T. Kuroda
Nagoya Math. J.
Vol. 37 (1970), 5–12

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF PARABOLIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

TADASHI KURODA

Dedicated to Professor Katuzi Ono on his 60th birthday

1. Let R^n be the n-dimensional Euclidean space, each point of which is denoted by its coordinate $x = (x_1, \cdots, x_n)$. The variable t is in the real half line $[0, \infty)$. We consider a differential operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c - \frac{\partial}{\partial t} \tag{1}$$

in the $(n+1)$-dimensional Euclidean half space $R^n \times (0, \infty)$ and assume that the matrix (a_{ij}) is positive definite in $R^n \times (0, \infty)$. Suppose that for coefficients of L there exist constants $K_1(>0)$, $K_2(\geqq 0)$, $K_3(>0)$ and $\lambda \in [0, 1]$ such that

$$|a_{ij}| \leqq K_1(|x|^2 + 1)^{1-\lambda}, \qquad 1 \leqq i, \ j \leqq n,$$
$$|b_i| \leqq K_2(|x|^2 + 1)^{1/2}, \qquad 1 \leqq i \leqq n,$$
$$|c| \leqq K_3(|x|^2 + 1)^{\lambda}.$$

Besala and Fife [1] investigated the asymptotic behavior of solutions of the Cauchy problem for such a parabolic differential operator L under a non-negative Cauchy data not identically equal to zero.

One of their result is as follows:

Let a continuous function $u(x, t)$ in $R^n \times [0, \infty)$ have the following properties;

i) *$Lu \leqq 0$ in $R^n \times (0, \infty)$ in the usual sense,*

ii) *$u(x, 0)$ is non-negative and not identically equal to zero*

and

iii) *there exist positive constants μ and ν such that*

Received December 18, 1968

$$u(x,t) \geqq \begin{cases} -\mu e^{\nu(|x|^2+1)^\lambda}, & \lambda \in (0,1], \\ -\mu(|x|^2+1)^\nu, & \lambda = 0 \end{cases}$$

in $R^n \times (0,\infty)$.

If there exist a sufficiently large constant α *and a positive* β *satisfying*

$$4\alpha^2\lambda^2(|x|^2+1)^{2\lambda-2}\sum_{i,j=1}^n a_{ij}x_ix_j - 4\alpha\lambda(\lambda-1)(|x|^2+1)^{\lambda-2}\sum_{i,j=1}^n a_{ij}x_ix_j$$
$$-2\alpha\lambda(|x|^2+1)^{\lambda-1}\sum_{i=1}^n(a_{ii}+b_ix_i)+c \geqq \beta$$

in $R^n \times (0,\infty)$, *then* $u(x,t)$ *grows exponentially as* t *tends to* ∞ *and this exponential growth of* $u(x,t)$ *is uniform with respect to* $x \in R^n$.

In their proof of this result, the condition that α is sufficiently large, is essential. In this note we shall give a rather simple condition than that of Besala-Fife under a somewhat different condition for coefficients of the operator L.

2. In the following we assume that coefficients of the operator L in (1) satisfy the following condition in $R^n \times (0,\infty)$ for some $\lambda \in (0,1]$:

$$(2)\qquad \begin{cases} k_1(|x|^2+1)^{1-\lambda}|\xi|^2 \leqq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leqq K_1(|x|^2+1)^{1-\lambda}|\xi|^2 \\ \qquad\qquad\qquad \text{for any real vector } \xi = (\xi_1, \cdots, \xi_n), \\ |b_i| \leqq K_2(|x|^2+1)^{1/2}, \qquad 1 \leqq i \leqq n, \\ -k_3(|x|^2+1)^\lambda + k_3' \leqq c \leqq K_3(|x|^2+1)^\lambda, \end{cases}$$

where $k_1(>0)$, K_1, $K_2(\geqq 0)$, $k_3(>0)$, $k_3'(\geqq 0)$ and $K_3(>0)$ are constants.

First we construct a function of the form $H(x,t) = \exp\{-\alpha(t)(|x|^2+1)^\lambda + \beta(t)\}$ satisfying $LH \geqq 0$ in $R^n \times (0,\infty)$, where $\alpha(t)$ and $\beta(t)$ are positive and differentiable once in $(0,\infty)$.

Obviously the condition (2) implies

$$\begin{aligned} \frac{LH}{H} &\geqq 4\alpha^2(t)\lambda^2k_1(|x|^2+1)^{\lambda-1}|x|^2 - 2\alpha(t)\lambda nK_1 \\ &\quad - 2\alpha(t)\lambda nK_2(|x|^2+1)^\lambda - k_3(|x|^2+1)^\lambda + k_3' \\ &\quad + \alpha'(t)(|x|^2+1)^\lambda - \beta'(t) \\ &\geqq (|x|^2+1)^\lambda[4\alpha^2(t)\lambda^2k_1 - 2\alpha(t)\lambda nK_2 - k_3 + \alpha'(t)] \\ &\quad - 2\alpha(t)\lambda nK_1 + k_3' - 4\alpha^2(t)\lambda^2k_1 - \beta'(t). \end{aligned}$$

We can easily verify that the function

$$\alpha(t)=\frac{\gamma_0}{\lambda\sqrt{k_1}}\frac{1}{e^{4\gamma_0\lambda\sqrt{k_1}t}-1}+\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1},\quad \gamma_0=\left(k_3+\frac{n^2K_2^2}{4k_1}\right)^{1/2}$$

is a solution of the differential equation

$$4\alpha^2(t)\lambda^2k_1-2\alpha(t)\lambda nK_2-k_3+\alpha'(t)=0$$

of the Riccati type in $(0,\infty)$ and that for this $\alpha(t)$ the function

$$\beta(t)=\left\{\frac{n\gamma_0}{\sqrt{k_1}}(K_1+K_2)-\frac{n^2K_2}{2k_1}(K_1+K_2)-k_3+k_3'\right\}t$$
$$-\frac{n(K_1+K_2)}{2\lambda k_1}\log(e^{4\gamma_0\lambda\sqrt{k_1}t}-1)+\frac{\gamma_0}{\lambda\sqrt{k_1}}\frac{1}{e^{4\gamma_0\lambda\sqrt{k_1}t}-1}$$

satisfies

$$-2\alpha(t)\lambda nK_1+k_3'-4\alpha^2(t)\lambda^2k_1-\beta'(t)=0$$

in $(0,\infty)$. Hence we see $LH\geqq 0$ in $R^n\times(0,\infty)$ for the function

$$H(x,t)=(e^{4\gamma_0\lambda\sqrt{k_1}t}-1)^{-\frac{n(K_1+K_2)}{2\lambda k_1}}\exp\left\{\frac{\gamma_0}{\lambda\sqrt{k_1}}\frac{1}{e^{4\gamma_0\lambda\sqrt{k_1}t}-1}\right\}\times$$
$$\times\exp\left\{-\left(\frac{\gamma_0}{\lambda\sqrt{k_1}}\frac{1}{e^{4\gamma_0\lambda\sqrt{k_1}t}-1}+\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\right)(|x|^2+1)^{\lambda}\right.\tag{3}$$
$$\left.+\left[\frac{n\gamma_0}{\sqrt{k_1}}(K_1+K_2)-\frac{n^2K_2}{2k_1}(K_1+K_2)-k_3+k_3'\right]t\right\},$$

where $\gamma_0=\left(k_3+\dfrac{n^2K_2^2}{4k_1}\right)^{1/2}$. Since

$$\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}>0\quad\text{and}\quad\frac{\gamma_0}{\lambda\sqrt{k_1}}\{1-(|x|^2+1)^{\lambda}\}<0,\quad x\neq 0,$$

it holds that

$$\lim_{t\downarrow 0}H(x,t)=0\quad\text{for }x\neq 0.\tag{4}$$

3. Suppose that the function $u(x,t)$ non-negative and continuous in $R^n\times[0,\infty)$ has the following property:

$$\begin{cases}\text{i)}\quad Lu\leqq 0\ \text{in}\ R^n\times(0,\infty)\ \text{in the usual sense,}\\ \text{ii)}\quad u(x,0)\,(\geqq 0)\ \text{is not identically equal to zero.}\end{cases}\tag{5}$$

Here L is a differential operator of the form (1) with coefficients satisfying (2) and

$$(6)\qquad -2\Big(\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)\lambda K_1 n-4\Big(\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)^2\lambda^2 k_1+k_3'>0,$$

$$\gamma_0=\Big(k_3+\frac{n^2K_2^2}{4k_1}\Big)^{1/2}.$$

We can find a positive number ε so small that

$$(7)\qquad -2\Big(\varepsilon+\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)\lambda K_1 n-4\Big(\varepsilon+\frac{\gamma_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)^2\lambda^2 k_1+k_3'>0.$$

Let T be a positive number such that

$$0<\frac{\gamma_0}{\lambda\sqrt{k_1}}\,\frac{1}{e^{2\gamma_0\lambda\sqrt{k_1}T}-1}<\varepsilon.$$

From the assumption for $u(x,t)$ we see by the strong maximum principle due to Nirenberg [5] that $u(x,t)>0$ in $R^n\times(0,\infty)$. So the value $m=\min\limits_{\substack{|x|=r\\ t\in[\delta,T]}} u(x,t)$ is positive for an arbitrary $r(>0)$ and for any $\delta(>0)$ fixed sufficiently small. We may assume that $\frac{T}{2}<T-\delta$. For these r and δ, clearly $0<M_1=\max\limits_{\substack{|x|=r\\ t\in[\delta,T]}} H(x,t-\delta)<\infty$, where H is the function given by (3). Put

$$w(x,t)=\frac{m}{M_1}H(x,t-\delta)-u(x,t).$$

Evidently we have $Lw\geqq 0$ in $\Omega\times(\delta,T)$, where Ω is the set of all points $x\in R^n$ such that $r<|x|$. Moreover, $w(x,t)$ is continuous on $\bar{\Omega}\times[\delta,T]$, $w(x,\delta)\leqq 0$ for $|x|\geqq r$ and $w(x,t)\leqq 0$ for $|x|=r$ and $t\in[\delta,T]$. Bodanko's maximum principle [2] implies that $w(x,t)\leqq 0$ in $\bar{\Omega}\times[\delta,T]$. Therefore we get

$$\frac{m}{M_1}H(x,T-\delta)\leqq u(x,T)$$

for $|x|\geqq r(>0)$. As is seen easily, there is a positive constant M_2 such that $M_2H(x,T-\delta)\leqq u(x,T)$ in $|x|\leqq r$. Hence by putting $M_3=\min\Big(\frac{m}{M_1},\ M_2\Big)$ we have $M_3H(x,T-\delta)\leqq u(x,T)$ at every point $x\in R^n$. Since $\frac{T}{2}<T-\delta$, we obtain

$$\begin{aligned} u(x,T) &\geqq M_3 H(x, T-\delta) \\ &\geqq M_4 \exp\left\{-\left(\varepsilon + \frac{\gamma_0}{2\lambda\sqrt{k_1}} + \frac{nK_2}{4\lambda k_1}\right)(|x|^2+1)^\lambda\right\} \end{aligned}$$

in R^n for some positive constant M_4.

4. Now we can prove the following theorem.

THEOREM 1. *Let L be a parabolic differential operator of the form* (1) *with coefficients satisfying* (2) *and* (6). *Assume that the function $u(x,t)$ continuous in $R^n \times [0,\infty)$ satisfies* (5) *and $u(x,t) \geqq -\mu e^{\nu(|x|^2+1)^\lambda}$ for some positive constants μ and ν. Then $u(x,t)$ grows to infinity exponentially as t tends to ∞ and this exponential growth of $u(x,t)$ is uniform in any compact subset of R^n.*

Proof. Bodanko's maximum principle shows that $u(x,t) \geqq 0$ in $R^n \times [0,\infty)$. As was shown in §3, for a positive number ε satisfying (7) there exist a positive number T and a positive constant M_4 such that

$$\begin{aligned} u(x,T) &\geqq M_4 \exp\left\{-\left(\varepsilon + \frac{\gamma_0}{2\lambda\sqrt{k_1}} + \frac{nK_2}{4\lambda k_1}\right)(|x|^2+1)^\lambda\right\} \\ &\equiv M_4 H_0(x), \qquad \text{say.} \end{aligned}$$

From (7) we can take a positive number β_0 which satisfies

$$-2\left(\varepsilon + \frac{\gamma_0}{2\lambda\sqrt{k_1}} + \frac{nK_2}{4\lambda k_1}\right)\lambda K_1 n - 4\left(\varepsilon + \frac{\gamma_0}{2\lambda\sqrt{k_1}} + \frac{nK_2}{4\lambda k_1}\right)^2 \lambda^2 k_1 + k_3' - \beta_0 > 0.$$

Putting

$$h(x,t) = M_4 H_0(x) e^{\beta_0(t-T)}$$

and $v(x,t) = u(x,t) - h(x,t)$ in $R^n \times (T,\infty)$, we see

$$\begin{aligned} Lv &\leqq -Lh \\ &= -h[4\alpha_0^2\lambda^2(|x|^2+1)^{2\lambda-2}\sum_{i,j=1}^n a_{ij}x_i x_j \\ &\quad - 4\alpha_0\lambda(\lambda-1)(|x|^2+1)^{\lambda-2}\sum_{i,j=1}^n a_{ij}x_i x_j \\ &\quad - 2\alpha_0\lambda(|x|^2+1)^{\lambda-1}\sum_{i=1}^n (a_{ii} + b_i x_i) + c - \beta_0] \end{aligned}$$

in $R^n \times (T,\infty)$, where $\alpha_0 = \varepsilon + \dfrac{\gamma_0}{2\lambda\sqrt{k_1}} + \dfrac{nK_2}{4\lambda k_1}$. Hence it follows from (2) that

$$Lv \leqq -h[(|x|^2+1)^\lambda\{4\alpha_0^2\lambda^2k_1 - 2\alpha_0\lambda K_2 n - k_3\}$$
$$- 2\alpha_0\lambda K_1 n - 4\alpha_0^2\lambda^2 k_1 + k_3' - \beta_0].$$

Evidently α_0 and β_0 satisfy

$$4\alpha_0^2\lambda^2k_1 - 2\alpha_0\lambda K_2 n - k_3 > 0 \text{ and } -2\alpha_0\lambda K_1 n - 4\alpha_0^2\lambda^2 k_1 + k_3' - \beta_0 > 0.$$

Therefore we have $Lv \leqq 0$ in $R^n \times (T, \infty)$. Further, we see $v(x, T) = u(x, T) - M_4 H_0(x) \geqq 0$. Applying Bodanko's maximum principle again, we can see $v(x, t) \geqq 0$ in $R^n \times [T, \infty)$, so

$$M_4 H_0(x) e^{\beta_0(t-T)} \leqq u(x, t) \text{ in } R^n \times [T, \infty).$$

From this we get the assertion of Theorem 1.

Example. Consider an operator

$$L_0 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + (-k^2|x|^2 + l) - \frac{\partial}{\partial t} \tag{8}$$

in $R^n \times (0, \infty)$. Let $u(x, t)$ continuous in $R^n \times [0, \infty)$ satisfy $L_0 u \leqq 0$ and $u(x, t) \geqq -\mu e^{\nu|x|^2}$ in $R^n \times (0, \infty)$ for some positive μ, ν and let $u(x, 0)$ be non-negative and not identically equal to zero. The condition (2) is satisfied for $\lambda = 1$, $k_1 = K_1 = 1$, $K_2 = 0$, $k_3 = k^2$ and $k_3' = k^2 + l$. Theorem 1 implies that, if the condition $l > kn$ corresponding to (6) is fulfilled, then $u(x, t)$ grows exponentially to infinity as t tends to infinity. This fact was essentially proved by Szybiak [6] although his theorem is false as Besala and Fife pointed out. Szybiak missed the condition $l > kn$ out of the statement of his theorem.

5. Assume $l < kn$ in (8). In this case, Krzyżański [4] proved the following by constructing the fundamental solution of the Cauchy problem for the equation $L_0 u = 0$: Let u be the solution of the Cauchy problem

$$L_0 u = 0 \quad \text{in } R^n \times (0, \infty),$$
$$u(x, 0) = f(x)$$

for the Cauchy data $f(x)$ bounded in R^n. Then $u(x, t)$ tends to zero uniformly in $x \in R^n$ as t tends to infinity.

Recently Chen [3] treated an analogous problem for an operator of a general form and proved the following fact.

Let the differential operator L in (1) *satisfy the condition*

https://doi.org/10.1017/S0027763000013234 Published online by Cambridge University Press

$$(9)\quad \begin{cases} k_1(|x|^2+1)^{1-\lambda}|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq K_1(|x|^2+1)^{1-\lambda}|\xi|^2 \text{ for real vector } \xi, \\ |b_i| \leqq K_2(|x|^2+1)^{1/2}, \quad 1 \leqq i \leqq n, \\ c \leqq -k_3(|x|^2+1)^{\lambda} + k_3' \end{cases}$$

for some $\lambda \in (0,1]$, $k_1(>0)$, K_1, $K_2(\geqq 0)$, $k_3(>0)$ *and* k_3'. *Further, let* $u(x,t)$ *be a solution of the Cauchy problem* $Lu=0$ *in* $R^n \times (0,\infty)$, $u(x,0)=f(x)$ *in* R^n *and satisfy* $|u(x,t)| \leqq \mu e^{\nu(|x|^2+1)^{\lambda}}$ *for some constants* μ *and* ν *positive. If* $f(x)$ *is bounded in* R^n *and if*

$$\frac{1}{2K_1}[2K_1(1-\lambda)-k_1 n](\sqrt{n^2K_2^2+4K_1k_3}-nK_2)+k_3'<0,$$

then $u(x,t)$ *tends to zero uniformly in* $x \in R^n$ *as* t *tends to infinity.*

6. Here we shall discuss the case when $\lambda \in [1,\infty)$ in Chen's theorem. Let L be an operator of the form (1) with coefficients satisfying (9) for $\lambda \in [1,\infty)$. For $H(x,t)=\exp\{-\alpha(t)(|x|^2+1)^{\lambda}+\beta(t)\}$ with $\alpha(t)(>0)$ and $\beta(t)$ differentiable once in $(0,\infty)$ we have

$$\begin{aligned}\frac{LH}{H} &\leqq (|x|^2+1)^{\lambda}[4\lambda^2K_1\alpha^2(t)+2\lambda nK_2\alpha(t)-k_3+\alpha'(t)] \\ &\quad -2\lambda k_1 n\alpha(t)+k_3'-4\lambda^2K_1\alpha^2(t)-\beta'(t).\end{aligned}$$

Hence, if we take

$$(10)\qquad \alpha(t)=\gamma\ \tanh 4\lambda^2K_1\gamma t$$

and

$$\beta(t)=[-2\lambda k_1 n\gamma-4\lambda^2K_1\gamma^2+k_3']t+\frac{k_1 n}{2\lambda K_1}\log\frac{e^{8\lambda^2K_1\gamma t}}{e^{8\lambda^2K_1\gamma t}+1}-\frac{2\gamma}{e^{8\lambda^2K_1\gamma t}+1}$$

for the positive root γ of the quadratic equation $4\lambda^2K_1X^2+2\lambda nK_2X-k_3=0$, then we get $LH\leqq 0$ in $R^n\times(0,\infty)$. Clearly $H(x,0)=e^{\beta(0)}=e^{-\gamma-\frac{k_1 n}{2\lambda K_1}\log 2}$. Putting $w_{\pm}(x)=Me^{-\beta(0)}H(x,t)\pm u(x,t)$ for $M=\sup_{x\in R^n}|f(x)|$, where $u(x,t)$ is a solution of the Cauchy problem $Lu=0$ in $R^n\times(0,\infty)$, $u(x,0)=f(x)$ for the bounded Cauchy data $f(x)$ and satisfies $|u(x,t)|\leqq \mu e^{\nu(|x|^2+1)^{\lambda}}$ for some positive μ and ν, we have $Lw_{\pm}\leqq 0$ in $R^n\times(0,\infty)$ and $w_{\pm}(x,0)\geqq 0$. From Bodanko's maximum principle in the case of $\lambda\in[1,\infty)$ we get $w_{\pm}(x,t)\geqq 0$ in $R^n\times[0,\infty)$, so

$$|u(x,t)| \leqq Me^{-\beta(0)}H(x,t)$$
$$\leqq Me^{-\beta(0)}e^{\beta(t)} \leqq Me^{-\beta(0)}e^{(-2\lambda k_1 n\gamma - 4\lambda^2 K_1\gamma^2 + k_3')t}$$

in $R^n \times [0,\infty)$. Therefore, if

$$-2\lambda k_1 n\gamma - 4\lambda^2 K_1\gamma^2 + k_3' < 0, \tag{11}$$

then $u(x,t)$ decays to zero exponentially as t tends to infinity. Thus we have the following

THEOREM 2. *Let L be a differential operator of the form* (1) *with coefficients satisfying* (9) *for some $\lambda \in [1,\infty)$ and let $u(x,t)$ be a solution of the Cauchy problem*

$$Lu = 0 \quad \textit{in } R^n \times (0,\infty),$$
$$u(x,0) = f(x) \quad \textit{in } R^n$$

for a bounded continuous Cauchy data $f(x)$ in R^n. Assume that there exist positive constants μ and ν such that $|u(x,t)| \leqq \mu e^{\nu(|x|^2+1)^\lambda}$ in $R^n \times [0,\infty)$. If the condition (11) *is valid, then $u(x,t)$ decays to zero exponentially as t tends to infinity and this decay of $u(x,t)$ is uniform in R^n.*

7. We apply Theorem 2 to the operator (8). In this case we may take $\lambda = 1$, $k_1 = K_1 = 1$, $K_2 = 0$, $k_3 = k^2$, $k_3' = k^2 + l$ and γ in (10) equal to $\frac{k}{2}$. So (11) reduces to $kn > l$. This is nothing but the result of Krzyżański stated in §5.

REFERENCES

[1] P. Besala-P. Fife, The unbounded growth of solutions of linear parabolic differential equations, Ann. Scoula Norm. Sup. Pisa III, **20** (1966), 719–732.

[2] W. Bodanko, Sur le problème de Cauchy et les problèmes de Fourier pour les equations paraboliques dans un domaine non borné, Ann. Polo. Math., **18** (1966), 79–94.

[3] L.S. Chen, Note on the behavior of solutions of parabolic equations with unbounded coefficients, Nagoya Math. Journ., **37** (1969), 1~4.

[4] M. Krzyżański, Evaluations des solutions de l'equation lineaire de type paraboliques à coefficients non-borné, Ann. Polo. Math., **11** (1961–62), 253–260.

[5] L. Nirenberg, A strong maximum principle for parabolic equations, Comm. Pure Appl. Math., **6** (1953), 167–177.

[6] A. Szybiak, On the asymptotic behavior of the solutions of the equation $\Delta u - \frac{\partial u}{\partial t} + c(x)u = 0$, Bull. Acad. Polo. Sci., **7** (1959), 183–186.

Mathematical Institute,
Tohoku University,
Sendai, Japan.