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SOME FIXED AND COMMON FIXED POINT THEOREMS IN METRIC SPACES

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Let (X, d) be a metric space and T_i $(i=1, 2)$ be self mappings of X. The purpose of this paper is to investigate the fixed and common fixed points of T_i, when the pair T_i $(i=1, 2)$ satisfies a condition of the following type:

(1) $$d(T_1x, T_2y) \le \Psi(d(x, T_1x), d(y, T_2y), d(x, y)) \qquad x, y \in X,$$

where Ψ is some real valued function defined on a subset of $R\times R\times R$ (R=reals). Special cases of (1) have been discussed by Rakotch [5] and more recently by Boyd and Wong [1], Fukushime [2], Kannan [3], Maki [4], Reich [6, 7], Sehgal [8], Singh [9], Srivastava and Gupta [10] and others. The results presented here generalize some of the results of these authors.

Throughout this paper, (X, d) is a complete metric space, Q is the closure of the set $\{d(x, y): x, y \in X\}$ and $P=Q\times Q\times Q$. A function $\Psi: P\to R^+$ (non-negative reals) is right continuous iff (a_{n1}, a_{n2}, a_{n3}), $(a_1, a_2, a_3) \in P$ and $a_{nk} \downarrow a_k$, $k=1, 2, 3$ ($\downarrow$= decreasing), then $\Psi(a_{n1}, a_{n2}, a_{n3}) \to \Psi(a_1, a_2, a_3)$. The function Ψ will be called symmetric iff $\Psi(a, b, c)=\Psi(b, a, c)$ for all $(a, b, c) \in P$.

Further, the mappings T_i $(i=1, 2)$ satisfy a (I_1, I_2, Ψ, k) functional inequality iff for each i $(i=1, 2)$, there is a mapping $I_i: T_i\times X \to I^+$ (positive integers) such that if $n(x)=I_1(T_1, x)$ and $m(x)=I_2(T_2, x)$, then

(2) $$d(T_1^{n(x)}x, T_2^{m(y)}y) \le k\Psi(d(x, T_1^{n(x)}x), d(y, T_2^{m(y)}y), d(x, y)),$$

for all $x, y \in X$, where k is some real constant, and $\Psi: P \to R^+$ is a symmetric right continuous function. If (2) holds for $k=1$, then (I_1, I_2, Ψ) will denote $(I_1, I_2, \Psi, 1)$.

THEOREM 1. *Let the mappings $T_i: X\to X$ $(i=1, 2)$, satisfy a (I_1, I_2, Ψ, k) functional inequality for some $k<1$. If (i) $\Psi(a, b, a) \le \max\{a, b\}$, $(a, b, a) \in P$, then there exists a $\xi \in X$ such that*

(3) $$T_1^{n(\xi)} = T_2^{m(\xi)}\xi = \xi.$$

If (ii) $\Psi(0, 0, a) \le a$ for each $a \in Q$, then ξ is unique satisfying (3).

Proof. Let $x_0 \in X$ and define $x_1=T_1^{n(x_0)}x_0$, $x_2=T_2^{m(x_1)}x_1$, and inductively

$$x_{2n} = T_2^{m(x_{2n-1})}x_{2n-1}, \qquad x_{2n+1} = T_1^{n(x_{2n})}x_{2n}.$$

Then,

$$d(x_{2n}, x_{2n+1}) \le k\Psi(d(x_{2n-1}, x_{2n}), (x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n})).$$

Since $k<1$, it follows by (*i*) that

$$d(x_{2n}, x_{2n+1}) \leq kd(x_{2n-1}. x_{2n}). \tag{4}$$

Similarly, we have

$$d(x_{2n-1}, x_{2n}) \leq kd(x_{2n-2}, x_{2n-1}). \tag{5}$$

Thus, $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of reals and it is obvious from (4) and (5) that $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) \to 0$ as $n \to \infty$. It follows therefore, that $\{x_n\}$ is a Cauchy sequence in X. Let $x_n \to \xi$. To show $T_1^{n(\xi)}\xi = T_2^{m(\xi)}\xi = \xi$, choose a subsequence $\{x_{2n(i)}\}$ of the sequence $\{x_{2n}\}$ such that $d(x_{2n(i)}, \xi) \downarrow 0$. Then

$$d(x_{2n(i)+1}, T_2^{m(\xi)}\xi) \leq k\Psi(d(x_{2n(i)}, x_{2n(i)+1}), d(\xi, T_2^{m(\xi)}\xi), d(x_{2n(i)}, \xi))$$

Therefore, as $i \to \infty$, we obtain

$$d(\xi, T_2^{m(\xi)}\xi) \leq k\Psi(0, d(\xi, T_2^{m(\xi)}\xi), 0) \leq kd(\xi, T_2^{m(\xi)}\xi),$$

that is $T_2^{m(\xi)}\xi = \xi$. Choosing a subsequence $\{x_{2n(k)+1}\}$ of the sequence $\{x_{2n+1}\}$ such that $d(x_{2n(k)+1}, \xi) \downarrow 0$, we obtain similarly $T_1^{n(\xi)}\xi = \xi$.

Now, suppose Ψ satisfies (*ii*) and there is a $u \in X$ such that $T_2^{m(u)}u = T_1^{n(u)}u = u$. Then

$$d(\xi, u) = d(T_1^{n(\xi)}\xi, T_2^{m(u)}u) \leq k\Psi(0, 0, d(\xi, u)) \leq kd(\xi, u).$$

Thus ξ is unique element satisfying (3).

If I_i $(i=1, 2)$ are the mappings introduced earlier, then we have

COROLLARY 1. *Let the mappings $T_i: X \to X$ $(i=1, 2)$ satisfy either of the following conditions*

$$d(T_1^{n(x)}x, T_2^{m(y)}y) \leq k \max\{d(x, T_1^{n(x)}x), d(y, T_2^{m(y)}y), d(x, y)\} \tag{6}$$

for some $k < 1$,

$$d(T_1^{n(x)}x, T_2^{m(y)}y) \leq \alpha d(x, T_1^{n(x)}x) + \beta d(y, T_2^{m(y)}y) + \gamma d(x, y), \tag{7}$$

for some non negative reals α, β, γ satisfying $\alpha+\beta+\gamma < 1$.

Then there exists a unique $\xi \in X$ such that $T_1^{n(\xi)}\xi = T_2^{m(\xi)}\xi = \xi$.

Proof. If (6) holds, set $\Psi(a, b, c) = \max\{a, b, c\}$ in Theorem 1. In case of (7), let $k = \alpha+\beta+\gamma$. Then (7) implies (6) and the desired result follows from previous part.

In the special case when I_1 and I_2 are constant mappings, we have

THEOREM 2. *Let for some positive integers m and n, the mappings $T_i: X \to X$ $(i=1, 2)$ satisfy for all $x, y \in X$,*

$$d(T_1^n x, T_2^m y) \leq k\Psi(d(x, T_1^n x), d(y, T_2^m y), d(x, y)) \tag{8}$$

where $k<1$ and the function $\Psi: P \to R^+$ is symmetric and right continuous. If Ψ satisfies condition (i) and (ii) of Theorem 1, then T_i $(i=1, 2)$ have a unique common fixed point $\xi \in X$.

Proof. By Theorem 1, there is a unique $\xi \in X$ such that $T_1^n\xi=T_2^m\xi=$. It now follows from (8) that ξ is the unique fixed point of T_1^n, in fact if $T_1^n u=u$ for some $u \in X$, then

$$d(u, \xi) = d(T_1^n u, T_2^m \xi) \leq k\Psi(0, 0, d(u, \xi)) \leq kd(u, \xi),$$

which implies $\xi=u$. Since $T_1^n(T_1\xi)=T_1\xi$, we have $T_1\xi=\xi$. Similarly, $T_2\xi=\xi$.

COROLLARY 2. *Let for some positive integers m and n, the mappings* $T_i: X \to X$ *satisfy the condition*

$$d(T_1^n x, T_2^m y) \leq k \max\{d(x, T_1^n x), d(y, T_2^m y), d(x, y)\} \tag{9}$$

for $k<1$ *and for all* $x, y \in X$. *Then* T_i $(i=1, 2)$ *have a unique common fixed point in X.*

COROLLARY 3. *If for some positive integers m and n, the mappings* $Ti: X \to X$ $(i=1, 2)$ *satisfy the inequality*

$$d(T_1^n x, T_2^m y) \leq \alpha d(x, T_1^n x)+\beta d(y, T_2^m y)+\gamma d(x, y) \tag{10}$$

for some non-negative reals α, β, γ *with* $\alpha+\beta+\gamma<1$, *then* T_i $(i=1, 2)$ *have a unique common fixed point in X.*

REMARKS. (*i*) If we set $n=m$ and $T_1=T_2=T$ in (10), then we obtain a result of Reich [6]. (ii) if $\gamma=0$ in (10) then Corollary 3 yields a recent result of Srivastava and Gupta [10].

It may be noted that if $k=1$ in (2), the conclusion is no longer valid in Theorem 1. However, if Ψ satisfies a condition similar to Rakotch [5] or Boyd and Wong [1], we could obtain a fixed point theorem for mappings T_i $(i=1, 2)$ satisfying a (I_1, I_2, Ψ) functional inequality. Such results will be published in a subsequent paper.

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