Can. J. Math., Vol. XXVII, No. 6, 1975, pp. 1185–1210

ON NILPOTENT PRODUCTS OF CYCLIC GROUPS—REEXAMINED BY THE COMMUTATOR CALCULUS

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1. Introduction. Ruth R. Struik investigated the nilpotent group $\bar{G}^n = G/G_{n+1}$ in [**11**; **12**], where G is a free product of a finite number of cyclic groups, not all of which are of infinite order, and G_m is the mth subgroup of the lower central series of G. Making use of the "collection process" first given by Philip Hall in [**8**], she determined $\bar{G}^n$ completely for $1 \leqq n \leqq p + 1$, where p is the smallest prime with the property that it divides the order of at least one of the free factors of G. However, she was unable to proceed beyond $n = p + 1$.

Rex S. Dark [**2**] found all $\bar{G}^n$ when the free factors have order p, a fixed prime. Anthony M. Gaglione [**3**] did so when these orders are p or ∞. But general results are not known yet. This paper aims to overcome in principle the obstacles which Struik encountered by giving a procedure, valid for n arbitrarily large, which expresses the elements of $\bar{G}^n$ uniquely by basic commutators. We shall call this procedure the "representation algorithm."

We will conclude this paper with an example in which we determine $\bar{G}^5$ for $G = \langle a, b; a^9, b^9 \rangle$. (Note that Struik could only find $\bar{G}^4$.)

We hope that general results obtained from the "representation algorithm" will be given in a future publication.

The "representation algorithm" is based on known methods of the commutator calculus. To describe it we will present a preliminary discussion of the commutator calculus. The notation and definitions in this discussion originate to a great extent from the listed references.

2. Preliminaries from the commutator calculus.

2(a). *The lower central series. Basic commutators.* Let G be a group. Let $a, b \in G$. Then the commutator

$$(2.1) \qquad A = (a, b) = a^{-1}b^{-1}ab.$$

We will write $a = A^L$ and $b = A^R$. Also the lower central series

$$(2.2) \qquad G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n \supseteq \ldots$$

is the sequence of subgroups given by

Definition 2.1. $G_1 = G$. G_n is generated by all commutators (a, b_{n-1}), where $a \in G$ and $b_{n-1} \in G_{n-1}$. In particular G_2 is called the commutator subgroup.

Received September 13, 1972 and in revised form, September 15, 1975.

https://doi.org/10.4153/CJM-1975-125-9 Published online by Cambridge University Press

We say that the element $c \neq 1$ has weight $n = W(c)$ if $c \in G_n$ but $c \notin G_{n+1}$. It is evident that $a \in G_n$, implies that if $W(a)$ is defined, then $W(a) \geqq n$.

The following properties of the lower central series are well-known [**7**; **10**; **16**; **17**]:

If $(a, b) \neq 1$ and $W((a, b))$ is defined, then

(2.3a) $\quad W((a, b)) \geqq W(a) + W(b)$

If $W(a_i) = w_1$, $W(b_j) = w_2$, then

(2.3b) $$\left(\prod_{i=1}^{I} a_i^{\alpha_i}, \prod_{j=1}^{J} b_j^{\beta_j}\right) \equiv \prod_{i=1}^{I} \prod_{j=1}^{J} [(a_i, b_j)]^{\alpha_i \beta_j} \bmod G_{w_1+w_2+1}$$

If $a \equiv c \bmod G_{W(a)+1}$, $b \equiv d \bmod G_{W(b)+1}$, then

(2.3c) $\quad (a, b) \equiv (c, d) \bmod G_{W(a)+W(b)+1}$.

The Jacobi identity

(2.3d) $\quad ((a, b), c)((b, c), a)((c, a), b) \equiv 1 \bmod G_{W(a)+W(b)+W(c)+1}$.

We proceed to define basic commutators according to the natural linear ordering given in [**14**]. We will need the properties of this ordering in our investigation of a group, G, which is the homorphic image of a free group, F, of finite rank, r. To distinguish between F and G, we shall call the weight in F of an element $a \in F$, its dimension and denote it by $D(a)$; we will reserve the phrase weight of b and the notation $W(b)$ for the weight in G of the element $b \in G$. ($G = F$ is a special case where dimension and weight have the same meaning.)

Definition 2.2. The basic commutators of dimension one are the free generators of the free group F in the order

(2.4) $\quad c_1 < c_2 < \ldots < c_r$.

(The word dimension is used here with the previous meaning according to a remark at the end of this definition.) Having defined and ordered basic commutators of dimension less than m, we use them to define and order basic commutators of dimension m. The basic commutators of dimension m are $c_k = (c_i, c_j)$ where c_i and c_j are basic commutators such that

(i) $D(c_i) + D(c_j) = m$,

(ii) $c_i > c_j$,

(iii) if $c_i = (c_s, c_t)$, then $c_j \geqq c_t$.

Let $c_{k_1} = (c_{i_1}, c_{j_1})$ and $c_{k_2} = (c_{i_2}, c_{j_2})$ such that $D(c_{k_1}) = D(c_{k_2})$. Then $c_{k_1} > c_{k_2}$ if $c_{i_1} > c_{i_2}$, or $c_{i_1} = c_{i_2}$ but $c_{j_1} > c_{j_2}$. A basic commutator of dimension m is greater than any of smaller dimension. Having ordered all basic commutators, we assume that their subscripts are chosen so that c_i is the ith basic commutator. (In this definition we are using the word dimension according to its general meaning since a basic commutator of dimension m is in F_m, but not in F_{m+1} [**7**; **10**].)

To proceed we introduce an auxiliary definition.

Definition 2.3. Let G have the presentation

(2.5) $$G = \langle c_1, c_2, \ldots, c_r; s_1, s_2, \ldots, s_t \rangle.$$

(Then G is the factor group F/N, where N is the normal closure of the subgroup of F generated by the words $s_1, s_2, \ldots, s_t$. In particular when $t = 1$ and $s_1 = 1$, then $G = F$.) Let the basic commutator c_m be the element of F of Definition 2.2, as well as its image in G under the homomorphism $F \rightarrow G = F/N$; we shall, however, always mean by the dimension of $c_m[= D(c_m)]$ the number of Definition 2.2. The element $a \in G_w$ is said to be basic commutator representable (b.c.-representable) if

(2.6) $$a \equiv c_{i_1}^{\epsilon_1} c_{i_2}^{\epsilon_2} \ldots c_{i_k}^{\epsilon_h} \bmod G_{w+1}$$

such that: (i) the c_{i_σ} are elements of G as well as basic commutators of dimension w, (ii) $c_{i_1} < \ldots < c_{i_h}$ if $h > 1$, and (iii) $\epsilon_1, \epsilon_2, \ldots, \epsilon_h$ are non-zero exponents. The product on the right-hand side of (2.6) will be called a basic commutator representation (or b.c.-representation).

Before going further we note an important inequality which is obvious from Definition 2.1. If $a \in F$ and $\bar{a}$ is its image under the homomorphism $F \rightarrow G = F/N$, then

(2.7) $$D(a) \leqq W(\bar{a})$$

when $W(\bar{a})$ is defined.

The name basic commutator is appropriate in the sense of the following well-known Theorem [**7**].

THEOREM 2.1. *Every group F_{m+1} is a normal subgroup of F_k where $1 \leqq k \leqq m$, and every factor group $\bar{F}_m = F_m/F_{m+1}$ is a free abelian group. The basic commutators of dimension $m (m \geqq 1)$ are mapped into a basis of $\bar{F}_m$ (under the homomorphism $F_m \rightarrow \bar{F}_m = F_m/F_{m+1}$) such that every element $a \in F$, which $\neq 1$, has a unique dimension and a unique b.c.-representation. If a, b are distinct basic commutators, then $D((a, b)) = D(a) + D(b)$. Moreover, F_m is the normal closure in F of that subgroup which is generated by the basic commutators of dimension m.*

By Definition 2.1 we have the following corollary for the group G presented in (2.5).

COROLLARY 2.1. *Every group G_{m+1} is a normal subgroup of G_k where $1 \leqq k \leqq m$, and every factor group $\bar{G}_m = G_m/G_{m+1}$ is an abelian group. The basic commutators of dimension m are mapped into generators of $\bar{G}_m$ (under the homomorphism $G_m \rightarrow \bar{G}_m = G_m/G_{m+1}$) such that every element a of weight $m > 0$ is b.c.-representable. Moreover, G_m is the normal closure in G of that subgroup which is generated by the basic commutators of dimension m.*

https://doi.org/10.4153/CJM-1975-125-9 Published online by Cambridge University Press

To compute a b.c.-representation of a group element, we make use of the well-known "collection process" [**7**; **8**] which is discussed in Subsection 2(b).

For the above properties of basic commutators, our natural linear ordering is not required [**7**; **10**]. It is, however, order preserving under commutation [**14**]. Before stating this result we give some preliminary definitions.

Definition 2.4. Let $a \in F$ and have dimension $n > 0$. The maximal component of a, $M(a)$, is the largest commutator in the b.c.-representation (2.6), i.e. $M(a) = c_{i_h}$.

Definition 2.5. Let $a, b \in F$. The inequalities $a > b$ and $a \geqq b$ will mean that $M(a) > M(b)$ and $M(a) \geqq M(b)$, respectively. We will also write $a \approx b$ and $a \not\approx b$ to stand for $M(a) = M(b)$ and $M(a) \neq M(b)$, respectively.

The following result [**14**] is of importance in this paper.

THEOREM 2.2. *Let the elements $a, b, c \in F$ be basic commutators such that $a > b$, $a \neq c$, and $b \neq c$. Then $(a, c) > (b, c)$.*

It is evident from Theorem 2.1 and Equations (2.3) that Theorem 2.2 has the alternate, more general formulation:

Let $a, b, c, \in F$, such that $a > b$, $c \neq 1$, $a \not\approx c$, $b \not\approx c$. Then $(a, c) > (b, c)$.

To apply Theorem 2.2 we shall need more machinery. We shall introduce for every basic commutator c its "regular sequence", $[c]$, i.e.,

$$(2.8) \qquad [c] = [d_1, d_2, \ldots, d_h].$$

Definition 2.6. The sequence on the right-hand side of (2.8) consists of c only when $D(c) = 1$. Having defined the regular sequences of all basic commutators of dimension $< n$, we define $[c]$ for $D(c) = n$. The sequence $[c] = [e_1, e_2, \ldots, e_q, c^R]$, where $[c^L] = [e_1, e_2, \ldots, e_q]$.

At this point we are ready to conclude Subsection 2(a) with an important lemma first given in [**16**].

LEMMA 2.1. *Let C and c be basic commutators such that* (i) $D(C) > 1$, (ii) $C > c$, (iii) $[C] = [d_1, d_2, \ldots, d_h]$. *Then $[M(C, c)] = [d_1, e_1, e_2, \ldots, e_h]$, such that $e_1 \leqq e_2 \leqq \ldots \leqq e_h$ is a rearrangement of $d_2, d_3, \ldots, d_h, c$.*

2(b). *The collection process.* The "collection process" was first given by Philip Hall [**8**]. Its use to represent group elements by basic commutators is discussed in [**7**]. We shall now generalize this discussion for application of the "collection process" to our "representation algorithm."

The "collection process" is based on (2.1) and the well-known identities [**7**; **10**]

$$\textbf{(2.9a)} \qquad (ab, c) = (a, c)((a, c), b)(b, c)$$

$$\textbf{(2.9b)} \qquad (a, bc) = (a, c)(a, b)((a, b), c).$$

Let

$$P_1(a, b) = (a, b) \qquad (2.10)$$
$$P_{\mu+1}(a, b) = ([P_\mu(a, b)], b) \text{ for } \mu = 1, 2, \ldots$$

Let m be a positive integer. The identities

$$ba^{-1} = a^{-1}b\left[\prod_{k=1}^{m} P_{2k}(b, a)\right]([P_{2m}(b, a)], a^{-1})\left[\prod_{k=1}^{m} P_{2k-1}(b, a)\right]^{-1} \qquad (2.11a)$$

$$b^{-1}a = a(b, a)^{-1}b^{-1} \qquad (2.11b)$$

$$b^{-1}a^{-1} = a^{-1}\left[\prod_{k=1}^{m} P_{2k-1}(b, a)\right]([P_{2m}(b, a)], a^{-1})^{-1} \times \left[\prod_{k=1}^{m} P_{2k}(b, a)\right]^{-1} b^{-1} \qquad (2.11c)$$

are easy consequences of (2.1), (2.9a), and (2.9b). (For details see section 11.1 of [7]). To proceed from these identities we now require the following three definitions:

Definition 2.7. The basic commutator c is a F-simple commutator if either (i) $D(c) = 1$, or (ii) $D(c) > 1$, but $D(c^R) = 1$.

Definition 2.8. Let N be a positive integer. Let $c_1, c_2, \ldots, c_{q(N)}$ be the basic commutators of dimension $\leqq N$. We shall call any element

$$\prod = \prod_{i=1}^{I} c_i^{\epsilon_i} \qquad (2.12)$$

a collected-I-word, where $1 \leqq I \leqq q(N)$. If $I = q(N)$, then (2.12) is said to be a basic N-word. If $w \in F$ and

$$w \equiv \left(\prod_{i=1}^{q(N)} c_i^{\epsilon_i}\right) \bmod F_{N+1} \qquad (2.13)$$

then the basic N-word, $\prod = \prod_{i=1}^{q(N)} c_i^{\epsilon_i}$ is referred to as the N-composite basic commutator representation (or N-*c.b.c.* representation) of w.

Definition 2.9. The generators $c_1, c_2, \ldots, c_r$ of F and their inverses c_1^{-1}, $c_2^{-1}, \ldots, c_r^{-1}$ are the 1-commutators. Suppose that we have defined the k-commutators for $1 \leqq k \leqq m$. An $(m + 1)$-commutator is any element $w = (u, v)$ such that u is a s-commutator, v is a t-commutator, and $s + t = m + 1$. (Note that a k-commutator $\in F_k$ by Definition 2.1 and inequality (2.3a).)

Let $w \in F$ and $w \neq 1$, i.e.,

$$w = \prod_{j=1}^{J} c_{k_j}^{\eta_j} \qquad (2.14)$$

where the c_{k_j} are among the generators $c_1, c_2, \ldots, c_r$. Let N be a given positive integer. Let $1 \leqq I \leqq q(N)$. Making repeated use of (2.1) and of the identities

(2.9) and (2.11), we find that w has the form

$$(2.15) \qquad w = \left(\prod_{i=1}^{I} c_i^{\epsilon_i}\right) f_I g_{N+1,\, I}$$

such that (i), (ii), and (iii) hold:

(i) If $I = q(N)$, then $f_I = 1$. If $I < q(N)$, then f_I is a word in basic commutators, c_k, with two properties:

(a) $c_I < c_k \leqq c_{q(N)}$

(b) If $D(c_k) > 1$, then $c_k^R \leqq c_I$.

(In particular by Definition 2.7, f_r is a word in F-simple commutators of dimension > 1.)

(ii) $g_{N+1,I}$ is a word in finitely many m-commutators so that each $m > N$.

(iii) If $I < q(N)$, then

$$(2.16) \qquad c_{I+1}^{\epsilon_{I+1}} f_{I+1} g_{N+1,\, I+1} = f_I g_{N+1,\, I}.$$

Having obtained the collected-I-word $(\prod_{i=1}^{I} c_i^{\epsilon_i})$ we thus find the collected-$(I+1)$-word $(\prod_{i=1}^{I+1} c_i^{\epsilon_i})$ by a rewriting of f_I.

We will refer to the computation of ϵ_I as the "collection of c_I." (See [7, Section 11.1.]) We note that (2.15) gives a N-*c.b.c.* representation of w for $I = q(N)$.

A generalization of the above computation of the "collected-I-words," $\prod_{i=1}^{I} c_i^{\epsilon_i}$, is required to arrive at the "representation algorithm." We will show later on that every F-simple commutator of dimension > 1 is a word in "auxiliary-simple" commutators, not all of which are F-simple. We will thus express f_r (see property (b) above) by "auxiliary-simple" commutators. For $N > 1$ we will compute in our generalization an "N-collected-auxiliary-commutator representation" of f_r instead of its "N-collected basic commutator representation." To describe the generalization in detail we need four additional definitions.

Definition 2.10. Let $c_{i_1} < c_{i_2} < \ldots < c_{i_k} < \ldots$ be the F-simple commutators of dimension >1 in the ordering of Definition 2.2. An auxiliary-simple commutator of class $\{k\}$ is a commutator, w, with the property that $w \approx c_{i_k}$. All of the classes $\{1\}, \{2\}, \ldots, \{k\}, \ldots$ are non-empty and consist of finitely many distinct elements given by a specified rule. (See the remark at the end of this definition.) The auxiliary-simple commutators are ordered as follows:

(i) If $d_\lambda \in \{k_1\}$, $d_\mu \in \{k_2\}$, and $k_1 < k_2$, then $d_\lambda <_a d_\mu$.

(ii) The distinct elements of a class $\{k\}$ are ordered in a specified manner.

(Note that we will use $<_a$, $\leqq_a$, $>_a$, $\geqq_a$ for the ordering of Definitions 2.10 and 2.11 to distinguish it from the ordering of Definition 2.2. Also note that the elements of the classes $\{k\}$ and their orderings will be specified in Definition 2.21. But specific rules are not needed in the present discussion of the collection process.)

Definition 2.11. Suppose that d is an auxiliary-simple commutator of class

$\{k\}$. Then d has pseudo-dimension $D_p(d) = D(c_{i_k})$. The auxiliary commutators of pseudo-dimension 2 are the auxiliary-simple commutators of pseudo-dimension 2; these commutators are ordered according to Definition 2.10. Having defined and ordered the auxiliary commutators of pseudo-dimension $< m$ but > 1, we use them to define and order the ones of pseudo-dimension m. The set of auxiliary commutators of pseudo-dimension m consists of two subsets:

I. The auxiliary-simple commutators of pseudo-dimension m in the ordering of Definition 2.10.

II. The commutators $d_\zeta = (d_\xi, d_\eta)$ such that d_ξ and d_η are auxiliary commutators with the following three properties:

(i) $D_p(d_\xi) + D_p(d_\eta) = m$

(ii) $d_\xi >_a d_\eta$ and $D_p(d_\xi) \geqq D_p(d_\eta) > 1$

(iii) If d_ξ is not auxiliary-simple, then $d_\xi = (d_\alpha, d_\beta)$ and $d_\beta \leqq_a d_\eta$. (Here $D_p(d) = m$ means that d has pseudo-dimension m.)

An auxiliary commutator of pseudo-dimension m is $>_a$ any of smaller pseudo-dimension. An auxiliary-simple commutator of pseudo-dimension m is $>_a$ any non-simple-auxiliary commutator of the same pseudo-dimension. Let $d_{\zeta_1} = (d_{\xi_1}, d_{\eta_1})$ and $d_{\zeta_2} = (d_{\xi_2}, d_{\eta_2})$ be two non-simple-auxiliary commutators of pseudo-dimension m. Then $d_{\zeta_1} >_a d_{\zeta_2}$ if either $d_{\xi_1} >_a d_{\xi_2}$, or $d_{\xi_1} = d_{\xi_2}$ but $d_{\eta_1} >_a d_{\eta_2}$.

Having ordered all auxiliary commutators, we assume that their subscripts are chosen so that d_i is the ith auxiliary commutator.

(We note that the auxiliary commutators need not be basic commutators.)

Remark 2.1. We note by Definitions 2.1, 2.10, 2.11 and by inequality (2.3a) that $d_i \in F_m$ if $D_p(d_i) = m$.

Definition 2.12. Let $d_1, d_2, \ldots, d_{q_a(m)}$ be the auxiliary commutators of pseudo-dimension $\leqq m$ but > 1. Let $q_a(1) = 0$. If $f \in F$, $D(f) = m$,

$$\prod\nolimits_1 = \prod_{i=q_a(m-1)+1}^{q_a(m)} d_i^{\epsilon_i} \tag{2.17}$$

and

$$f \equiv \prod\nolimits_1 \bmod F_{m+1} \tag{2.18}$$

then $\prod_1$ is said to be an auxiliary commutator representation (or *a.c.* representation) of f. We shall call any element

$$\prod = \prod_{i=1}^{I} d_i^{\eta_i} \tag{2.19}$$

a collected-I_a-word, where $1 \leqq I \leqq q_a(m)$. If $I = q_a(m)$, then (2.19) is said to be an auxiliary m-word. If $w \in F_2$ and

$$w \equiv \left(\prod_{i=1}^{q_a(m)} d_i^{\eta_i}\right) \bmod F_{m+1} \tag{2.20}$$

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then the auxiliary m-word, $\prod_{i=1}^{q_a(m)} d_i^{\eta_i}$, is referred to as the m-composite auxiliary commutator representation (or m-*c.a.c.* representation) of w.

Definition 2.13. The auxiliary commutators $d_1, d_2, \ldots, d_{q_a(2)}$ of pseudo-dimension 2 and their inverses $d_1,^{-1} d_2^{-1}, \ldots, d_{q_a(2)}^{-1}$ are the auxiliary 2-commutators. Suppose that we have defined the auxiliary k-commutators for $2 \leqq k \leqq m$. The auxiliary $(m + 1)$-commutators are the elements of two categories:

(I) The auxiliary-simple commutators of pseudo-dimension $(m + 1)$ together with their inverses.

(II) All commutators $w = (u, v)$ such that u is an auxiliary s-commutator, v is an auxiliary t-commutator, s and $t \geqq 2$, and finally $s + t = m + 1$.

(Note that an auxiliary k-commutator $\in F_k$ by Definitions 2.1, 2.10, 2.13 and by inequality (2.3a).)

Having given Definitions 2.10-2.13, we are now ready to describe our generalization of the collection process in which we will work with auxiliary commutators just as we worked with basic commutators before. Let f be any word in auxiliary-simple commutators, $f \neq 1$. Let N be a given integer $\geqq 2$. Let $1 \leqq I \leqq q_a(N)$. Making repeated use of (2.1) and the identities (2.9) and (2.11), we find that f has the form

$$f = \left(\prod_{i=1}^{I} d_i^{\epsilon_i}\right) f_{I,a}\, g_{N+1,I,a} \tag{2.21}$$

such that (i), (ii) and (iii) hold:

(i) If $I = q_a(N)$, then $f_{I,a} = 1$. If $I < q_a(N)$, then $f_{I,a}$ is a word in auxiliary commutators, d_k, with two properties:

(a) $d_I <_a d_k \leqq_a d_{q_a(N)}$:

(b) If d_k is not auxiliary-simple, then $d_k^R \leqq_a d_I$.

(ii) $g_{N+1,I,a}$ is a word in finitely many auxiliary m-commutators so that each $m > N$.

(iii) If $I < q_a(N)$, then

$$d_{I+1}{}^{\epsilon_{I+1}} f_{I+1,a}\, g_{N+1,I+1,a} = f_{I,a}\, g_{N+1,I,a}. \tag{2.22}$$

Having obtained the collected-I_a-word, $(\prod_{i=1}^{I} d_i^{\epsilon_i})$, we thus find the collected-$(I + 1)_a$-word, $(\prod_{i=1}^{I+1} d_i^{\epsilon_i})$, by a rewriting of $f_{I,a}$.

We will refer to the computation of ϵ_i in (2.21) as the "a-collection of d_i." (This computation is described in Section 11.1 of [7] for basic commutators rather than the auxiliary commutators required here.) We note that (2.21) gives an N-*c.a.c.* representation of f for $I = q_a(N)$. Moreover, the N-*c.a.c.* representation of f becomes identical with the N-*c.b.c.* representation found in (2.15) in the special case where all auxiliary-simple commutators are also F-simple.

2(c). *Free generators of* G_2. The presentations (2.5) of the groups considered here have the form

(2.5a) $G = \langle c_1, c_2, \ldots, c_r; c_1^{\alpha_1}, c_2^{\alpha_2}, \ldots, c_r^{\alpha_r}\rangle$

where the r exponents α_i are nonnegative integers so that at least one among them does not vanish. Their commutator subgroups are known to be free [**4**; **5**] and sets of free generators are given for them below. For this purpose we require additional notation and definitions.

Remark 2.2. From now on $\prod_{i=1}^{I} c_{j_i}^{\eta_i} (j_i \in \{1, 2, \ldots, r\})$ will denote an element of F as well as its image in G under the homomorphism $F \to G$ of (2.5a).

Definition 2.14. (See Definition 2.9.) A 1-commutator $u = c_i^{\pm 1}$ $(i = 1, 2, \ldots, r)$ has generator sequence $\langle u\rangle = \langle c_i\rangle$ consisting of c_i. Suppose (i) e is a s-commutator with generator sequence $\langle e\rangle = \langle e_1, e_2, \ldots, e_s\rangle$ and, (ii) f is a t-commutator with generator sequence $\langle f\rangle = \langle f_1, f_2, \ldots, f_t\rangle$. Then (e, f) is a $(s + t)$-commutator with generator sequence $\langle (e, f)\rangle = \langle e_1, e_2, \ldots, e_s, f_1, f_2, \ldots, f_t\rangle$. (Note that $\langle c\rangle = [c]$ if c is a F-simple commutator.)

Definition 2.15. The generator c_i has order

$$O(c_i) = \text{order of } c_i \text{ in } G = \begin{cases} \alpha_i \text{ if } \alpha_i \neq 0 \\ \infty \text{ if } \alpha_i = 0 \end{cases}$$

Definition 2.16. Let c_i be a F-simple commutator. Then c_i is said to be G-simple if any generator, c_k, which occurs in $\langle c_i\rangle$ does so fewer than $O(c_k)$ times.

Definition 2.17. Let $t > 1$. A commutator

$$e = (\ldots (c_{i_1}^{\epsilon_1}, c_{i_2}^{\epsilon_2}), \ldots, c_{i_t}^{\epsilon_t})$$

is quasi-G-simple provided it has the following four properties:

(a) $c = (\ldots (c_{i_1}, c_{i_2}), \ldots, c_{i_t})$ is G-simple.
(b) The $\epsilon_j = \pm 1$ for $j = 1, 2, \ldots, t$.
(c) If $\epsilon_j = -1$, then $O(c_{i_j}) = \infty$.
(d) If $i_k = i_j$, then $\epsilon_k = \epsilon_j$.

We are ready to state a theorem of Gruenberg [**5**] which is a special case of Theorem 2.1 of [**4**]:

THEOREM 2.3. *The quasi-G-simple commutators are free generators of* G_2.

2(d). *The investigation of G reduced to a special case.* We will show in this subsection that it is sufficient to obtain the "representation algorithm" for the special groups, $G[p]$, of

Definition 2.18. Let p be a fixed prime. Suppose that every non-vanishing α_i in the presentation (2.5a) is a power of p. Then G will be denoted by $G[p]$.

The above conclusion arises in part from a well-known fact stated as

LEMMA 2.2 [7]. *Suppose c generates a cyclic group, C, of order $\alpha = p_1^{\eta_1}p_2^{\eta_2}\ldots p_k^{\eta_k}$, where the p_j are distinct primes and the η_j are positive integers. Then C is the direct product of the cyclic groups generated by*

$$c^{\alpha/p_1^{\eta_1}}, c^{\alpha/p_2^{\eta_2}}, \ldots, c^{\alpha/p_k^{\eta_k}}.$$

It follows from Lemma 2.2 that the group G given by (2.5a) is the free product of special abelian groups and has the alternative presentation

(2.5b) $\quad G = \langle e_1, e_2, \ldots, e_t; s_1, s_2, \ldots, s_u\rangle$

with three properties:

(I) Every e_i is a power of a c_j in (2.5a).

(II) The relators s_i belong to categories (a) and (b):

(a) s_i is the $([p(i)]^{\lambda_i})$ power of a generator e_j, where λ_i is a positive integer and $p(i)$ is a prime.

(b) $s_i = (e_u, e_v)$ such that e_u and e_v are powers of the same c_k.

(III) If the generators e_u and e_v commute, then their orders are relatively prime.

To make use of the representation (2.5b), we require the free group

(2.23) $\quad \mathscr{F} = \langle e_1, e_2, \ldots, e_t\rangle$.

Let $e_1, e_2, \ldots, e_{q(n)}$ be the basic commutators of dimension $\leqq n$ in $\mathscr{F}$, given by Definition 2.2. By Corollary 2.1, $\bar{G}^n = G/G_{n+1}$ consists of the images in $\bar{G}^n$ of the basic n-words

(2.24) $\quad \prod_{i=1}^{q(n)} e_i^{\epsilon_i}$.

(We are applying Definition 2.8 and Remark 2.2 to $\mathscr{F}$ in place of F.) Having given (2.24) we need additional definitions to continue.

Definition 2.19. Let

(2.25) $\quad p_1, p_2, \ldots, p_z$

be the distinct primes $p(i)$ which occur in the relators s_i of category (a) above. Let e be a f-commutator with generator sequence $\langle e\rangle = \langle e_{j_1}, e_{j_2}, \ldots, e_{j_f}\rangle$. (We are applying Definition 2.8 to $\mathscr{F}$ rather than F.) Then

e is said to be of type ∞, if $O(e_{j_u}) = \infty$ for $1 \leqq u \leqq f$;

e is said to be of type v, if it is not of type ∞ and every $O(e_{j_u})$ is either ∞ or a power of p_v, where $1 \leqq u \leqq f$ and p_v is among the primes (2.25) ($O(e_{j_u})$ denotes the order of e_{j_u} in G.);

e is said to be of mixed type, if it is not of one of the types $1, 2, \ldots z, \infty$.

Definition 2.20. $\mathscr{G}_\infty$ is the subgroup of G which is generated by the generators in (2.5b) of type ∞.

Let v be fixed, $1 \leqq v \leqq z$. $\mathscr{G}_v$ is the subgroup generated by the generators in (2.5b) of types v or ∞. Also for m a given positive integer, $\mathscr{G}_{vm}$ is the subgroup generated by those basic commutators, e_i, (in the notation of (2.24)) which have dimension m and are of type v.

Known properties of G which we will combine with (2.24) follow.

LEMMA 2.3. (See [**1**; **11**; and **17**]). *Let e be a f-commutator of mixed type. Let m be any positive integer. Then*

(2.26) $\quad e \equiv 1 \bmod G_m.$

The following result is a special case of Theorem 2.1 of [**3**].

THEOREM 2.4. *Let* $\overline{\mathscr{G}}_{\infty m} = (\mathscr{G}_\infty)_m/(\mathscr{G}_\infty)_{m+1}$. *Let* $\overline{\mathscr{G}}_{vm}(1 \leqq v \leqq z)$ *be the image of* $\mathscr{G}_{vm}$ *under the homomorphism* $\mathscr{G}_v \rightarrow \mathscr{G}_v/(\mathscr{G}_v)_{m+1} = \overline{(\mathscr{G}_v)^m}$. *Then* G_m/G_{m+1} *is the direct product of the groups*

$$\begin{cases} \overline{\mathscr{G}}_{\infty m}, \overline{\mathscr{G}}_{1m}, \overline{\mathscr{G}}_{2m}, \ldots, \overline{\mathscr{G}}_{zm} \text{ if } \mathscr{G}_\infty \text{ is non-empty.} \\ \overline{\mathscr{G}}_{1m}, \overline{\mathscr{G}}_{2m}, \ldots, \overline{\mathscr{G}}_{zm} \text{ if } \mathscr{G}_\infty \text{ is empty.} \end{cases}$$

At this point the words (2.24) can be examined. Let us divide the set of these words into equivalence classes according to the relation that

(2.27) $$\prod_1 = \prod_{i=1}^{q(n)} e^{\epsilon_{i1}} \sim \prod_2 = \prod_{i=1}^{q(n)} e_i^{\epsilon_{i2}}$$

if

(2.28) $$\Pi_1 \equiv \Pi_2 \bmod G_{n+1}.$$

It is evident that to determine $\overline{G^n}$ we only need a rule for giving representatives of our equivalence classes and a multiplication table for these representatives. Making use of Lemma 2.3 it is sufficient to find a rule which takes for representatives only words (2.24) with the property that $\epsilon_i = 0$ if e_i is of mixed type. But such class representatives can be constructed by Definitions 2.1, 2.19, 2.20, Corollary 2.1, and Theorem 2.4 from class representatives for the factor groups of the subgroups $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_z$. (The class representatives for $\overline{(\mathscr{G}_v)^n}$ are found by working with $\mathscr{G}_v$ in place of G.) Thus we may obtain a rule for $\overline{G^n}$ by giving rules for the $\overline{(\mathscr{G}_v)^n}$. A multiplication table for the representatives of the equivalence classes (2.27) can then be found by the "collection process" and our rule for $\overline{G^n}$.

We have thus shown that it is sufficient to obtain the "representation algorithm" for groups $G[p]$.

2(e) *The F-simple commutators expressed by auxiliary-simple commutators.* Auxiliary-simple commutators were introduced in Definition 2.10. This definition leads to a discussion of the collection process for auxiliary commutators without specifying the elements in the classes $\{k\}$ or the ordering of the elements in a class. We cannot proceed to the "representation algorithm," however, without a rule which gives the elements of every class $\{k\}$ and orders them.

To state this rule we first require

Remark 2.3. Consider the F-simple commutator, c_{i_k} ,with generator sequence

$$(2.29) \qquad \langle c_{i_k} \rangle = \langle c_{j_1}, c_{j_2}, \ldots, c_{j_f} \rangle$$

such that c_{i_k} is not of type ∞. (We are applying Definition 2.19 with the generators e_{j_u} of $\mathscr{F}$ replaced by the generators c_{j_u} of F.) Hence integers, m, exist which have properties (a), (b), (c), and (d):

(a) $1 \leqq m \leqq f$.

(b) $O(c_{j_m}) < \infty$.

(c) If $m < t \leqq f$, then $j_t \neq j_m$.

(d) If $j_m < j_t$, then c_{j_t} occurs in $\langle c_{i_k} \rangle$ fewer than $O(c_{j_t})$ times.

We are now ready for

Definition 2.21. Consider the F-simple commutator, c_{i_k}, of dimension > 1. (i) If c_{i_k} is G-simple, then $c_{i_k} = d(0, k)$ is in class $\{k\}$. In particular if c_{i_k} is of type ∞, then the class $\{k\}$ of Definition 2.10 consists only of c_{i_k}. (ii) If c_{i_k} is not of type ∞ and $\langle c_{i_k} \rangle$ is given by (2.29), then the class $\{k\}$ consists of (a) $d(0, k)$ if c_{i_k} is G-simple, and (b) the commutators, $d(m, k)$, constructed as follows from the integers, m, of Remark 2.3:

For $1 \leqq h \leqq f$ and $h \neq m$, let $d_{hmk} = c_{j_h}$. But let $d_{mmk} = (c_{jm})^{\gamma_m}$ where $\gamma_m = O(c_{jm})$. Then

$$(2.30) \qquad d(m, k) = (\ldots (d_{1mk}, d_{2mk}), \ldots, d_{fmk})$$

Having given the class $\{k\}$, let us order its elements: Suppose that $d(m_1, k)$ and $d(m_2, k) \in \{k\}$. Then $d(m_1, k) <_a d(m_2, k)$ if $m_1 < m_2$.

The usefulness of the auxiliary commutators as given by Definitions 2.10, 2.11, and 2.21 rests on three properties:

(I) Those auxiliary commutators which are not basic commutators are $= 1$ in G. (See also Definitions 2.2 and 2.16).

(II) The G-simple commutators are among the free generators of G_2 according to Theorem 2.3.

(III) The truth of

LEMMA 2.4. *The subgroup of F generated by the F-simple commutators of dimension > 1 is also generated by the auxiliary-simple commutators.*

To establish Lemma 2.4 we must first introduce a correspondence between F-simple commutators and auxiliary-simple commutators.

Definition 2.22. Suppose that c_{i_k} is a F-simple commutator which is not G-simple. According to Definitions 2.7, 2.14, and 2.16, the generator sequence (2.29) contains a unique element c_{j_μ} such that:

(i) c_{j_μ} occurs in $\langle c_{i_k} \rangle$ at least $\alpha = O(c_{j_\mu})$ times.

(ii) If $\mu < \nu \leqq f$, then c_{j_ν} occurs in $\langle c_{i_k} \rangle$ fewer than $O(c_{j_\nu})$ times.

The auxiliary-simple commutator, $\tilde{c}_{i_k}$, which corresponds to c_{i_k} is then given by

$$(2.31)\qquad \tilde{c}_{i_k} = \begin{cases} ((\ldots(c_{j_1}, c_{j_2}), \ldots, c_{j_{\mu-\alpha}}), c_{j_{\mu-\alpha+1}}{}^{\alpha}) \text{ if } \mu = f \text{ and } j_{\mu-\alpha+1} = j_\mu \\ (\ldots(c_{j_1}{}^{\alpha}, c_{j_2}), \ldots, c_{j_{\mu-\alpha+1}}) \text{ if } \mu = f \text{ and } j_{\mu-\alpha+1} \neq j_\mu \\ (\ldots((((\ldots(c_{j_1}, c_{j_2}), \ldots, c_{j_{\mu-\alpha}}), c_{j_{\mu-\alpha+1}}{}^{\alpha}), c_{j_{\mu+1}}), c_{j_{\mu+2}}), \ldots, c_{j_f}) \\ \qquad\qquad \text{if } \mu < f \text{ and } j_{\mu-\alpha+1} = j_\mu \\ (\ldots(((\ldots(c_{j_1}{}^{\alpha}, c_{j_2}), \ldots, c_{j_{\mu-\alpha+1}}), c_{j_{\mu+1}}), c_{j_{\mu+2}}), \ldots, c_{j_f}) \\ \qquad\qquad \text{if } \mu < f \text{ and } j_{\mu-\alpha+1} \neq j_\mu \end{cases}$$

When c_{i_k} is G-simple, however, then $\tilde{c}_{i_k} = c_{i_k}$.

Having introduced the $\tilde{c}_{i_k}$, we observe by Definitions 2.2, 2.7, 2.10, 2.14, 2.16, 2.21, 2.22, and by mathematical induction that Lemma 2.4 is a consequence of

LEMMA 2.5.

$$(2.32)\qquad \tilde{c}_{i_k} = uc_{i_k}v$$

in F, where u and v are either 1 *or are in the subgroup generated by* $c_{i_1}, c_{i_2}, \ldots, c_{i_{k-1}}$.

Evidently we only need to prove Lemma 2.5 for c_{i_k} not G-simple. For this purpose we shall require the auxiliary

LEMMA 2.6. (See [**17**], Lemma 4.3.) *Suppose that c and d are F-simple commutators such that $\langle c\rangle = \langle c_{j_1}, c_{j_2}, \ldots, c_{j_\omega}\rangle$, $\omega > 1$ and $d < c_{j_\omega}$. The identity*

$$(2.33)\qquad (c, d) = \Pi_1 M(c, d)\Pi_2$$

is then valid in F, where Π_1 and Π_2 are words in F-simple commutators, v_i, with the properties (i), (ii), *and* (iii) *below:*

(i) $1 < D(v_i) \leqq \omega + 1$.

(ii) *If $\langle v_i\rangle = \langle w_1, w_2, \ldots, w_\eta\rangle$, then $w_1, w_2, \ldots, w_\eta$ is a rearrangement of a subsequence of $c_{j_1}, c_{j_2}, \ldots, c_{j_\omega}, d$.*

(iii) $v_i < (c, d)$.

We are now ready for the

Proof of Lemma 2.5. We start from the two special cases (a) and (b) given in the notation of (2.10):

(a) $c_{i_k} = P_\mu(A, c_{j_f})$ where $\mu = O(c_{j_f})$

(b) $c_{i_k} = P_{\nu-1}((c_{j_1}, c_{j_2}), c_{j_1})$ where $\nu = O(c_{j_1})$

But

$$(2.34)\qquad \tilde{c}_{i_k} = \begin{cases} (A, c_{j_f}{}^{\mu}) & \text{in case (a)} \\ (c_{j_1}{}^{\nu}, c_{j_2}) & \text{in case (b)} \end{cases}$$

according to (2.31). Equation (2.32) is then obtained in the special cases by expressing $\tilde{c}_{ik}$ as a unique word in F-simple commutators through the application of (2.1), the identities (2.9b) and

$$(2.35) \qquad (M, b) = M^{-1} \prod_{i=1}^{I} [a_i(a_i, b)]^{\epsilon_i}$$

where $M = \prod_{i=1}^{I} a_i^{\epsilon_i}$. (Note that (2.35) is a consequence of (2.1).) We now observe by Definitions 2.7 and 2.16 that the smallest F-simple commutator which is not G-simple, belongs to a special case (a) or (b). To complete our proof by mathematical induction on the place of c_{i_k} in the ordering of Definition 2.2, it is therefore sufficient to establish the following proposition:

Suppose that (i) c_{i_K} is not G-simple, (ii) c_{i_K} does not belong to a special case (a) or (b), and (iii) c_{i_k} satisfies (2.32) when $1 \leqq k < K$. Then (2.32) also holds for $C = c_{i_K}$.

When C^L is not G-simple, then the conclusion of the proposition is found easily by identity (2.35), hypothesis, Definitions 2.2, 2.7, 2.16, 2.22, Lemma 2.6, and Theorem 2.2. When C^L is G-simple, however, then C has the form

$$(2.36) \qquad C = P_{\nu-1}([(\ldots(c_{j_1}, c_{j_2}), \ldots, c_{j_{f-\nu+1}})], c_{j_1})$$

by Definitions 2.7, 2.16, and hypothesis, where $\nu = O(c_{j_1})$. Consider

$$(2.37) \qquad C' = P_{\nu-1}([(\ldots(c_{j_1}, c_{j_2}), \ldots, c_{j_{f-\nu}})], c_{j_1})$$

By hypothesis $f - \nu > 1$ and c_{j_h} occurs in $\langle C \rangle$ fewer than $O(c_{j_h})$ times, where $2 \leqq h \leqq f - \nu + 1$. Also C' satisfies the relation

$$(2.32a) \qquad \tilde{C}' = u'C'v'$$

of the form (2.32) by Definitions 2.2, 2.7, 2.16 and hypothesis. Now $\tilde{C} = (\tilde{C}', c_{j_{f-\nu+1}})$ by Definition 2.22. Applying identity (2.35), Lemmas 2.1 and 2.6, Theorem 2.2, and Definitions 2.2 and 2.7 to the computation of $(u'C'v', c_{j_{f-\nu+1}})$, we obtain the conclusion of the proposition also for C^L G-simple.

Having established Lemmas 2.4 and 2.5, we have thus finished our preliminary discussion of the commutator calculus.

3. The representation algorithm.

3(a). *Formulation of the problem.* Let G have the presentation (2.5a). The factor group $\bar{G}^n = G/G_{n+1}$ (n a fixed positive integer) consists of the cosets of G_{n+1} in G. These cosets have basic n-words

$$(3.1) \qquad \prod_{i=1}^{q(n)} c_i^{\epsilon_i}$$

as their representatives according to Corollary 2.1 and Definition 2.8. (See Remark 2.2. Note that (2.5a) occurs in Subsection 2(c).)

In order to investigate the nilpotent group $\bar{G}^n$, we first analyze the groups $\bar{G}_m = G_m/G_{m+1}$ for $m = 1, 2, \ldots, n$. For this purpose let us consider the special basic m-words

$$(3.2) \qquad \prod_m = \prod_{i=q(m-1)+1}^{q(m)} c_i^{\epsilon_i}$$

where $q(0) = 0$. Let $\prod_m$ be a representative of a coset of $\bar{G}_m$. It is then evident by Corollary 2.1 and Definition 2.1, that the representatives of all of the cosets of $\bar{G}^n$ are those distinct basic n-words

$$(3.3) \qquad \prod_1 \prod_2 \ldots \prod_n$$

which have the property that every $\prod_m (1 \leqq m \leqq n)$ is in a complete set of coset-representatives of $\bar{G}_m$.

Therefore to determine $\bar{G}^n$, we will proceed as follows: First we will obtain a rule for choosing the representatives, $\prod_m$, in (3.2) of the cosets of G_m. Then we will compute a multiplication table for the group of coset representatives (3.3) found by our rule.

Accordingly, we will start the first task of choosing the coset representatives, $\prod_m$, in (3.2) by finding those $\prod_m$ which are in G_{m+1}. We will do this in the next subsection for $m > 1$.

3(b) *The relators of $\bar{G}_m$ expressed by relation commutators* ($m > 1$). We begin this subsection with two preliminary definitions.

Definition 3.1. Let $\prod = \prod_{j=1}^{l} c_{j_i}^{\epsilon_i}$. $\prod$ is said to be a relator in G if $\prod$ is mapped into the identity under the homorphism $F \to G$ of (2.5a). In particular, a relator $c_i^{\alpha_i} (\alpha_i \neq 0)$ which occurs in (2.5a) is said to be a defining relator.

Definition 3.2. (See Definitions 2.10, 2.11, 2.12, 2.21, and Remark 2.2.) The auxiliary-simple commutator, $d(m, k)$, is relation-simple if $m > 0$. (By Definition 2.21 any $d(0, k)$ is G-simple, but any relation-simple commutator is a relator in G.) The relation-simple commutators of pseudo-dimension 2 are the relation commutators of pseudo-dimension 2. Having defined the relation commutators of pseudo-dimension h, we define those of pseudo-dimension $(h + 1)$. Let d be an auxiliary commutator of pseudo-dimension $(h + 1)$. If d is auxiliary-simple, then d is a relation commutator provided it is relation-simple. If d is not auxiliary-simple, then d is a relation commutator provided at least one among d^L and d^R is a relation commutator. (Note by Theorem 2.3 that an auxiliary commutator is a relator in G if and only if it is a relation commutator.)

The auxiliary h-word ($h > 1$)

$$(3.4) \qquad \prod = \prod_{i=1}^{q_a(h)} d_i^{\eta_i}$$

is said to be a relation h-word provided $\eta_i = 0$ when d_i is not a relation commutator. If $w \in F_2$ and w has the h-c.a.c. representation $\prod$, then $\prod$ is said to be a h-composite relation commutator (h-c.r.c.) representation of w provided $\prod$ is a relation h-word.

https://doi.org/10.4153/CJM-1975-125-9 Published online by Cambridge University Press

Having given our preliminary definitions, we are ready to state the important

LEMMA 3.1. *Let $m > 1$. Let $\prod_m$ be a basic m-word of the form* (3.2). $\prod_m \in G_{m+1}$ *if and only if it has a m-composite relation commutator representation.*

It is evident from Definitions 2.1, 2.12, and 3.2 that $\prod_m \in G_{m+1}$ if $\prod_m$ has an m-c.r.c. representation. Thus we only need to prove

LEMMA 3.2. *If $\prod_m \in G_{m+1}$, then $\prod_m$ has an* m-c.r.c. *representation.*

The homomorphism $F_{m+1} \to G_{m+1}$ induced by the presentation (2.5a), is onto by Definition 2.1. Hence there exists an element $f_{m+1} \in F_{m+1}$ such that $\prod_m f_{m+1}$ is a relator in G in the sense of Definition 3.1. Thus it is sufficient to prove

LEMMA 3.3. *Let $w \in F$. Suppose that $D(w) = m$ and w is a relator in G. Then w has an* m-c.r.c. *representation.*

In the following we will establish Lemma 3.3 through the use of the collection process. For this purpose we require an alternative presentation of G which we will construct from (2.5a). Let ρ be the number of those exponents α_i in (2.5a) which do not vanish. ($0 < \rho \leqq r$ by hypothesis.) Let H be the free group

(3.5) $$H = \langle a_1, a_2, \ldots, a_{r+\rho}\rangle.$$

Then G is the homomorphic image of H obtained by the presentation

(3.6) $$G = \langle a_1, a_2, \ldots, a_{r+\rho}; s_1, s_2, \ldots, s_{2\rho}\rangle$$

where the s_i are given in Definition 3.3. below. This definition also expresses the generators a_j of H (or G) as words in the generators c_i of F (or G). Also Definition 3.3 shows how to obtain the presentation (3.6) from (2.5a) by application of Tietze transformations [**10**].

Definition 3.3. If the generator c_1 in (2.5a) has infinite order, let $a_1 = c_1$. If c_1 has finite order α_1, let $a_1 = c_1^{\alpha_1}$ and $a_2 = c_1$; also let $s_1 = a_1$ and $s_2 = a_1 a_2^{-\alpha_1}$. Suppose that we have introduced $(h + v)$ generators $a_1, a_2, \ldots, a_{h+v}$ as words in the generators $c_1, c_2, \ldots, c_h$ and have specified $s_1, s_2, \ldots, s_{2v}$ when $v > 0$. Then $a_{h+v+1} = c_{h+1}$, if c_{h+1} has infinite order. But $a_{h+v+1} = c_{h+1}^{\alpha_{h+1}}$, $a_{h+v+2} = c_{h+1}$, $s_{2v+1} = a_{h+v+1}$, and $s_{2v+2} = a_{h+v+1} a_{h+v+2}^{-\alpha_{h+1}}$, if c_{h+1} has finite order.

Evidently F has the presentation

(3.7) $$F = \langle a_1, a_2, \ldots, a_{r+\rho}; s_2, s_4, \ldots, s_{2\rho}\rangle.$$

Note that we are proceeding according to Remark 3.1 in analogy to Remark 2.2.

Remark 3.1. $\prod_{i=1}^{I} a_{j_i}^{\epsilon_i}$ denotes an element of H as well as its image in G or F under the homorphisms (3.6) or (3.7).

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Having given the presentation (3.6), we must divide the k-commutators in H into two categories before applying the collection process to the proof of Lemma 3.3.

Definition 3.4. (We are applying Definition 2.9 to H in place of F.) The 1-commutator $a_i^{\pm 1}$ is in category I, if a_i (as a word in the c_j) is not a defining relator; $a_i^{\pm 1}$ is in category II, if a_i (as a word in the c_j) is a defining relator. Let d be a k-commutator, where $k > 1$. d is in category II, if at least one among d^L and d^R is in category II; d is in category I, if it is not in category II.

We are finally ready to apply the collection process to relators, w. (See Definition 3.1 and Lemma 3.3.) It is well-known that w is a product of conjugates of defining relators [**10**]. Hence w has the form

$$(3.8) \qquad w = \prod_{j=1}^{J} (w_j^{-1} a_{i_j} w_j)^{\epsilon_j}$$

by the substitutions of Definition 3.3, where (i) the $\epsilon_j = \pm 1$, (ii) the a_{i_j} are generators of H of category II, (iii) the w_j are words in generators of H of category I. Computing the m-c.b.c. representation of w given by (3.8) as a first step in the proof of Lemma 3.3, we find

LEMMA 3.4. *Let* $a_1, a_2, \ldots, a_{q(m)}$ *be the basic commutators of dimension* $\leqq m$ *in* H. *Let*

$$(3.9) \qquad g = \prod_{j=1}^{q(m)} a_j^{\eta_j}$$

be the m-c.b.c. *representation of* w *in* H. *Then* g *has the property that* $\eta_j = 0$, *if* a_j *is in category I. Hence* $h = g^{-1}w$ *is a relator in* G *and also* $\in F_{m+1}$, *by* (3.8) *and Definitions* 2.1, 2.8, 3.1, 3.3, *and* 3.4.

Proof. Let us compute g by the collection process as discussed in subsection 2b. Consider

$$(3.10) \qquad g_1 = \prod_{j=1}^{q(m)} a_j^{\eta_{j1}}$$

where

$$(3.11) \qquad \eta_{j1} = \begin{cases} \eta_j \text{ if } a_j \text{ is in category I} \\ 0 \text{ if } a_j \text{ is in category II.} \end{cases}$$

Then g_1 is an m-c.b.c. representation of 1 in F. (See (3.8) and Definitions 2.1, 2.2, 2.8, 3.3, and 3.4.) Hence all $\eta_{j1} = 0$ by Definition 2.1 and Theorem 2.1. We then obtain our conclusion according to (3.11).

To establish Lemma 3.3 as a consequence of Lemma 3.4 we require the auxiliary

LEMMA 3.5. *Let* $u = \prod_{j=q(1)+1}^{q(m)} a_j^{\epsilon_i}$ *be a basic m-word in H. Then u is a word in F-simple commutators of dimension* > 1, *when u is rewritten as a word in* $c_1, c_2, \ldots, c_r$. *Hence u is also a word in auxiliary-simple commutators.*

To derive this lemma we first express the generators of H by generators of F according to Definition 3.3. We then apply the techniques of the proof of Lemma 2.5 repeatedly to express u as a word in F-simple commutators of dimension > 1. Finally making use of Lemma 2.4 we find that u is a word in auxiliary-simple commutators.

To apply Lemmas 3.4 and 3.5 to the proof of Lemma 3.3 we need additional terminology.

Definition 3.5. The relation commutators of pseudo-dimension 2 and their inverses are the relation 2-commutators. Suppose that we have defined the relation k-commutators for $2 \leqq k \leqq m$. The relation $(m + 1)$-commutators are the elements of two categories:

(I) The relation-simple commutators of pseudo-dimension $(m + 1)$ together with their inverses.

(II) All auxiliary $(m + 1)$-commutators $w = (u, v)$ such that u is a relation s-commutator, v is a relation t-commutator, s and $t \geqq 2$, and finally $(s + t) = m + 1$. (See Definition 2.13.)

We are finally ready for the

Proof of Lemma 3.3.

$$w \equiv g \bmod F_{m+1}$$

$$g = \prod_{j=1}^{J} d_{i_j}^{\eta_j} \tag{3.12}$$

by Lemmas 3.4 and 3.5, where g is a relator and the d_{i_j} are auxiliary-simple commutators of pseudo-dimension > 1. Let us rewrite g by the collection process as described in subsection 2.b. We then find in the notation of (2.21) that

$$g = \prod_{i=1}^{q_a(m)} d_i^{\epsilon_i} g_{m+1, q_a(m), a} \tag{3.13}$$

where $g_{m+1,q_a(m),a}$ is a word in finitely many auxiliary s-commutators, u_z, so that each $s > m$. But all relation commutators and relation s-commutators are relators in G by Definitions 3.1, 3.2, and 3.5. Hence

$$g_1 = \prod_{i=1}^{q_a(m)} d_i^{\epsilon_{i1}} g_{m+1,1} \tag{3.14}$$

is a relator in G where

$$\epsilon_{i1} = \begin{cases} \epsilon_i \text{ if } d_i \text{ is not a relation commutator} \\ 0 \text{ if } d_i \text{ is a relation commutator} \end{cases} \tag{3.15}$$

and $g_{m+1,1}$ is obtained from $g_{m+1,q_a(m),a}$ by replacing those auxiliary s-commutators, u_z, which are relation s-commutators, by the identity. Thus g_1 is a word in G-simple commutators by Definitions 2.2, 2.8, 2.12, 2.13, 2.16, 2.21, 3.2, and 3.5. But the G-simple commutators are according to Theorem 2.3, free generators of subgroups of F as well as of G, in the sense of Remark 2.2. Since g_1 is a relator in G, we conclude that

$$(3.16)\qquad 1 \equiv \prod_{i=1}^{q_a(m)} d_i^{\epsilon_{i1}} \bmod F_{m+1}$$

Hence all $\epsilon_{i1} = 0$ by Theorem 2.1 and Definition 2.1. Applying Definition 3.2 we then obtain our conclusion, i.e., $\prod_{i=1}^{q_a(m)} d_i^{\epsilon_i}$ is an m-c.r.c. representation of g as well as of w.

We have now established Lemmas 3.1, 3.2, and 3.3. By the discussion of Subsection 3(a) and by Corollary 2.1, we easily obtain

THEOREM 3.1. (See Definition 2.8.) *Let* $m > 1$. $\bar{G}_m = G_m/G_{m+1}$ *is the abelian group generated by* $c_{q(m-1)+1}, c_{q(m-1)+2}, \ldots, c_{q(m)}$ *subject to the additional relations that*

$$(3.17)\qquad \prod = \prod_{i=q(m-1)+1}^{q(m)} c_i^{\epsilon_i} = 1$$

in $\bar{G}_m$ *if and only if* $\prod$ *has an* m-c.r.c. *representation as an element of* F.

3(c) $\bar{G}_m = G_m/G_{m+1}$ *determined by ideal theory for* $m > 1$. We will obtain a rule for choosing the representatives of the cosets of $\bar{G}_m (m > 1)$ by ideal theory [**13**].

We start out by dividing the relators which occur in (3.17) into $[q(m) - q(m-1)]$ relator-classes $\mathscr{C}_{q(m-1)+1}, \mathscr{C}_{q(m-1)+2}, \ldots, \mathscr{C}_{q(m)}$.

Definition 3.6. $\prod = \prod_{i=q(m-1)+1}^{q(m)} c_i^{\epsilon_i}$ is a relator in $\bar{G}_m$ if $\prod \equiv 1 \bmod G_{m+1}$. ($\prod$ is a relator in $\bar{G}_m$, according to Theorem 3.1, if and only if it has an m-c.r.c. representation.)

Definition 3.7. Consider the basic m-word

$$(3.18)\qquad w = \prod_{i=1}^{q(m)} c_i^{\epsilon_i}.$$

w is in the class $\mathscr{D}_j (1 < j \leqq q(m))$, if it has two properties: (i) $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_{j-1} = 0$, and (ii) If $\epsilon_j = 0$, then $w = 1$ in F.

If $w \in \mathscr{D}_j$, then the exponent ϵ_j in (3.18) which we shall denote by $E_j(w)$ is said to be the minimal exponent of w.

The element w of class $\mathscr{D}_j$ is in the relator-class $\mathscr{C}_j (q(m-1) < j \leqq q(m))$, if it is a relator in $\bar{G}_m$. We will refer to the elements of $\mathscr{C}_j$ as relators of $\mathscr{C}_j$.

It is evident that every relator in $\bar{G}_m$ is in a unique $\mathscr{C}_j$. A class $\mathscr{C}_j$ may, however, consist only of 1.

Definition 3.8. The class $\mathscr{C}_j$ is said to be trivial if it contains only the identity. Any class $\mathscr{C}_j$ which contains a non-identity element is said to be non-trivial.

Before proceeding to Lemma 3.6, let us state the classical

Definition 3.9. [**13**]. Let $\mathscr{R}$ be the ring of integers. The subset $\mathscr{I} \subset \mathscr{R}$ is said to be an ideal in $\mathscr{R}$ if it satisfies two requirements: (I) If a and $b \in \mathscr{I}$, then $(a + b) \in \mathscr{I}$. (II) If $a \in \mathscr{I}$ and $\alpha \in \mathscr{R}$, then $(\alpha a) \in \mathscr{I}$.

LEMMA 3.6. *Suppose that* $\mathscr{C}_j$ $(q(m - 1) < j \leqq q(m))$ *contains the relators*

$$(3.19a) \quad w_1 = \prod_{i=j}^{q(m)} c_i^{\epsilon_{i1}}$$

and

$$(3.19b) \quad w_2 = \prod_{i=j}^{q(m)} c_i^{\epsilon_{i2}}$$

such that $\epsilon_{j1} + \epsilon_{j2} \neq 0$. *Then*

$$(3.20) \quad w_3 = \prod_{i=j}^{q(m)} c_i^{\alpha\epsilon_{i1}}$$

and

$$(3.21) \quad w_i = \prod_{i=j}^{q(m)} ci^{\epsilon_{i1}+\epsilon_{i2}}$$

are elements of $\mathscr{C}_j$, *where the integer* $\alpha \neq 0$. *Hence the set of minimal exponents,* $E_j(w)$, *of elements of* $\mathscr{C}_j$ *constitutes an ideal,* $\mathscr{I}_j$, *in* $\mathscr{R}$.

This lemma is an immediate consequence of Definitions 3.6–3.9 and of Theorem 3.1.

To proceed from Lemma 3.6 we now require a classical property of $\mathscr{R}$.

THEOREM 3.2 [**13**]. *Let* $\mathscr{I}$ *be an ideal in* $\mathscr{R}$. *Then* $\mathscr{I}$ *is a principal ideal, i.e., it consists of all integral mulitples* βA *of the unique generating element* A, *where* $A \geqq 0$.

By Definitions 3.6–3.9, Lemma 3.6, and Theorem 3.2, we easily find

LEMMA 3.7. *Let* $q(m - 1) < j \leqq q(m)$ *where* $m > 1$. *Suppose that the class* $\mathscr{C}_j$ *is non-trivial. The ideal* $\mathscr{I}_j$ *is then generated by a positive integer,* A_{jj}. *Hence there exists a relator*

$$(3.22) \quad R_j = \prod_{i=j}^{q(m)} c_i^{A_{ji}}$$

of class $\mathscr{C}_j$ *such that* $E_j(R_j) = A_{jj}$.

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Definition 3.10. If $\mathscr{C}_j$ is non-trivial, then the R_j of Lemma 3.7 is said to be the representative of the relator-class $\mathscr{C}_j$. A trivial relator-class, $\mathscr{C}_j$, has representative $R_j = 1$ for which we take $A_{jj} = A_{j,j+1} = \ldots = A_{j,q(m)} = 0$. (Note that the choice of R_j is often not unique.)

Making use of Corollary 2.1, Definitions 3.7 and 3.10, and Lemma 3.7, let us characterize the relators in $\bar{G}_m$ and also determine the elements of $\bar{G}_m$.

LEMMA 3.8. $\prod = (\prod_{i=q(m-1)+1}^{q(m)} c_i^{\epsilon_i})$ is a relator in $\bar{G}_m$ if and only if there exist integers $\delta_{q(m-1)+1}, \delta_{q(m-1)+2}, \ldots, \delta_{q(m)}$ such that $\prod \equiv (\prod_{i=q(m-1)+1}^{q(m)} R_i^{\delta_i})$ mod F_{m+1} or such that $\epsilon_i = \sum_{j=q(m-1)+1}^{i} \delta_j A_{ji}$ for $q(m-1) < i \leqq q(m)$.

THEOREM 3.3. *A complete set of representatives of the cosets of* G_m (mod G_{m+1}) *consists of those elements*

$$\left(\prod_{i=q(m-1)+1}^{q(m)} c_i^{\epsilon_i}\right)$$

which have the following property: $0 \leqq \epsilon_i < A_{ii}$ *for* $q(m-1) < i \leqq q(m)$, *if* $\mathscr{C}_i$ *is non-trivial.*

We have completed our investigation of $\bar{G}_m$. We are thus ready to determine $\bar{G}^n = G/G_{n+1}$ where n is a given positive integer.

3(d) *The group* $\bar{G}^n = G/G_{n+1}$. The representatives of the cosets of G (mod G_{n+1}) were discussed in subsection 3(a). To determine them we must still examine $\bar{G}_1 = G/G_2$. It is evident from Definition 2.1 and Corollary 2.1 that $\bar{G}_1$ is the abelian group generated by $c_1, c_2, \ldots, c_r$ subject to the additional relations that $c_1^{\alpha_1} = c_2^{\alpha_2} = \ldots = c_r^{\alpha_r} = 1$, where the α_i are given in the presentation (2.5a) of G. Applying Theorem 3.3 and recalling the discussion of the basic n-words (3.3), we immediately obtain Theorem 3.4 below. To state it, however, we require the auxiliary

Definition 3.11. Let $1 \leqq j \leqq r = q(1)$. The relator-class $\mathscr{C}_j$ is trivial and consists only of the identity, if $\alpha_j = 0$. The relator-class $\mathscr{C}_j$ is non-trivial and consists of all powers of $c_j^{\alpha_j}$, if $\alpha_j \neq 0$. If $\mathscr{C}_j$ is non-trivial, then $A_{jj} = \alpha_j$.

THEOREM 3.4. *A complete set of representatives of the cosets of* G (mod G_{n+1}) *consists of those elements* $(\prod_{i=1}^{q(n)} c_i^{\epsilon_i})$ *which have the following property*: $0 \leqq \epsilon_i < A_{ii}$ *for* $1 \leqq i \leqq q(n)$, *if* $\mathscr{C}_i$ *is non-trivial.*

Having found the elements of $\bar{G}^n$ it remains to compute a multiplication table for this group. We will see that this can be done by the "representation algorithm" given below. This algorithm finds the representative, given by Theorem 3.4, of that coset which contains a specified freely reduced word in the generators $c_1, c_2, \ldots, c_r$.

To describe this algorithm, we must assign to relator-classes, $\mathscr{C}_j$, not only their representatives, R_j (given by Definitions 3.7 and 3.10), but we must also assign to them elements $\tilde{R}_j$ of F, which are relators in G. (See Definition 3.1.) We do this in

Definition 3.12. (See Definitions 2.2, 2.12, 3.2, 3.7, 3.8, 3.10, 3.11.) If $\mathscr{C}_j$ is trivial, then $\tilde{R}_j = R_j = 1$. If $\mathscr{C}_j$ is non-trivial and $1 \leqq j \leqq r$, then $\tilde{R}_j = R_j = c_j^{\alpha_i}$. If $\mathscr{C}_j$ is non-trivial and $r = q(1) < j \leqq q(n)$, then $\tilde{R}_j$ is a $(D(c_j))$-composite relation commutator representation of R_j. (Note that $\tilde{R}_j$ always exists by Theorem 3.1. However, the choice of $\tilde{R}_j$ need not be unique. That the $\tilde{R}_j$ are relators in G is evident from Definitions 2.21, 3.1, and 3.2.)

We are now ready to consider a freely reduced word $w = \prod_{i=1}^{l} c_{j_i}^{\eta_i} \neq 1$, where the c_{j_i} are among the generators $c_1, c_2, \ldots, c_r$. This word, when thought of as an element of G, is in a coset of G (mod G_{n+1}). This coset has a representative of the form $\prod_{i=1}^{q(n)} c_i^{\epsilon_i}$, as given in Theorem 3.4. In the "representation algorithm," we will compute the exponents $\epsilon_1, \epsilon_2, \ldots, \epsilon_{q(n)}$ in order of increasing subscripts.

To find $\epsilon_j (1 \leqq j \leqq q(n))$, we proceed as follows. By means of the algorithm to be presented, we express w in the form

$$w = u_{j-1} v_{j-1} f_{j-1} \tag{3.23}$$

such that (3.23) has the three properties:

(I):

$$u_{j-1} = \begin{cases} \prod_{i=1}^{j-1} c_i^{\epsilon_i} & \text{if } 1 < j \leqq q(n) \\ 1 & \text{if } j = 1 \end{cases} \tag{3.24}$$

(II): $v_0 = w$. In general v_{j-1} either $= 1$ in F or $v_{j-1} \geqq c_j$. (See remark 2.2, Definitions 2.4, 2.5, and Theorem 2.1.)

(III): $f_{j-1} \in G_{n+1}$, when f_{j-1} is considered as an element of G.

We then obtain the exponent ϵ_j from the expression (3.23) in the two steps below:

(A): Making use of property II, we rewrite v_{j-1}, considered as a word in F, in the form

$$v_{j-1} = \left(\prod_{i=j}^{q(n)} c_i^{\epsilon_{ij}} \right) h_j \tag{3.25}$$

through the collection process, where $h_j \in F_{n+1}$. (See Theorem 2.1 and Subsection 2(b))

(B): We take $\epsilon_j = \epsilon_{jj}$ and $\rho_j = 0$, when the relator-class $\mathscr{C}_j$ is trivial. When $\mathscr{C}_j$ is non-trivial, however, then ϵ_j and ρ_j are the unique solutions of the equations

$$\begin{aligned} \epsilon_{jj} &= \rho_j A_{jj} + \epsilon_j \\ 0 &\leqq \epsilon_j < A_{jj}. \end{aligned} \tag{3.26}$$

(See Definitions 3.7, 3.8, 3.11, and Lemma 3.7.)

Having completed steps (A) and (B), we express the word w in the form

(3.27) $$w = u_j v_j f_j$$

valid in F, such that

(3.28)
$$\begin{aligned} u_j &= u_{j-1}\, c_j^{\epsilon_j} \\ v_j &= \tilde{R}_j^{-\rho_j}\, v_{j1} \\ v_{j1} &= c_j^{-\epsilon_j}\, v_{j-1}\, h_j^{-1} \\ f_j &= v_{j1}^{-1}\, \tilde{R}_j^{\rho_j}\, v_{j1}\, h_j\, f_{j-1}. \end{aligned}$$

Making use of equations (3.22), (3.25), (3.26), (3.27), and (3.28), of Definitions 2.1 and 3.12, and recalling that v_{j-1} has property (II) by hypothesis, we find that v_j also has property (II) for $1 \leqq j < q(n)$ and $f_j \in G_{n+1}$, when considered as an element of G. Thus having expressed w in the form (3.27) for $1 \leqq j < q(n)$, we are ready to compute ϵ_{j+1} by the above steps (A) and (B).

We refer to our computation of the $q(n)$ exponents ϵ_j by successive steps of two kinds as the "representation algorithm." We note that $v_{q(n)} \equiv 1 \bmod F_{n+1}$ by equations (3.22), (3.25), (3.26), (3.27), (3.28), and by Definition 3.12. Recalling equations (3.24), (3.26), and (3.27), we conclude that this algorithm yields a coset representative as given by Theorem 3.4.

Our algorithm evidently provides a means for obtaining a multiplication table of $\bar{G}^n$; i.e., multiplication is carried out by finding the coset representatives of products of freely reduced words.

Thus we have completed both of the tasks stated in Subsection 3(a).

3(e) *Concluding remarks.* We conclude this section with two remarks.

Remark 3.2. It was shown in Subsection 2(d) that the group $\bar{G}^n$ can be found from the groups $\overline{(\mathscr{G}_1)^n}$, $\overline{(\mathscr{G}_2)^n}$, ..., $\overline{(\mathscr{G}_z)^n}$. (See Definition 2.20 and Theorem 2.4.) Theorem 3.4 and the "representation algorithm" are, however, given here for groups G with presentations (2.5a) in general, rather than the special groups $G[p]$ of Definition 2.18. This was done since the assumption that G is a special group $G[p]$ does not simplify either the results of this section or their proofs.

Nevertheless, the methods of Subsection 2(d) have considerable value for two purposes:

(I) The simplification of the practical computation of a given group $\bar{G}^n$.

(II) The derivation of general results to be obtained in future investigations from our present conclusions.

Remark 3.3. Let $n > 1$. To enumerate the elements of $\bar{G}^n$ according to Theorem 3.4, we must tabulate the representatives, R_j, of the non-trivial relator-classes, $\mathscr{C}_j$, which are such that $r = q(1) < j \leqq q(n)$. For the "representation algorithm" we also need the corresponding relation $D(c_j)$-words, $\tilde{R}_j$.

We will now give a brief outline of a procedure for computing the above R_j and $\tilde{R}_j$. This procedure consists of satisfying conditions (i), (ii), and (iii) below. It will be applied in the example of Section 4.

Let $m > 1$. Let

$$(3.29) \qquad d_{k_1} <_a d_{k_2} <_a \ldots <_a d_{k_{q_r(m)}}$$

be the relation commutators of pseudo-dimension $\leqq m$ given by Definition 3.2. Consider the relation m-word

$$(3.30) \qquad \prod = \prod_{i=1}^{q_r(m)} d_{k_i}^{\eta_i}.$$

Then

$$(3.31) \qquad \prod_1 = \prod_{i=r+1}^{q(m)} c_i^{\epsilon_i} \equiv \prod \bmod F_{m+1}$$

such that the exponents $\epsilon_i (1 \leqq i \leqq q(m))$ are functions

$$(3.32) \qquad \epsilon_i = \phi_i(\eta_1, \eta_2, \ldots, \eta_{q_r(m)})$$

which are determined uniquely by the collection process.

Let c_j be a basic commutator of dimension $m > 1$. By the preceding discussion we note the following:

The relator-class $\mathscr{C}_j$ consists of the elements $\prod_1$ corresponding to the exponents $\eta_1, \eta_2, \ldots, \eta_{q_r(m)}$ which are such that

$$(3.33) \qquad \text{(i) } \phi_j(\eta_1, \eta_2, \ldots, \eta_{q_r(m)}) \neq 0$$

and

$$(3.34) \qquad \text{(ii) } \phi_i(\eta_1, \eta_2, \ldots \eta_{q_r(m)}) = 0$$

for $r = q(1) < i < j$.

$\mathscr{C}_j$ is trivial if conditions (i) and (ii) have no solutions.

Furthermore if $\mathscr{C}_j$ is non-trivial, then R_j is an element $\prod_1$ of $\mathscr{C}_j$ which satisfies the additional condition

(iii) ϵ_j is the smallest positive value of the function ϕ_j of the integral exponents η_i subject to the requirements of (3.33) and (3.34).

The relation m-word $\tilde{R}_j$ is then the word (3.30) which is such that the exponents η_i are those found for R_j by the above conditions (i), (ii), and (iii).

4. An example. In this example we will determine $\bar{G}^5$ for

$$(4.1) \qquad G = \langle c_1, c_2; c_1^9, c_2^9 \rangle.$$

G is evidently a homomorphic image of $F = \langle c_1, c_2 \rangle$ which has 14 basic commutators of dimension $\leqq 5$, i.e., the commutators $c_1 < c_2 < \ldots < c_{14}$ in the ordering of Definition 2.2. We note by Definition 3.2 that G gives rise to $\sigma(k)$

relation commutators of pseudo-dimension k such that $\sigma(2) = 2$, $\sigma(3) = 4$, $\sigma(4) = 9$, and $\sigma(5) = 24$.

To determine $\bar{G}^5$, we must tabulate the representatives, R_j, of the relator-classes, $\mathscr{C}_j$. For this purpose we first compute twelve functions $\phi_j(\eta_1, \eta_2, \ldots, \eta_{39})$ according to Remark 3.3 where $3 \leqq j \leqq 14$. These functions are as follows:

$$\phi_3 = 9\eta_1 + 9\eta_2$$
$$\phi_4 = 36\eta_2 + 9\eta_3 + 9\eta_4$$
$$\phi_5 = 36\eta_1 + 9\eta_5 + 9\eta_6$$
$$\phi_6 = 84\eta_2 + 36\eta_4 + 9\eta_{10} + 9\eta_{11}$$
$$\phi_7 = 36\eta_3 + 36\eta_5 + 9\eta_{12} + 9\eta_{13}$$
$$\phi_8 = 84\eta_1 + 36\eta_6 + 9\eta_{14} + 9\eta_{15}$$
$$\phi_9 = 84\eta_2 + 72\eta_3 + 36\eta_8 + 324\eta_9 + 36\eta_{10} + 9(\eta_{16} + \eta_{17} + \eta_{18} + \eta_{21}) + 81(\eta_{19} + \eta_{20} + \eta_{22} + \eta_{23}) + 162\eta_2(\eta_2 - 1)$$
$$\phi_{10} = 204\eta_1 + 120\eta_3 + 36\eta_5 + 36\eta_7 - 324\eta_9 + 9(\eta_{24} + \eta_{25} + \eta_{26} + \eta_{29}) + 81(\eta_{27} + \eta_{28} + \eta_{30} + \eta_{31}) + 162\eta_1(\eta_1 - 1) + 324\eta_1\eta_2$$
$$\phi_{11} = 126\eta_2 + 84\eta_4 + 36\eta_{11} + 9(\eta_{32} + \eta_{33})$$
$$\phi_{12} = 84\eta_5 + 36\eta_{10} + 36\eta_{12} + 9(\eta_{34} + \eta_{35})$$
$$\phi_{13} = 84\eta_3 + 36\eta_{13} + 36\eta_{14} + 9(\eta_{36} + \eta_{37})$$
$$\phi_{14} = 126\eta_1 + 84\eta_6 + 36\eta_{15} + 9(\eta_{38} + \eta_{39})$$

Making use of the functions, ϕ_j, exhibited above, we obtain the R_j in the manner of Remark 3.3. We then find the following:

$$R_3 = c_3^9,\ R_4 = c_4^9,\ R_5 = c_5^9,\ R_6 = c_6^3 c_8^{-3},\ R_7 = c_7^9,$$
$$R_8 = c_8^9,\ R_9 = c_9^9,\ R_{10} = c_{10}^3 c_{11}^{-3} c_{13}^3,\ R_{11} = c_{11}^9,$$
$$R_{12} = c_{12}^3 c_{14}^{-3},\ R_{13} = c_{13}^9,\ R_{14} = c_{14}^9.$$

Having the above R_j at our disposal, we finally determine $\bar{G}^5$ by Theorem 3.4. This yields the following result:

A complete set of coset representatives for $G(\text{mod } G_6)$ consists of those basic 5-words

$$\prod_{i=1}^{14} c_i^{\epsilon_i} \tag{4.2}$$

which are such that

$$\begin{aligned} &0 \leqq \epsilon_i < 9 \quad \text{for } i = 1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14 \\ &0 \leqq \epsilon_i < 3 \quad \text{for } i = 6, 10, 12. \end{aligned} \tag{4.3}$$

https://doi.org/10.4153/CJM-1975-125-9 Published online by Cambridge University Press

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https://doi.org/10.4153/CJM-1975-125-9 Published online by Cambridge University Press