SELF-POLAR DOUBLE CONFIGURATIONS IN PROJECTIVE GEOMETRY

I. A GENERAL CONDITION FOR SELF-POLARITY

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Let

$$X \equiv [x_{\alpha\beta}] \equiv [a_{\alpha\beta\delta}x_\delta], \qquad \begin{matrix} \alpha = 1, \cdots, p \\ \beta = 1, \cdots, q \\ \delta = 0, \cdots, n \end{matrix}$$

be a $p \times q$ matrix of linear forms in the $n+1$ coordinates in a projective space Π_n. Then points which satisfy the q equations

(1) $$\lambda_\alpha x_{\alpha\beta} \equiv \lambda_\alpha a_{\alpha\beta\delta}x_\delta = 0$$

in general span a space Π_{n-q}, but will span a space Π_{n-q+1} if a set $\boldsymbol{\mu}, = \{\mu_\beta\}$, of multipliers can be found such that

(2) $$\mu_\beta \lambda_\alpha a_{\alpha\beta\delta} = 0.$$

Such a set $\boldsymbol{\mu}$ can be found if and only if the equations

(3) $$||\lambda_\alpha a_{\alpha\beta\delta}|| = 0$$

have solutions. (I.e., all $p \times q$ determinants vanish in the $q \times (n+1)$ matrix $[\lambda_\alpha a_{\alpha\beta\delta}]$ of linear forms in the parameters λ_α). Let $\boldsymbol{l} = \{l_\alpha\}$ be a set $\boldsymbol{\lambda}$ which satisfies equations (3); then $\boldsymbol{l}$ determines, as solutions of the linear equations

$$\mu_\beta(l_\alpha a_{\alpha\beta\delta}) = 0,$$

a set $\boldsymbol{m}, = \{m_\beta\}$, such that

$$\boldsymbol{l}^T X \boldsymbol{m} \equiv 0 \text{ in } x_\delta,$$

i.e., such that the points which satisfy the p equations

$$X\boldsymbol{m} = \boldsymbol{0}$$

span a space Π_{n-p+1}.

Thus the spaces Π_{n-q+1} and Π_{n-p+1} associated in this way with the matrix X occur in pairs.

In particular, if $n = p+q-3$ (and the geometry is over the field of complexes), there is a finite number, $N, = \binom{p+q-2}{p-1}$, of homogeneous sets λ satisfying the equations (3). Thus, for $n = p+q-3$, the matrix X determines a set of N pairs of spaces h_i, k_i of dimensions $p-2$ and $q-2$ respectively. It can be proved that h_i and k_j have a common point except when $i = j$, so that X determines a "double-N of Π_{p-2}'s and Π_{q-2}'s in Π_{p+q-3}" (Room [1] p. 72).

The spaces Π_{p-2} and Π_{q-2} in Π_{p+q-3} are of dual dimensions, and the configuration is therefore "formally" self-dual, but it will be "intrinsically" self-dual [1] only if a quadric can be found with regard to which each k_i is the polar of the corresponding h_i. If the linear forms in X are not specially selected, then there is no such quadric except in the cases: $p = 2$, $q = n+1$ (the double-N consists of the vertices and prime (hyperplane) faces of a simplex), and $p = 3$, $q = 3 = n$ (the double-six of lines in Π_3) (Room [1], p. 77).

Up until the present the only non-trivial case of a special selection of linear forms in X which determines a self-polar double-N is that discovered by Coble ([2], p. 447) for $p = 3$, $q = n$ — a special double-$\frac{1}{2}n(n+1)$ of lines and secunda in Π_n, depending on n^2-n-8 fewer parameters than the general double -$\frac{1}{2}n(n+1)$.

The object of this paper is to establish intrinsically self-dual forms for general values of p and q, and this Part of the paper is devoted to the theorem on which later parts depend, namely:

THEOREM I. *A sufficient condition that the double-N determined by the matrix*

$$X = [x_{\alpha\beta}] = [a_{\alpha\beta\delta}x_\delta] \qquad \begin{array}{l} \alpha = 1, \cdots, p \\ \beta = 1, \cdots q \\ \delta = 0, \cdots, p+q-3 \end{array}$$

should be intrinsically self-dual is that, of the quadratic forms

$$\begin{vmatrix} x_{ij} & x_{ij'} \\ x_{i'j} & x_{i'j'} \end{vmatrix}$$

determined by the 2×2 *minors in* X, *only* $\frac{1}{2}(p+q)(p+q-3)$ *are linearly independent.*

There are $\binom{p}{2}\binom{q}{2}$ of these quadratic forms, and, since they are forms in $p+q-2$ homogeneous coordinates, at most $\frac{1}{2}(p+q)(p+q-3)+1$ are linearly independent. The condition for the double-N to be intrinsically self-dual is that the forms shall be linearly dependent on one less than this number.

[1] The terms "formally" and "intrinsically" self-dual are due to Coble [2] p. 436.

When $p = 2$, $q = n+1$ the number of quadratic forms in $\boldsymbol{X}$ is $\frac{1}{2}q(q-1)$, whilst $\frac{1}{2}(p+q)(p+q-3)+1 = \frac{1}{2}(q^2+q)$. There are therefore $q(=n+1)$ fewer linearly independent quadratic forms determined by $\boldsymbol{X}$ than the required number, corresponding to the existence of $n+1$ linearly independent quadrics with regard to which the simplex is self-polar.

When $p = q = n = 3$, the number of quadratic forms in $\boldsymbol{X}$ is 9, whilst the total number of linearly independent forms is 10; the difference, one, between these numbers corresponds to the existence of the single "Schur" quadric with regard to which the double-six of lines in Π_3 is self-polar. (cf. Baker [3], p. 187).

Theorem I is a consequence of the following lemmas all of which are either statements of basic relations or are capable of immediate verification.

LEMMA 1. If $\boldsymbol{L}$ and $\boldsymbol{M}$ are any non-singular matrices of respectively $p\times p$ and $q\times q$ constants and

$$\boldsymbol{X}' = \boldsymbol{LXM},$$

then

(i) $\boldsymbol{X}'$ determines the same double-N as $\boldsymbol{X}$,

(ii) each of the $\binom{p}{2}\binom{q}{2}$ quadratic forms determined by the 2×2 minors of $\boldsymbol{X}'$ is a linear combination of those determined by $\boldsymbol{X}$.

LEMMA 2. If only $\frac{1}{2}(p+q)(p+q-3)$ of the quadratic forms determined by $\boldsymbol{X}$ are linearly independent, then the $\binom{p}{2}\binom{q}{2}$ quadratic forms are all apolar to the same quadratic form.

LEMMA 3. A quadratic form which factorizes as uv is apolar to a quadratic form S if and only if the primes (hyperplanes) $u = 0$, $v = 0$ are conjugate with regard to the tangential quadric $S = 0$.

LEMMA 4. A Π_{p-2} and a Π_{q-2} in Π_{p+q-3} given respectively by $u_2 = \cdots = u_q = 0$ and $v_2 = \cdots = v_p = 0$ are polars with regard to the tangential quadric $S = 0$ if and only if all $(p-1)(q-1)$ quadratic forms $u_s v_r$ are apolar to S.

LEMMA 5. If sets of parameters $\boldsymbol{l}$, $\boldsymbol{m}$ are such that $\boldsymbol{l}^T\boldsymbol{X}\boldsymbol{m} = 0$, and

$$\boldsymbol{X}' = \begin{bmatrix} \boldsymbol{l}^T & \\ \boldsymbol{o} & \boldsymbol{1}_{p-1} \end{bmatrix} \quad \boldsymbol{X} \begin{bmatrix} \boldsymbol{m} & \boldsymbol{o}^T \\ & \boldsymbol{1}_{q-1} \end{bmatrix},$$

where $l_1 m_1 \neq 0$,

then

$$\boldsymbol{X}' = \begin{bmatrix} 0 & u_2 & \cdots & u_q \\ v_2 & x_{22} & \cdots & x_{2q} \\ v_p & x_{p2} & \cdots & x_{pq} \end{bmatrix}$$

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where $u_s = l_\alpha x_{\alpha s}$, $v_r = m_\beta x_{r\beta}$, and the spaces Π_{p-2} and Π_{q-2} whose equations are respectively $u_2 = \cdots = u_q = 0$ and $v_2 = \cdots = v_p = 0$ are a pair of corresponding spaces in the double-N determined by $\boldsymbol{X}$.

LEMMA 6. If only $\frac{1}{2}(p+q)(p+q-3)$ of the quadratic forms determined by $\boldsymbol{X}$ are linearly independent and S is the quadratic form to which they are all apolar, then the quadratic forms $u_s v_r$ defined in Lemma 5, since they are linear combinations of these (Lemma 1(ii)) are all apolar to S. That is, the Π_{p-2} and Π_{q-2} of the pair determined in Lemma 5 are polars with regard to the tangential quadric $S = 0$, and in consequence all pairs of corresponding spaces in the double-N are polars with regard to $S = 0$.

The existence of the form S apolar to the $\binom{p}{2}\binom{q}{2}$ quadratic forms $x_{ij}x_{i'j'} - x_{i'j}x_{ij'}$ is therefore a sufficient condition for the double-N to be intrinsically self-dual.

References

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[3] Baker, H. F., *Principles of Geometry*, Vol. III (Cambridge U.P., 1923).

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