SELF-POLAR DOUBLE CONFIGURATIONS IN PROJECTIVE GEOMETRY

I. A GENERAL CONDITION FOR SELF-POLARITY

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Let

\[ X \equiv \begin{bmatrix} x_{\alpha\beta} \equiv [a_{\alpha\beta}] \end{bmatrix}, \]

\( \alpha = 1, \ldots, p \)

\( \beta = 1, \ldots, q \)

\( \delta = 0, \ldots, n \)

be a \( p \times q \) matrix of linear forms in the \( n+1 \) coordinates in a projective space \( \Pi_n \). Then points which satisfy the \( q \) equations

\[ \lambda_{\alpha} x_{\alpha\beta} \equiv \lambda_{\alpha} a_{\alpha\beta} x_{\delta} = 0 \]

in general span a space \( \Pi_{n-q} \), but will span a space \( \Pi_{n-q+1} \) if a set \( \mu, \{\mu_{\beta}\} \), of multipliers can be found such that

\[ \mu_{\beta} \lambda_{\alpha} a_{\alpha\beta} = 0. \]

Such a set \( \mu \) can be found if and only if the equations

\[ || \lambda_{\alpha} a_{\alpha\beta} || = 0 \]

have solutions. (I.e., all \( p \times q \) determinants vanish in the \( q \times (n+1) \) matrix \( [\lambda_{\alpha} a_{\alpha\beta}] \) of linear forms in the parameters \( \lambda_{\alpha} \). Let \( I = \{l_{\alpha}\} \) be a set \( \lambda \) which satisfies equations (3); then \( I \) determines, as solutions of the linear equations

\[ \mu_{\beta}(l_{\alpha} a_{\alpha\beta}) = 0, \]

a set \( m, \{m_{\beta}\} \), such that

\[ l^\top X m \equiv 0 \text{ in } x_{\delta}, \]

i.e., such that the points which satisfy the \( p \) equations

\[ X m = 0 \]

span a space \( \Pi_{n-p+1} \).

Thus the spaces \( \Pi_{n-q+1} \) and \( \Pi_{n-p+1} \) associated in this way with the matrix \( X \) occur in pairs.

In particular, if \( n = p+q-3 \) (and the geometry is over the field of 65
complexes), there is a finite number, \( N = \binom{p+q-2}{p-1} \), of homogeneous sets \( \mathcal{X} \) satisfying the equations (3). Thus, for \( n = p + q - 3 \), the matrix \( X \) determines a set of \( N \) pairs of spaces \( h_i, k_j \) of dimensions \( p-2 \) and \( q-2 \) respectively. It can be proved that \( h_i \) and \( k_j \) have a common point except when \( i = j \), so that \( X \) determines a "double-N of \( \Pi_{p-2}'s \) and \( \Pi_{q-2}'s \) in \( \Pi_{p+q-3} \)" (Room [1] p. 72).

The spaces \( \Pi_{p-2} \) and \( \Pi_{q-2} \) in \( \Pi_{p+q-3} \) are of dual dimensions, and the configuration is therefore "formally" self-dual, but it will be "intrinsically" self-dual\(^1\) only if a quadric can be found with regard to which each \( k_j \) is the polar of the corresponding \( h_i \). If the linear forms in \( X \) are not specially selected, then there is no such quadric except in the cases: \( p = 2, q = n+1 \) (the double-N consists of the vertices and prime (hyperplane) faces of a simplex), and \( p = 3, q = 3 = n \) (the double-six of lines in \( \Pi_3 \)) (Room [1], p. 77).

Up until the present the only non-trivial case of a special selection of linear forms in \( X \) which determines a self-polar double-N is that discovered by Coble ([2], p. 447) for \( p = 3, q = n \) — a special double-\( \frac{1}{2} n(n+1) \) of lines and secunda in \( \Pi_n \), depending on \( n^2 - n - 8 \) fewer parameters than the general double-\( \frac{1}{2} n(n+1) \).

The object of this paper is to establish intrinsically self-dual forms for general values of \( p \) and \( q \), and this Part of the paper is devoted to the theorem on which later parts depend, namely:

**Theorem I.** A sufficient condition that the double-N determined by the matrix

\[
X = [x_{\alpha \beta}] = [a_{\alpha \beta}\delta x_\delta]
\]

\( \alpha = 1, \cdots, p \)
\( \beta = 1, \cdots, q \)
\( \delta = 0, \cdots, p+q-3 \)

should be intrinsically self-dual is that, of the quadratic forms

\[
\begin{vmatrix}
  x_{ii} & x_{ij}' \\
  x_{ij}' & x_{jj}'
\end{vmatrix}
\]

determined by the \( 2 \times 2 \) minors in \( X \), only \( \frac{1}{2}(p+q)(p+q-3) \) are linearly independent.

There are \( \binom{p}{2}\binom{q}{2} \) of these quadratic forms, and, since they are forms in \( p+q-2 \) homogeneous coordinates, at most \( \frac{1}{2}(p+q)(p+q-3)+1 \) are linearly independent. The condition for the double-N to be intrinsically self-dual is that the forms shall be linearly dependent on one less than this number.

\(^1\) The terms "formally" and "intrinsically" self-dual are due to Coble [2] p. 436.
When \( p = 2, q = n + 1 \) the number of quadratic forms in \( X \) is \( \frac{1}{2}q(q-1) \), whilst \( \frac{1}{2}(p+q)(p+q-3)+1 = \frac{1}{2}(q^2+q) \). There are therefore \( q(=n+1) \) fewer linearly independent quadratic forms determined by \( X \) than the required number, corresponding to the existence of \( n+1 \) linearly independent quadrics with regard to which the simplex is self-polar.

When \( p = q = n = 3 \), the number of quadratic forms in \( X \) is 9, whilst the total number of linearly independent forms is 10; the difference, one, between these numbers corresponds to the existence of the single "Schur" quadric with regard to which the double-six of lines in \( \Pi_3 \) is self-polar. (cf. Baker [3], p. 187).

Theorem I is a consequence of the following lemmas all of which are either statements of basic relations or are capable of immediate verification.

**Lemma 1.** If \( L \) and \( M \) are any non-singular matrices of respectively \( p \times p \) and \( q \times q \) constants and

\[ X' = LXM, \]

then

(i) \( X' \) determines the same double-\( N \) as \( X \),

(ii) each of the \( \binom{p}{2} \binom{q}{2} \) quadratic forms determined by the \( 2 \times 2 \) minors of \( X' \) is a linear combination of those determined by \( X \).

**Lemma 2.** If only \( \frac{1}{2}(p+q)(p+q-3) \) of the quadratic forms determined by \( X \) are linearly independent, then the \( \binom{p}{2} \binom{q}{2} \) quadratic forms are all apolar to the same quadratic form.

**Lemma 3.** A quadratic form which factorizes as \( uv \) is apolar to a quadratic form \( S \) if and only if the primes (hyperplanes) \( u = 0, v = 0 \) are conjugate with regard to the tangential quadric \( S = 0 \).

**Lemma 4.** A \( \Pi_{p-2} \) and a \( \Pi_{q-2} \) in \( \Pi_{p+q-3} \) given respectively by \( u_2 = \cdots = u_q = 0 \) and \( v_2 = \cdots = v_p = 0 \) are polars with regard to the tangential quadric \( S = 0 \) if and only if all \( (p-1)(q-1) \) quadratic forms \( u_i, v_r \) are apolar to \( S \).

**Lemma 5.** If sets of parameters \( l, m \) are such that \( l^TXm = 0 \), and

\[ X' = \begin{bmatrix} l^T \\ o & 1_{p-1} \end{bmatrix} X \begin{bmatrix} m \\ o^T & 1_{q-1} \end{bmatrix}, \]

where \( l_1m_1 \neq 0 \), then

\[ X' = \begin{bmatrix} 0 & u_2 & \cdots & u_q \\ v_2 & x_{22} & \cdots & x_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ v_p & x_{p2} & \cdots & x_{pq} \end{bmatrix} \]
where \( u_s = l_x x_{a_2}, v_r = m_\beta x_{r_\beta} \), and the spaces \( \Pi_{p-2} \) and \( \Pi_{q-2} \) whose equations are respectively \( u_2 = \cdots = u_a = 0 \) and \( v_2 = \cdots = v_p = 0 \) are a pair of corresponding spaces in the double-\( N \) determined by \( X \).

**Lemma 6.** If only \( \frac{1}{2}(p+q)(p+q-3) \) of the quadratic forms determined by \( X \) are linearly independent and \( S \) is the quadratic form to which they are all apolar, then the quadratic forms \( u_s v_r \) defined in Lemma 5, since they are linear combinations of these (Lemma 1(ii)) are all apolar to \( S \). That is, the \( \Pi_{p-2} \) and \( \Pi_{q-2} \) of the pair determined in Lemma 5 are polars with regard to the tangential quadric \( S = 0 \), and in consequence all pairs of corresponding spaces in the double-\( N \) are polars with regard to \( S = 0 \).

The existence of the form \( S \) apolar to the \( \binom{p}{2} \binom{q}{2} \) quadratic forms \( x_{ij} x_{i'j'} - x_{i'j} x_{ij'} \) is therefore a sufficient condition for the double-\( N \) to be intrinsically self-dual.

**References**


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