $\delta(A)$, is the smallest non-negative integer n such that $\mathcal{R}(A^n) = \mathcal{R}(A^{n+1})$. A closed operator $A \in \mathcal{C}(H)$ is said to be *Drazin invertible* if there exists a bounded operator $X \in \mathcal{B}(H)$ with $\mathcal{R}(X) \subseteq \mathcal{D}(A)$ such that

$$XAX = X, \quad AXu = XAu \quad \text{for all } u \in \mathcal{D}(A)$$

and

$$A^k(I - AX) = 0 \quad \text{for some non-negative integer } k.$$

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The smallest non-negative integer k that satisfies the preceding equation is called the *Drazin index of* A and is denoted by $i(A)$. From the theory of the Drazin inverse of closed linear operators [**10**] we know that an operator $A \in \mathcal{C}(H)$ has a Drazin inverse with $i(A) = k$ if and only if $\alpha(A) = \delta(A) = k$ and $H = \mathcal{R}(A^k) \oplus \mathcal{N}(A^k)$.

Next, using Theorem 2.4 and the characterization of unbounded Fredholm operators given in [**11**, Theorem 1.1], we characterize unbounded B-Fredholm operators of index 0 as the direct sum of a Drazin invertible closed operator and a finite-rank operator.

Theorem 2.6. *Let $T \in \mathcal{C}(H)$ with $\mathcal{D}(T)$ dense in H. Then T is B-Fredholm with $d = \operatorname{dis}(T)$ and such that $\operatorname{ind}(T) = 0$ if and only if $T = A + F$, where A is a Drazin invertible closed linear operator with Drazin index $i(A) = d$, $\mathcal{D}(A) = \mathcal{D}(T)$, and F is a finite-rank operator.*

Proof. Assume that T is B-Fredholm. From Theorem 2.4 we have $H = M \oplus N$, $N \subseteq \mathcal{N}(T^d)$, $N \not\subseteq \mathcal{N}(T^{d-1})$ and, relative to this direct sum, we have the decomposition $T = T_0 \oplus T_1$, where T_0 is a Fredholm closed operator densely defined on M to M such that $\operatorname{ind}(T_0) = 0$ and T_1 is a nilpotent operator. Now, from [**11**, Theorem 1.1] it follows that $T_0 = B + G$, where B is a linear closed operator, $\mathcal{D}(B) = \mathcal{D}(T_0)$, $\mathcal{R}(B) = M$, $\mathcal{N}(B) = \{0\}$ and $G : M \to M$ is a finite-rank operator. Then $T = (B \oplus T_1) + (G \oplus 0)$. Now define $A = B \oplus T_1$ and $F = G \oplus 0$. Clearly, $F : H \to H$ is a finite-rank operator. We easily see that the operator $A : H \to H$ is closed with $\mathcal{D}(A) = \mathcal{D}(T)$,

$$\mathcal{N}(A^d) = \{0\} \oplus \mathcal{N}(T_1^d) = \{0\} \oplus N = \mathcal{N}(A^{d+1}),$$

$\mathcal{N}(A^{d-1}) \neq \mathcal{N}(A^d)$ because $N \not\subseteq \mathcal{N}(T^{d-1})$, and

$$\mathcal{R}(A^d) = M \oplus \mathcal{R}(T_1^d) = M \oplus \{0\} = \mathcal{R}(A^{d+1}).$$

Hence, $\alpha(A) = \delta(A) = d$ and $H = \mathcal{R}(A^d) \oplus \mathcal{N}(A^d)$. Thus, A is Drazin invertible with $i(A) = d$.

Conversely, let us assume that $T = A + F$, where A is a Drazin invertible closed operator with $\mathcal{D}(A) = \mathcal{D}(T)$ such that $i(A) = d$ and F is a finite-rank operator. Then A is a B-Fredholm operator and $\operatorname{ind}(A) = 0$. Thus, by Proposition 2.5, it follows that $T = A + F$ is a B-Fredholm operator and $\operatorname{ind}(T) = 0$. □

3. Properties of B-Fredholm closed operators

Let $T \in \operatorname{BF}(H)$ and let

$$\rho_{\mathrm{BF}}(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \in \operatorname{BF}(H)\}$$

be the B-Fredholm resolvent of T. We denote by $\rho(T)$ the usual resolvent set of T.

Theorem 3.1. *Let $T \in \mathcal{C}(H)$ be densely defined and such that $\rho(T) \neq \emptyset$. The following conditions are equivalent:*

(a) *T is B-Fredholm;*

(b) *T^m is B-Fredholm for each $m \geqslant 1$.*

Moreover, $\operatorname{ind}(T^m) = m \cdot \operatorname{ind}(T)$.

Proof. We only need to prove that (a) $\Longrightarrow$ (b). Assume that T is B-Fredholm and let $d = \mathrm{dis}(T)$. Since $T \in \text{q-}\Phi(d)$ and $\rho(T) \neq \emptyset$, from [**4**, Proposition 2.4] it follows that, for each $m \geqslant 1$, T^m is quasi-Fredholm of degree

$$d_m = \mathrm{dis}(T^m) = \begin{cases} 1 & \text{if } m \geqslant d, \\ k \text{ with } (k-1)m \leqslant d \leqslant km & \text{if } m \leqslant d. \end{cases}$$

Suppose that $m \leqslant d$. Let us prove that $\dim \mathcal{N}(T^m) \cap \mathcal{R}(T^{km}) < \infty$ and $\operatorname{codim} \mathcal{R}(T^m) + \mathcal{N}(T^{km}) < \infty$.

By applying Theorem 2.4, we obtain $T = T_0 \oplus T_1$, where T_0 is a closed Fredholm operator densely defined on M, T_1 is a nilpotent operator on N of degree d and $\mathrm{ind}(T) = \kappa(T_0)$. Since $km \geqslant d$ we obtain $T^{(k+1)m} = T_0^{(k+1)m} \oplus 0$. Now, from [**5**, Theorem IV.2.7] it follows that $T_0^{(k+1)m}$ is a Fredholm operator and, consequently, $\dim \mathcal{N}(T_0^{(k+1)m}) < \infty$ for each $m \geqslant 1$. Now, using [**6**, Lemma 3.1] we get

$$\begin{aligned} \dim \mathcal{N}(T^m) \cap \mathcal{R}(T^{km}) &= \dim \frac{\mathcal{N}(T^{(k+1)m})}{\mathcal{N}(T^{km})} \\ &= \dim \frac{\mathcal{N}(T_0^{(k+1)m}) + N}{\mathcal{N}(T_0^{km}) + N} \\ &= \dim \frac{\mathcal{N}(T_0^{(k+1)m})}{\mathcal{N}(T_0^{km})} < \infty. \end{aligned}$$

On the other hand, since T^* is also B-Fredholm and $d = \mathrm{dis}(T^*)$, using an argument similar to that above we can see that

$$\dim \mathcal{N}(T^{*m}) \cap \mathcal{R}(T^{*km}) < \infty.$$

It follows from [**8**, Proposition 3.3.5] that

$$\operatorname{codim} \mathcal{R}(T^m) + \mathcal{N}(T^{km}) = \dim \mathcal{N}(T^{*m}) \cap \mathcal{R}(T^{*km}) < \infty.$$

Finally, since $T^m = T_0^m \oplus T_1^m$ with T_0^m a Fredholm operator densely defined on M and T_1^m a nilpotent operator on N, and from the fact that $\kappa(T_0^m) = m \cdot \kappa(T_0)$, we conclude that $\mathrm{ind}(T^m) = \kappa(T_0^m) = m \cdot \mathrm{ind}(T)$.

Now we consider the case when $m \geqslant d$. We know that $\mathrm{dis}(T^m) = 1$ and $T^m = T_0^m \oplus 0$. Proceeding as in the above case we can prove that $\dim \mathcal{N}(T^m) \cap \mathcal{R}(T^m) < \infty$ and $\operatorname{codim} \mathcal{R}(T^m) + \mathcal{N}(T^m) < \infty$. $\square$

Given a polynomial $p(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i I)^{m_i}$, define $p(T) = \prod_{i=1}^n (T - \lambda_i I)^{m_i}$. It is well known that $\mathcal{D}(p(T)) = \mathcal{D}(T^s)$, where $s = m_1 + \cdots + m_n$ and that if T is closed and densely defined with $\rho(T) \neq \emptyset$, then the operator $p(T)$ is closed and densely defined.

Lemma 3.2. *Let $\{\lambda_i\}_{1 \leqslant i \leqslant n}$ be a sequence of distinct complex numbers, let $\{m_i\}_{1 \leqslant i \leqslant n}$ be a sequence of positive integers and let d be a positive integer. Then*

$$\mathcal{N}\biggl(\prod_{i=1}^n (T - \lambda_i I)^{m_i}\biggr) \cap \mathcal{R}\biggl(\prod_{i=1}^n (T - \lambda_i I)^{m_i d}\biggr) = \sum_{i=1}^n \mathcal{N}((T - \lambda_i I)^{m_i}) \cap \mathcal{R}((T - \lambda_i I)^{m_i d}).$$

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Proof. We prove the inclusion '$\subseteq$' by induction on n. The case when $n = 1$ is obvious. Assume that the inclusion is true for k and let

$$u \in \mathcal{N}\biggl(\prod_{i=1}^{k+1}(T-\lambda_i I)^{m_i}\biggr) \cap \mathcal{R}\biggl(\prod_{i=1}^{k+1}(T-\lambda_i I)^{m_i d}\biggr).$$

Then

$$(T-\lambda_{k+1}I)^{m_{k+1}}u \in \mathcal{N}\biggl(\prod_{i=1}^{k}(T-\lambda_i I)^{m_i}\biggr) \cap \mathcal{R}\biggl(\prod_{i=1}^{k}(T-\lambda_i I)^{m_i d}\biggr).$$

By the induction assumption, $(T-\lambda_{k+1}I)^{m_{k+1}}u = \sum_{i=1}^{k} v_i$ with $v_i \in \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d})$. Now, for any i, $1 \leqslant i \leqslant k$, since the polynomials $(\lambda-\lambda_i)^{m_i}$ and $(\lambda-\lambda_{k+1})^{m_{k+1}}$ are relatively prime, there exist polynomials $q_{1i}(\lambda)$, $q_{2i}(\lambda)$ such that

$$q_{1i}(\lambda)(\lambda-\lambda_i)^{m_i} + q_{2i}(\lambda)(\lambda-\lambda_{k+1})^{m_{k+1}} = 1.$$

Hence, we deduce that if $x \in \mathcal{D}(T^p)$ for sufficiently large p, then

$$q_{1i}(T)(T-\lambda_i I)^{m_i}x + q_{2i}(T)(T-\lambda_{k+1}I)^{m_{k+1}}x = x.$$

Therefore, since $v_i \in \mathcal{N}((T-\lambda_i I)^{m_i})$ we see that $v_i = q_{2i}(T)(T-\lambda_{k+1}I)^{m_{k+1}}v_i$. Define $w_i = q_{2i}(T)v_i$ for all i, $1 \leqslant i \leqslant k$ and $w_{k+1} = u - \sum_{i=1}^{k} w_i$. We observe that

$$w_i \in \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d}), \quad 1 \leqslant i \leqslant k.$$

Moreover,

$$(T-\lambda_{k+1}I)^{m_{k+1}}w_{k+1} = (T-\lambda_{k+1}I)^{m_{k+1}}u - \sum_{i=1}^{k}(T-\lambda_{k+1}I)^{m_{k+1}}q_{2i}(T)v_i.$$

Hence,

$$(T-\lambda_{k+1}I)^{m_{k+1}}w_{k+1} = (T-\lambda_{k+1}I)^{m_{k+1}}u - \sum_{i=1}^{k} v_i = 0.$$

Thus, $w_{k+1} \in \mathcal{N}((T-\lambda_{k+1}I)^{m_{k+1}})$. Since, for all i, $1 \leqslant i \leqslant k$, $\mathcal{N}((T-\lambda_i I)^{m_i}) \subseteq \mathcal{R}((T-\lambda_{k+1}I)^{m_{k+1}d})$, we see that $w_i \in \mathcal{R}((T-\lambda_{k+1}I)^{m_{k+1}d})$ and so we conclude that $w_{k+1} \in \mathcal{R}((T-\lambda_{k+1}I)^{m_{k+1}d})$. Thus,

$$u = \sum_{i=1}^{k+1} w_i \in \sum_{i=1}^{k+1} \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d}).$$

Now we prove the reverse inclusion by induction. The case $n = 1$ is obvious. Assume that the inclusion is true for k and let

$$u \in \sum_{i=1}^{k+1} \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d}).$$

Then $u = v_{k+1} + w$ with $v_{k+1} \in \mathcal{N}((T - \lambda_{k+1} I)^{m_{k+1}}) \cap \mathcal{R}((T - \lambda_{k+1} I)^{m_{k+1} d})$ and

$$w \in \sum_{i=1}^{k} \mathcal{N}((T - \lambda_i I)^{m_i}) \cap \mathcal{R}((T - \lambda_i I)^{m_i d}).$$

By the induction assumption, we have

$$w \in \mathcal{N}\biggl(\prod_{i=1}^{k} (T - \lambda_i I)^{m_i}\biggr) \cap \mathcal{R}\biggl(\prod_{i=1}^{k} (T - \lambda_i I)^{m_i d}\biggr).$$

Hence,

$$\prod_{i=1}^{k+1} (T - \lambda_i I)^{m_i} u = \prod_{i=1}^{k+1} (T - \lambda_i I)^{m_i} v_{k+1} + \prod_{i=1}^{k+1} (T - \lambda_i I)^{m_i} w = 0.$$

On the other hand, we can deduce that there exist polynomials $q_1(\lambda)$ and $q_2(\lambda)$ such that if $x \in \mathcal{D}(T^p)$ for sufficiently large p, then

$$q_1(T)(T - \lambda_{k+1} I)^{m_{k+1} d} x + q_2(T) \prod_{i=1}^{k} (T - \lambda_i I)^{m_i d} x = x.$$

We observe that $u \in \mathcal{D}(T^p)$ for all $p \geqslant 0$ and then

$$u = q_1(T)(T - \lambda_{k+1} I)^{m_{k+1} d} w + q_2(T) \prod_{i=1}^{k} (T - \lambda_i I)^{m_i d} (v_{k+1} + w).$$

Hence,

$$u = q_1(T)(T - \lambda_{k+1} I)^{m_{k+1} d} w + q_2(T) \prod_{i=1}^{k} (T - \lambda_i I)^{m_i d} v_{k+1}.$$

From this, and the fact that $v_{k+1} \in \mathcal{R}((T - \lambda_{k+1} I)^{m_{k+1} d})$, we see that

$$u \in \mathcal{R}\biggl(\prod_{i=1}^{k+1} (T - \lambda_i I)^{m_i d}\biggr).$$

Consequently,

$$u \in \mathcal{N}\biggl(\prod_{i=1}^{k+1} (T - \lambda_i I)^{m_i}\biggr) \cap \mathcal{R}\biggl(\prod_{i=1}^{k+1} (T - \lambda_i I)^{m_i d}\biggr).$$

□

Theorem 3.3. *Let $T \in \mathcal{C}(H)$ with dense domain and such that $\rho(T) \neq \emptyset$, and let $p(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)^{m_i}$ be a polynomial of degree s with complex coefficients. Then*

$$0 \in \rho_{\mathrm{BF}}(p(T)) \iff \lambda_1, \dots, \lambda_n \in \rho_{\mathrm{BF}}(T).$$

Moreover, in this case $\operatorname{ind}(p(T)) = \sum_{i=1}^{n} m_i \cdot \operatorname{ind}(T - \lambda_i I)$.

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Proof. Suppose that $p(T)$ is B-Fredholm. We shall prove that $(T-\lambda_i I)$ is B-Fredholm for all i, $1 \leqslant i \leqslant n$. Since $p(T)$ is quasi-Fredholm, from [**4**, Theorem 3.1], it follows that $T-\lambda_i I$ is quasi-Fredholm for $1 \leqslant i \leqslant n$. Now, using [**4**, Proposition 2.4], we find that $(T-\lambda_i I)^{m_i}$ is quasi-Fredholm for all i, $1 \leqslant i \leqslant n$. Let $d = \operatorname{dis}(p(T))$ and $d_i = \operatorname{dis}((T-\lambda_i I)^{m_i})$ for all i, $1 \leqslant i \leqslant n$. We have

$$\dim \mathcal{N}(p(T)) \cap \mathcal{R}((p(T))^d) < \infty \quad \text{and} \quad \operatorname{codim} \mathcal{R}(p(T)) + \mathcal{N}((p(T))^d) < \infty$$

because $p(T)$ is B-Fredholm. By applying Lemma 3.2 we obtain

$$\mathcal{N}(p(T)) \cap \mathcal{R}((p(T))^d) = \sum_{i=1}^{n} \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d}).$$

Hence, it follows that, for all i with $1 \leqslant i \leqslant n$, we have

$$\dim \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d}) < \infty.$$

By [**4**, Lemma 2.2] we have $d_i = \operatorname{dis}((T-\lambda_i I)^{m_i}) \leqslant d$ for all i, $1 \leqslant i \leqslant n$, and, thus,

$$\dim \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d_i}) < \infty.$$

On the other hand, applying [**8**, Proposition 3.3.5], we get

$$\begin{aligned} \operatorname{codim} \mathcal{R}(p(T)) + \mathcal{N}((p(T))^d) &= \dim(\mathcal{R}(p(T)) + \mathcal{N}((p(T))^d))^{\perp} \\ &= \dim \mathcal{R}((p(T))^{*d}) \cap \mathcal{N}(p(T)^*) < \infty. \end{aligned}$$

Now, using the symmetry between the conditions on $p(T)$ and on $p(T)^*$, and arguing as above, for $1 \leqslant i \leqslant n$, we obtain

$$\dim \mathcal{R}((T-\lambda_i I)^{*m_i d_i}) \cap \mathcal{N}((T-\lambda_i I)^{*m_i}) < \infty.$$

Hence, we conclude that, for all $1 \leqslant i \leqslant n$,

$$\dim(\mathcal{R}((T-\lambda_i I)^{m_i}) + \mathcal{N}((T-\lambda_i I)^{m_i d_i}))^{\perp} < \infty.$$

Thus, we have proved that $(T-\lambda_i I)^{m_i}$ is B-Fredholm for $1 \leqslant i \leqslant n$. Using Theorem 3.1, we conclude that $(T-\lambda_i I)$ is B-Fredholm for $1 \leqslant i \leqslant n$. Conversely, suppose that $(T-\lambda_i I)$ is B-Fredholm for all i, $1 \leqslant i \leqslant n$. Since each $(T-\lambda_i I)$ is quasi-Fredholm, using [**4**, Theorem 3.1], we conclude that $P(T)$ is quasi-Fredholm. Moreover, if $d_i = \operatorname{dis}((T-\lambda_i I)^{m_i})$ for $1 \leqslant i \leqslant n$ then $d = \operatorname{dis}(p(T)) = \max\{d_i : 1 \leqslant i \leqslant n\}$. If it happens that $\dim \mathcal{N}(p(T)) \cap \mathcal{R}(p(T)^d) = \infty$, then from Lemma 3.2 it would follow that, for some i, $1 \leqslant i \leqslant n$, $\dim \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d}) = \infty$ and we should arrive at a contradiction of the fact that $\dim \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d_i}) < \infty$ for all $1 \leqslant i \leqslant n$.

Analogously, if we suppose that $\operatorname{codim} \mathcal{R}(p(T)) + \mathcal{N}(p(T)^d) = \infty$, we arrive at a contradiction.

Now assume that $p(T)$ is B-Fredholm with $d = \operatorname{dis}(p(T))$. Since

$$\mathcal{N}(p(T)) \cap \mathcal{R}((p(T))^d) = \sum_{i=1}^{n} \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d}),$$

and since the polynomials $(\lambda - \lambda_i)^{m_i}_{1 \leqslant i \leqslant n}$ are relatively prime, we have

$$\dim \mathcal{N}(p(T)) \cap \mathcal{R}((p(T))^d) = \sum_{i=1}^{n} \dim \mathcal{N}((T-\lambda_i I)^{m_i}) \cap \mathcal{R}((T-\lambda_i I)^{m_i d}).$$

Similarly, we have

$$\begin{aligned} \operatorname{codim} \mathcal{R}(p(T)) + \mathcal{N}((p(T))^d) &= \dim \mathcal{N}(p(T^*)) \cap \mathcal{R}((p(T^*))^d) \\ &= \sum_{i=1}^{n} \dim \mathcal{N}((T^* - \bar{\lambda}_i I)^{m_i}) \cap \mathcal{R}((T^* - \bar{\lambda}_i I)^{m_i d}) \\ &= \sum_{i=1}^{n} \operatorname{codim} \mathcal{R}((T-\lambda_i I)^{m_i}) + \mathcal{N}((T-\lambda_i I)^{m_i d}). \end{aligned}$$

Therefore, observing that $d_i = \operatorname{dis}((T-\lambda_i I)^{m_i}) \leqslant d$ for all $1 \leqslant i \leqslant n$, we get

$$\operatorname{ind}(p(T)) = \sum_{i=1}^{n} \operatorname{ind}((T-\lambda_i I)^{m_i}) = \sum_{i=1}^{n} m_i \cdot \operatorname{ind}(T-\lambda_i I).$$

□

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