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AUTOMORPHISMS OF PERMUTATIONAL WREATH PRODUCTS

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Abstract

Ore (1942) studied the automorphisms of finite monomial groups and Holmes (1956, pp. 23–93) has given the form of the automorphisms of the restricted monomial groups in the infinite case. The automorphism group of a standard wreath product has been studied by Houghton (1962) and Segal (1973, Chapter 4). Monomial groups and standard wreath products are both special cases of permutational wreath product. Here we investigate the automorphisms of the permutational wreath product and consider to what extent the results holding in the special cases remain true for the general construction. Our results extend those of Bunt (1968).

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1. The structure of the automorphism group

Given non-trivial groups A and B and a subgroup H of B, let $X = B/H$ be the set of right cosets of H in B. Let F be the group of all functions from X to A with multiplication $fg(x) = f(x)g(x)$ for all $f, g \in F$ and $x \in X$. F is the unrestricted direct product of isomorphic copies of A, there being one copy of A for each $x \in X$. If $f \in F$ and $b \in B$, we define $f^b \in F$ by: $f^b(x) = f(xb^{-1})$, for all $x \in X$. Then the split extension W of F by B, with the relation $b^{-1}fb = f^b$ for all $f \in F$, $b \in B$, is called the unrestricted permutational wreath product of A and B with respect to H. We write $W = A\,\mathrm{Wr}(H)B$.

A function $f \in F$ is said to have finite support if $f(x) = 1$ for all but a finite number of $x \in X$. If E is the subgroup of F consisting of all the elements $f \in F$ of finite support, then the split extension V of E by B, with the relation $b^{-1}fb = f^b$ for all $f \in E$, $b \in B$, is the restricted permutational wreath product of A and B with respect to H. We write $V = A\,\mathrm{wr}\,(H)\,B$. We write PWP for permutational wreath product. The subgroups F and E are called the base groups of the associated permutational wreath products.

If $H = 1$, our PWP is simply the standard wreath product, and if B is a normal subgroup of the symmetric group on X, then the PWP is a monomial group in the sense of Crouch (1955).

Suppose the group B acts transitively on a set X. If $H = \mathrm{stab}\,x_0$ is the stabilizer of $x_0 \in X$, then X can be identified with the set B/H and we shall write $W = A\,\mathrm{Wr}\,B$ and $V = A\,\mathrm{wr}\,B$ for the PWP's $W = A\,\mathrm{Wr}\,(H)\,B$ and $V = A\,\mathrm{wr}\,(H)\,B$, respectively.

Two facts concerning the standard wreath products are (i) that the base group is characteristic, except in a few cases (see Neumann, 1964), and (ii) that all the complements of the base group are conjugate in the unrestricted case (see Houghton 1962; 1972). With these two facts, the automorphism group of the unrestricted standard wreath product W of two groups A and B can be completely described. Houghton (1962) shows that the automorphism group can be expressed as a product KI_1B^*, where K is the subgroup of $\mathrm{Aut}(W)$ consisting of those automorphisms which leave B elementwise fixed, I_1 is the subgroup of $\mathrm{Aut}(W)$ consisting of those inner automorphisms corresponding to transformation by elements of the base group F, and B^* is a subgroup of $\mathrm{Aut}(W)$ isomorphic to $\mathrm{Aut}(B)$.

We now investigate the automorphisms of the PWP. We write $\bar{W}$ to denote W or V and $\bar{F}$ to denote F or E as appropriate.

Let $\mathrm{Aut}_{\bar{F}}(\bar{W}) = \{\alpha \in \mathrm{Aut}(\bar{W}) : \bar{F}^{\alpha} = \bar{F}\}$. The elements of $\mathrm{Aut}_{\bar{F}}(\bar{W})$ inducing the identity on $\bar{W}/\bar{F} \cong B$ will induce monomorphisms from B to $\bar{W}$. If α is such an automorphism, then B^{α} is of the form $\{bf_b : b \in B\}$ with $f_b \in \bar{F}$, and is a complement of $\bar{F}$ in $\bar{W}$. Let $B(\bar{W})$ consist of all $\alpha \in \mathrm{Aut}_{\bar{F}}(\bar{W})$ which send B to a conjugate B^g, for some $g \in F$, that is, $B(\bar{W}) = \{\alpha \in \mathrm{Aut}_{\bar{F}}(\bar{W}) : B^{\alpha} = B^g \text{ for some } g \in F\}$.

We remark that in the restricted case we do not require B^{α} to be conjugate to B in $\bar{W} = V$.

In this paper we will show that $B(\bar{W})$ is an extension of KI_1 by J, where K and I_1 are certain subgroups of $B(\bar{W})$ and J is a certain subgroup of $\mathrm{Aut}(B)$, the description of which will be given. We consider the question of when $B(\bar{W}) = \mathrm{Aut}_{\bar{F}}(\bar{W})$, which leads us to consider conditions for all the complements of $\bar{F}$ to be conjugate in W. We note that $B(\bar{W}) = \mathrm{Aut}(\bar{W})$ (or $\mathrm{Aut}_{\bar{F}}(\bar{W})$ when $\bar{F}$ is not a characteristic subgroup) in the standard wreath product. We will show that the same is true for the monomial groups in many cases. The question of when $\mathrm{Aut}_{\bar{F}}(\bar{W}) = \mathrm{Aut}(\bar{W})$ will not be considered in detail here, but we note that the base group of a general PWP is not in general characteristic. In Mohammadi Hassanabadi (1976) we have used the methods of Neumann (1964) to show that the base group of a monomial group is always characteristic except when $|X| = 2$ and A is a special dihedral group, that is, A is a dihedral group whose normal abelian subgroup of index 2 contains unique square roots of all its elements.

In the standard wreath product of two groups A and B, all the automorphisms $\beta \in \operatorname{Aut}(B)$ can be extended to automorphisms of the wreath product by defining $(bf)^{\beta^*} = b^{\beta} f^{\beta^*}$, where $f^{\beta^*}(x) = f(x^{\beta^{-1}})$ for all $x \in B$. In the PWP this no longer holds, in general.

Let $\alpha \in \operatorname{Aut}(A)$ and define $\alpha^*\colon \bar{W} \to \bar{W}$ by: $(bf)^{\alpha^*} = bf^{\alpha^*}$ for all $b \in B$, $f \in \bar{F}$, where $f^{\alpha^*}(x) = (f(x))^{\alpha}$ for all $x \in X$. Then $\alpha^* \in \operatorname{Aut}(\bar{W})$. The group A^* of all such automorphisms α^* is isomorphic to $\operatorname{Aut}(A)$. The subgroup of F consisting of all constant functions, that is, functions $f \in F$ with $f(x) = f(x')$ for all $x, x' \in X$, is called the diagonal subgroup of F and is denoted by D.

THEOREM 1.1. (a) *$B(\bar{W})$ is an extension of KI_1 by J, where*

(i) *K is the subgroup of $B(\bar{W})$ consisting of those automorphisms of $\bar{W}$ which leave B elementwise fixed;*

(ii) *in the unrestricted case I_1 is the subgroup of $B(W)$ corresponding to the inner automorphisms induced by elements of F;*

(ii)′ *in the restricted case, I_1 is the subgroup of $B(V)$ corresponding to the inner automorphisms i_g induced by the elements g of F and such that, $g^{-b} g \in E$ for all $b \in B$;*

(iii) *J is the subgroup of* $\operatorname{Aut}(B)$ *consisting of all the automorphisms of B, which can be induced by elements of $B(\bar{W})$.*

(b) *In the unrestricted case, the subgroup K is a split extension of K_D by A^*, where $A^* \cong$* $\operatorname{Aut}(A)$ *is as defined earlier, and K_D consists of all elements of K which leave the diagonal subgroup D elementwise fixed.*

PROOF. (a) If $\alpha \in B(\bar{W})$, then $B^{\alpha} \cong B$. For $b \in B$, $b^{\alpha} \equiv b' \pmod{\bar{F}}$, where $b' \in B$, say, $b^{\alpha} = b' f_{b'}$ and $f_{b'} \in \bar{F}$. Define $\beta\colon B \to B$ by: $b^{\beta} = b'$ for all $b \in B$. Then since $\bar{F}^{\alpha} = \bar{F}$ and $\bar{W}/\bar{F} \cong B$, we have, for each $b \in B$, $(b\bar{F})^{\alpha} = b^{\alpha} \bar{F} = b' f_{b'} \bar{F} = b' \bar{F} = b^{\beta} \bar{F}$, and since $\alpha \in \operatorname{Aut}(\bar{W})$, it follows that $\beta \in \operatorname{Aut}(B)$. We say β is the automorphism of B induced by α. Now define a map $\varphi\colon B(\bar{W}) \to \operatorname{Aut}(B)$ by: for $\alpha \in B(\bar{W})$, let $(\alpha)\varphi =$ the automorphism of B induced by α. It follows that φ is a homomorphism. Let $M = \operatorname{Ker}(\varphi)$ be the kernel of φ; then $M = \{\gamma \in B(\bar{W})\colon b^{\alpha} \in b\bar{F} \text{ for all } b \in B\}$. If J is the image of $B(\bar{W})$ under φ, then $B(\bar{W})$ is an extension of M by J.

Take $\gamma \in M$, and let $\gamma|_B = \sigma$. Then $B^{\sigma} = \{bf_b\colon b \in B\}$, and so there exists an element $g \in F$ such that, $b\sigma = g^{-1} b g$ for all $b \in B$ and with $g^{-b} g \in E$ for all $b \in B$ in the restricted case. If $i_{g^{-1}}$ is the inner automorphism defined by $w^{i_{g^{-1}}} = gwg^{-1}$ for all $w \in \bar{W}$, then $b^{\gamma i_{g^{-1}}} = b$ for all $b \in B$. Thus $\gamma i_{g^{-1}} \in K$, and so $\gamma \in KI_1$. Hence $M = KI_1$.

The proof of (b) is the same as for the standard wreath product and is given in Houghton (1962).

2. The complements of the base groups

To consider the question of when $B(\bar{W}) = \mathrm{Aut}_{\bar{F}}(\bar{W})$, we need to decide when all the complements of the base group $\bar{F}$ are conjugate. A subgroup C of $\bar{W}$ is a complement of $\bar{F}$, if $\bar{W} = C\bar{F} = B\bar{F}$ and $\bar{F} \cap C = 1$. So in general, a complement of $\bar{F}$ in $\bar{W}$ has the form $\{bf_b \colon b \in B)$ with $f_b \in \bar{F}$. It follows since $bf_b\, cf_c = bcf_b^c f_c \in C$, that $f_{bc} = f_b^c f_c$ for all $b, c \in B$.

Given a complement C of $\bar{F}$ in $\bar{W}$, C is isomorphic to B, via an isomorphism $\sigma\colon b \to bf_b$, for all $b \in B$. A map $\sigma\colon b \to bf_b$, for all $b \in B$, is an isomorphism if and only if $f_{bc} = f_b^c f_c$ for all $b, c \in B$. Such a map will be called a section.

Let $\bar{W} = A\,\mathrm{Wr}(H)B$ or $A\,\mathrm{wr}(H)B$. Given a section $\sigma\colon b \to bf_b$, put $h\alpha = f_h(H)$ for all $h \in H$. Then α is a map from H to A, and for $h, h' \in H$, we have

$$(hh')\,\alpha = f_{hh'}(H) = f_h^{h'}(H) f_{h'}(H) = f_h(Hh'^{-1}) f_{h'}(H) = f_h(H) f_{h'}(H) = h\alpha . h'\alpha.$$

So $\alpha \in \mathrm{Hom}(H, A)$, where $\mathrm{Hom}(H, A)$ denotes the set of all homomorphisms from H to A.

Let $\sigma\colon b \to bf_b$ be any section and let $g \in F$. Then $\sigma'\colon b \to g^{-1} bf_b g$ for all $b \in B$ is also a section. We say σ and σ' are conjugate. Similarly if α and α' are homomorphisms from H to A related by $h\alpha' = a^{-1}.h\alpha.a$ for some $a \in A$, we say α and α' are conjugate. Conjugate sections give rise to conjugate homomorphisms. In fact Houghton (1975) has the following result.

THEOREM 2.1. *The set of conjugacy classes of complements of the base group F in the wreath product $W = A\,\mathrm{Wr}(H)B$ is bijective with the set of conjugacy classes of homomorphisms from H to A.*

Thus the complements of the base group F are all conjugate and therefore conjugate to B if and only if $\mathrm{Hom}(H, A)$ is trivial, in which case $\mathrm{Aut}_{\bar{F}}(\bar{W}) = B(\bar{W})$.

In the restricted case we have a different situation. Let $C = \{bf_b \colon b \in B\}$ be a complement of E in V (then of course C is a complement of F in W). The question of which conjugacy classes of $\mathrm{Hom}(H, A)$ will arise in correspondence to these complements was asked by Houghton (1975). Let L be the set of all conjugacy classes of $\mathrm{Hom}(H, A)$ which can arise. If A is abelian, L is a subgroup of the group $\mathrm{Hom}(H, A)$. In general, we shall say L is trivial or write $L = 1$ if L consists of a single conjugacy class. Houghton (1975) showed that if the kernels of all homomorphisms $\alpha\colon H \to A$ are normal in the normalizer N of H in B, and if N/H is infinite, then L is trivial. We note that for $B(V) = \mathrm{Aut}_E(V)$ to be true it is enough to have L trivial.

THEOREM 2.2. *Let $B = A_X$ or $S_{(X)}$, the alternating or finitary permutation group of an infinite set X. Then L is trivial.*

PROOF. Let $x_0 \in X$ be a fixed point of X, and let $H = \operatorname{stab} x_0$. Take α to be any element of L and suppose $\sigma\colon b \to bf_b$ is a section corresponding to α. Then we have $f_b \in E$ for all $b \in B$, and $f_h(H) = h\alpha$ for all $h \in H$. Take any element h in H and let $x, y \in X$ be fixed by h. Then $b = (x_0 xy) \in B$, and we have $hb = bh$, and so $f_{bh} = f_{hb}$. Then $f_b^h f_h(x) = f_h^b f_b(x)$, that is, $f_b(x) f_h(x) = f_h(x_0) f_b(x)$, which gives

$$f_h(x) = f_b(x)^{-1} f_h(x_0) f_b(x).$$

So $f_h(x)$ is conjugate to $f_h(x_0)$ for all $x \in X \backslash \operatorname{supp}(h)$. Since $f_h \in E$, we have $f_h(x) = 1$ for almost all $x \in X$, and since the support of h is finite, $f_h(x) = 1$ for almost all $x \in X \backslash \operatorname{supp}(h)$. So there exists $x \in X \backslash \operatorname{supp}(h)$ such that $f_h(x) = 1$. Thus $f_h(x_0) = f_h(H) = h\alpha = 1$. Since h was an arbitrary element of H, we get $H\alpha = 1$. Hence α is trivial and so L is trivial.

We note that Theorem 2.2 is true for $B = {}^{\alpha}S_X = \{s \in S_X : |\operatorname{supp}(s)| < \alpha\}$ where S_X is the symmetric group on X and α is any cardinal number with $\aleph_0 < \alpha < |X|$. In fact it is true for any transitive group B acting on X such that the centraliser of each element of H intersects infinitely many right cosets of H in B.

We recall from Theorem 2.1 that if $\operatorname{Hom}(H, A) = 1$ then all the complements of the base group F are conjugate in W. Also that if $L = 1$, then all the complements of E are conjugate in W. In the case of standard wreath product $H = 1$ and so trivially $\operatorname{Hom}(H, A) = 1$. At the other extreme, considering the situation for the monomial groups, if $H = \operatorname{stab} x_0$ for some fixed $x_0 \in X$, then $\operatorname{Hom}(H, A) = 1$ unless A contains elements of order 2, see Scott (1964, pp. 315–318).

When $L = 1$, all the complements of E are conjugate to B in W. We now consider the question of when all the complements of E are conjugate in V.

Let X be a set and B a group acting on X transitively. A subset S of X is called almost invariant with respect to B if Sb and S differ by a finite number of elements, for all $b \in B$. The number of ends of X with respect to B is the least upper bound of the number of infinite almost invariant subsets in partitions of X into almost invariant subsets of X with respect to B (see Houghton, 1972, 1973, 1975; Cohen, 1972).

Suppose $L = 1$ and let C be a complement of E in V. Then C is a complement of F in W, so there exists $g \in F$ such that $C = B^g$. So the complements of E in V are all of the form B^g with $g \in F$ and $B^g \leqslant V$, that is, $g^{-b} g \in E$ for all $b \in B$.

The proof of the following lemma is the same as of Lemma 4.3 of Houghton (1972).

LEMMA 2.3. *Suppose $\{a_i : i \in I\}$ is the set of distinct values taken by $g \in F$. If $X_i = \{x \in X : g(x) = a_i\}$, then $g^{-b} g \in E$, for all $b \in B$, if and only if $\bigcup_{i \in I} X_i b \cap (X \backslash X_i)$ is finite for all $b \in B$.*

THEOREM 2.4. *Every complement of E in V is conjugate to B in V if and only if $L = 1$ and $e(X) = 0$ or 1, where $e(X)$ means the number of ends of X with respect to B.*

PROOF. Let C be a complement of E in V, then as above $C = B^g$ for some $g \in F$. Let $\{X_i: i \in I\}$ be the partition of X corresponding to g as in Lemma 2.3. Now $e(X) = 0$ if and only if X is finite and $V = W$. Suppose $e(X) = 1$. If all X_i are finite then I is infinite and taking infinite unions of X_i's we can partition X into two infinite almost invariant subsets. So there exists i such that X_i is infinite and $X_i' = X \setminus X_i$ is finite. Let $k \in F$ be defined by $k(X) = a_i$. Then

$$k^{-1}g(x) = k(x)^{-1}g(x) = 1$$

unless $x \in X_i'$, which is finite. So $k^{-1}g \in E$ and $C = B^g = B^{kk^{-1}g} = B^{k^{-1}g}$.

Conversely if $e(X) > 1$, then there exists an infinite almost invariant subset S of X such that $S' = X \setminus S$ is infinite. Let $g \in F$ be such that $g(S) = a$, $g(S') = 1$. Then B^g is a complement of E in V. But if $B^g = B^k$ for some $k \in E$, then $B^{gk^{-1}} = B$, that is $(gk^{-1})^b = gk^{-1}$ for all $b \in B$, that is, $gk^{-1} \in D$. This gives $g \in DE$ which is a contradiction. So B^g is not conjugate to B in V.

Finally in this section we note that in the case where $B(\bar{W}) \neq \mathrm{Aut}_{\bar{F}}(\bar{W})$, then for $\gamma \in M = \{\alpha \in B(\bar{W}): b^\alpha \in b\bar{F}$ for all $b \in B\}$, $\gamma|_B = \sigma$ is a section. This belongs to a conjugacy class of sections, all of which arise as the restrictions of γi_g to B, where g runs over F. All elements of $\mathrm{Hom}(H, A)$ correspond to sections, but not all elements of $\mathrm{Hom}(H, A)$ necessarily correspond to automorphisms.

If $b \to bf_g$ is a section with $f_b \in Z(\bar{F})$ for all $b \in B$, then defining γ by: $(bf)^\gamma = bf_b f$, for all $b \in B$, $f \in \bar{F}$, gives an automorphism of $\bar{W}$ of the above type. Thus if $Z(A)$ is the centre of A, then all the elements of $\mathrm{Hom}(H, Z(A))$ correspond to automorphisms of $\bar{W}$. In general each automorphism $\gamma \in M$ corresponds to a homomorphism $\alpha \in \mathrm{Hom}(H, A)$, and the set of all those automorphisms $\gamma \in M$ which correspond to the trivial homomorphism form a subgroup (KI_1) of M. However, this may not be normal in M.

As we mentioned at the beginning, not all the automorphisms $\beta \in \mathrm{Aut}(B)$ extend to automorphisms of the general PWP $\bar{W} = A \,\mathrm{Wr}\, B$ or $A \,\mathrm{wr}\, B$. Let $B^* = \{\beta^* \in \mathrm{Aut}(\bar{W}): \beta^*|_B = \beta \in \mathrm{Aut}(B)\}$, that is, B^* is isomorphic to the subgroup of $\mathrm{Aut}(B)$ whose elements extend to automorphisms of $\bar{W}$. We note that $B^* \leqslant \mathrm{Aut}(\bar{W})$. We now consider the Subgroups B^*, I_1 and K of $\mathrm{Aut}(\bar{W})$.

3. The subgroups B^* and I_1

The subgroup B^*

Let J be the subgroup of $\mathrm{Aut}(B)$ consisting of those automorphisms of B induced by automorphisms of $B(\bar{W})$. In general, B^* is not isomorphic to J;

however if $B^* \cong J$, then $B(\bar{W}) = KI_1 B^*$. In the standard wreath product, this is the case. In fact $J \cong B^* \cong \mathrm{Aut}(B)$, and $\mathrm{Aut}(\bar{W}) = KI_1 B^*$. The following is well known (see Wielandt, 1959/60).

THEOREM 3.1. *Let $|X| \neq 6$ and suppose $A_X \leqslant B \leqslant S_X$. Then every automorphism of B is induced by an inner automorphism of S_X, that is, if $\delta \in \mathrm{Aut}(B)$, then there exists an element $s \in S_X$ such that $b^\delta = b^s$ for all $b \in B$.*

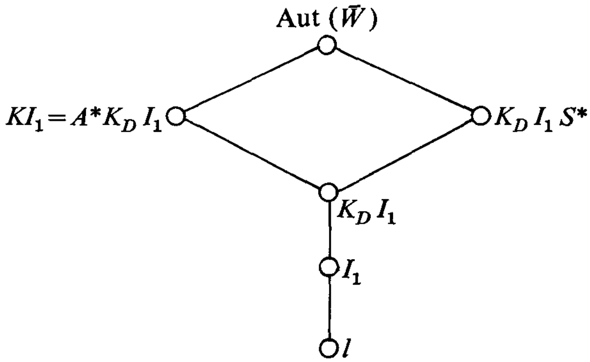
Thus if $|X| \neq 6$, we immediately have the following.

COROLLARY 3.2. *If $A_X \leqslant B$, then $B^* \cong \mathrm{Aut}(B)$.*

PROOF. Let $\beta \in \mathrm{Aut}(B)$, then as in Theorem 3.1, there exists an element $s \in S_X$ such that $b^\beta = b^s$ for all $b \in B$. Then define β^*: $\bar{W} \to \bar{W}$ by: $(bf)^{\beta^*} = b^s f^s$ for all $bf \in \bar{W}$, where $f^s(x) = f(x^{s^{-1}})$ as usual. We have $f^s \in \bar{F}$ (s being a permutation of X), and so β^* is well defined. It follows easily that $\beta^* \in \mathrm{Aut}(\bar{W})$.

Let $x_0 \in X$ and $H = \mathrm{stab}\, x_0$, then for monomial groups we have:

THEOREM 3.3. *If $|X| \neq 2$ or 6 and if A is any group with $\mathrm{Hom}(H, A)$, (or L in the restricted cases) trivial then $\mathrm{Aut}(\bar{W}) = KI_1 S^*$ is a split extension of KI_1 by S^*, where K and I_1 are as before and $S^* \cong \mathrm{Aut}(S_X) \cong S_X$. Also in the unrestricted case we have:*



where $A^* \cong \mathrm{Aut}(A)$ and $K_D = \{\gamma \in K : f^\gamma = f, \text{ for all } f \in D\}$.

Considering B^* in general, we have:

THEOREM 3.4. *Given $\beta \in \mathrm{Aut}(B)$, then β is induced by an automorphism of $\bar{W}$, if there exists $\alpha \in \mathrm{Aut}(\bar{F})$ such that $(f^b)^\alpha = (f^\alpha)^{b^\beta}$ for all $f \in \bar{F}$ and $b \in B$ (*).*

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Conversely, if A is abelian and $\beta \in \mathrm{Aut}(B)$ is induced by an automorphism $\beta^* \in \mathrm{Aut}_{\bar{F}}(\bar{W})$, then there exists $\alpha \in \mathrm{Aut}(\bar{F})$ such that the relation (*) holds.

PROOF. Suppose there exists $\alpha \in \mathrm{Aut}(\bar{F})$ with (*) holding. Then define $\beta^* : \bar{W} \to \bar{W}$ by: $(bf)^{\beta^*} = b^\beta f^\alpha$ for all $bf \in \bar{W}$. Then β^* is easily seen to be a well-defined automorphism of $\bar{W}$.

Conversely, suppose A is abelian and $\beta \in \mathrm{Aut}(B)$ is induced by $\beta^* \in \mathrm{Aut}_{\bar{F}}(\bar{W})$. Then $(bf)^{\beta^*} = b^\beta f_{b\beta} f^\alpha$, for all $f \in \bar{F}$, $b \in B$, where $f_{b\beta} \in \bar{F}$. $\beta^* \in \mathrm{Aut}_{\bar{F}}(\bar{W})$ and so $\beta^*|_B = \alpha \in \mathrm{Aut}(\bar{F})$. Also for $f \in \bar{F}$ and $b \in B$,

$$(f^b)^\alpha = (f^b)^{\beta^*} = (b^{-1} f b)^{\beta^*} = f_{b\beta}^{-1} (f^\alpha)^{b\beta} f_{b\beta} = (f^\alpha)^{b\beta}$$

since A and therefore $\bar{F}$ is abelian. So (*) holds and the proof is complete.

COROLLARY 3.5. *If A is abelian, then B^* is isomorphic to J and hence $B(\bar{W}) = KI_1 B^*$*

From the proof of Theorem 3.4 we have:

COROLLARY 3.6. *An element $\beta \in \mathrm{Aut}(B)$ extends to an automorphism $\beta^* \in \mathrm{Aut}_{\bar{F}}(\bar{W})$ if and only if there exists $\alpha \in \mathrm{Aut}(\bar{F})$ such that*

$$(f^b)^{\beta^*} = (f^b)^\alpha = (f^\alpha)^{b\beta} \quad \textit{for all } f \in \bar{F},\ b \in B.$$

The following generalizes Lemma 1.4 of Bunt (1968). We recall that we are only considering transitive action of B.

THEROEM 3.7. *Let $\beta \in \mathrm{Aut}(B)$ and $H = \mathrm{stab}\, x_0$. If $H^\beta = H^b$ for some $b \in B$, then β extends to $\beta^* \in \mathrm{Aut}(\bar{W})$.*

PROOF. If $y = x_0 b$, then $H^\beta = H^b = \mathrm{stab}\, y$. Define a map $\sigma: X \to X$ by: $x\sigma = x_0 c^{\beta^{-1}}$, where $x = yc \in X$ for some $c \in B$. It is not difficult to verify that σ is a well-defined permutation on X. Also $\beta^*: \bar{W} \to \bar{W}$ defined by: $(bf)^{\beta^*} = b^\beta f^{\beta^*}$ for all $bf \in \bar{W}$, where $f^{\beta^*}(x) = f(x\sigma)$ for all $x \in X$, is a well-defined automorphism of $\bar{W}$ which is an extension of β.

The subgroup I_1

Firstly note that as in the standard wreath product, we have $Z(\bar{W}) = Z(D)$, where, as before, D is the diagonal subgroup of $\bar{F}$ and $Z(G)$ means the centre of G.

(i) Define a map $\varphi: F \to I_1$ by: $g\varphi = i_g$ for all $g \in F$. This gives an epimorphism with kernel $Z(D)$. So in the unrestricted case, the subgroup $I_1 = \{i_g : g \in F\}$ is isomorphic to $F/Z(D)$.

(ii) In $V = A \operatorname{wr} B = BE$ (suppose X is infinite). $I_1 = \{i_g : g \in F$ and $g^{-b} g \in E$ for all $b \in B\}$. By Lemma 2.3, I_1 is determined by the number of ends of X with respect to B. In particular, let $e(X) = 1$ and suppose $i_g \in I_1$ for some $g \in F$, then the partition $\{X_i : i \in I\}$ of X corresponding to g has only one infinite component and so g is constant almost everywhere on X. Thus in this case I_1 is isomorphic to $DE/Z(D)$, since $g \in Z(D)$ induces the identity automorphism. This is the case for the monomial group $V = S_X E$. However, for the monomial groups $V = S_{(X)} E$ and $A_X E$, if $g \in F$ then $g^{-s} g \in E$ for all $s \in S_{(X)}$ or A_X and hence in these cases, again $I_1 = \{i_g : g \in F\} \cong F/Z(D)$.

4. The subgroup K

We recall that $K = \{\gamma \in \operatorname{Aut}(\bar{W}) : b^\gamma = b$ for all $b \in B\}$. If $b \in B$, $\gamma \in K$ and $f \in \bar{F}$, then $(f^b)^\gamma = (f^\gamma)^b$. Let $\alpha \in \operatorname{Aut}(\bar{F})$ be such that $(f^\alpha)^b = (f^b)^\alpha$, for all $b \in B$ and $f \in \bar{F}$. Then α can be extended to $\gamma \in K$ by defining $(bf)^\gamma = bf^\alpha$, for all $bf \in \bar{W}$. Hence the subgroup K is isomorphic to the group of those automorphisms of the base group $\bar{F}$ which commute with the inner automorphisms induced by elements of B. We note that A^* is always a subgroup of K.

We shall consider the structure of K in the case of the restricted PWP. Let $x_0 \in X$ be a fixed point and $H = \operatorname{stab} x_0$ as usual. Let

$$A_1 = \{f \in E : f(x) = 1 \text{ for all } x \neq x_0\}.$$

If $f \in A_1$ and $h \in H$, then $f^h(x_0) = f(x_0 h^{-1}) = f(x_0)$ and $f^h(x) = f(xh^{-1}) = 1$ for $x \neq x_0$, since then $xh^{-1} \neq x_0$. So we have $f^h = f$ for all $f \in A_1$ and $h \in H$.

Since $E = \langle f^b : f \in A_1,\ b \in B \rangle$, we can express $g \in F$ by $g = f_1^{b_1} f_2^{b_2} \ldots f_n^{b_n}$, where $f_1, f_2, \ldots, f_n \in A_1$ and $b_1, b_2, \ldots, b_n$ are in distinct cosets of H in B. For $\gamma \in K$, we have

$$g^\gamma = (f_1^{b_1})^\gamma (f_2^{b_2})^\gamma \ldots (f_n^{b_n})^\gamma = (f_1^\gamma)^{b_1} (f_2^\gamma)^{b_2} \ldots (f_n^\gamma)^{b_n}.$$

Thus γ is determined by its effect on A_1, and so to determine γ we only have to consider the image of elements of A_1 under γ.

Lemma 4.1. *For $f \in A_1$ and $\gamma \in K$, f^γ is constant on all the H-orbits of X.*

Proof. For $h \in H$, $f \in A_1$, we have $f^h = f$, as explained above. Take $\gamma \in K$, then $f^\gamma = (f^h)^\gamma = (f^\gamma)^h$ for all $h \in H$, and so $f^\gamma(x) = f^\gamma(xh^{-1})$ for all $h \in H$, hence the result.

Segal (1973, Chapter 4) has investigated the structure of the subgroup K for the restricted standard wreath product. Here we have a more general situation. In his case the subgroup $H (= \operatorname{stab} x_0)$ of B is trivial, and so there are no orbits involved.

As we noted earlier, each $\gamma \in K$ is determined by its effect on A_1 and for $f \in A_1$, f^γ is constant on all the H-orbits of X. Also, since we are considering the restricted PWP, f^γ takes value 1 on all the infinite H-orbits of X, for all $f \in A_1$. Let T be a transversal of the finite H-orbits and let, End(A) be the set of all endomorphisms of A.

For each $x \in X$ define a map $e_x: A \to A$ by: $(f(x_0))e_x = f^\gamma(x)$ for all $f \in A_1$. It is clear that $e_x \in \mathrm{End}(A)$ and γ is determined uniquely by $\tilde{\gamma} = (e_x)_{x \in T}$.

Routine arguments yield the following results.

LEMMA 4.2. *Let $\gamma \in K$ and $\tilde{\gamma} = (e_x)_{x \in T}$.*
(I) *If $a \in A$, then $ae_x = 1$ for all but a finite number of $x \in T$.*
(II) *If $x, y \in T$, $x \neq y$, then $[Ae_x, Ae_y] = 1$.*
(III) *If $x \in T$ and the B-stabilizer of x is not contained in H, then Ae_x is abelian.*

LEMMA 4.3. *Let $(e_x)_{x \in T}$ satisfy* (I), (II) *and* (III) *of Lemma* 4.2. *Then $(e_x)_{x \in T} = \tilde{\gamma}$ for some $\gamma \in \mathrm{End}(E)$ with $(f^b)^\gamma = (f^\gamma)^b$ for all $f \in E$ and $b \in B$.*

If the elements $(e_x)_{x \in T}$ are such that $(e_x)_{x \in T} = \tilde{\gamma}$ and γ is invertible, then $\gamma \in K$. Hence K corresponds to all elements $(e_x)_{x \in T}$ which satisfy (I), (II) *and* (III) *of lemma* 4.2 *and such that γ is invertible, where $(e_x)_{x \in T} = \tilde{\gamma}$.*

COROLLARY 4.4. *If all the H-orbits of X are infinite except $\bar{x}_0$ which is $\{x_0\}$, then $K \cong \mathrm{Aut}(A)$. Taking X infinite, this is the case for monomial groups.*

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