Appendix A

Partial Waves and Phase Shift Function

This appendix is devoted to a review of the partial wave expansion, comparing to the description of scattering in terms of a phase shift function. We start with the familiar scattering amplitude,

$$f(\theta) = \frac{1}{2ik} \sum_{\ell} (2\ell + 1) \left[e^{2i\delta_\ell} - 1 \right] P_\ell(\cos\theta), \tag{A.1}$$

where $P_\ell(z)$ is the ℓ-th Legendre polynomial, and the δ_ℓ are known as phase shifts. They are often determined from the imposition of boundary conditions on certain solutions to differential equations.

For large values of ℓ and small angles, one has the asymptotic relation

$$\begin{aligned} P_\ell(\cos\theta) &= J_0\left(2(\ell + \tfrac{1}{2}) \sin \tfrac{1}{2}\theta\right) + \tfrac{1}{4} \sin^2 \tfrac{1}{2}\theta^2 + \cdots \\ &= J_0\left((\ell + \tfrac{1}{2})\theta\right) + \mathcal{O}(\theta^2) \end{aligned} \tag{A.2}$$

with $J_0(z)$ the zeroth-order Bessel function. For large values of the angular momenta, we also have the semiclassical correspondence of angular momenta,

$$kb \leftrightarrow \ell + \tfrac{1}{2}, \tag{A.3}$$

with b the impact parameter. Replacing now the sum over ℓ in Eq. (A.1) by an integral over impact parameter b, and identifying the phase shift function $\chi(b)$ as

$$\chi(b) \leftrightarrow 2\delta_\ell, \tag{A.4}$$

with b and ℓ related as in Eq. (A.3), we arrive at

$$f(\theta) = ik \int_0^\infty J_0(2kb \sin \tfrac{1}{2}\theta) \left(1 - e^{i\chi(b)}\right) b \, \mathrm{d}b. \tag{A.5}$$

This can be rewritten in several similar forms. With the momentum transfer

$$q = 2k \sin \tfrac{1}{2}\theta, \tag{A.6}$$

we find

$$f(q) = ik \int_0^\infty J_0(qb)\left(1 - e^{i\chi(b)}\right) b\,\mathrm{d}b \tag{A.7}$$

$$= \frac{ik}{2\pi} \int \mathrm{d}^2 b\, e^{-i\boldsymbol{q}\cdot\boldsymbol{b}} \left(1 - e^{i\chi(b)}\right), \tag{A.8}$$

which is the form discussed in Chapter 2. Away from the forward direction, one can drop the “1,” and get

$$f(q) = \frac{k}{2\pi i} \int \mathrm{d}^2 b\, e^{-i\boldsymbol{q}\cdot\boldsymbol{b}} e^{i\chi(b)}. \tag{A.9}$$