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The magnetohydrodynamic equations in terms of waveframe variables

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Generalising the Elsässer variables, we introduce the Q-variables. These are more flexible than the Elsässer variables, because they also allow us to track waves with phase speeds different than the Alfvén speed. We rewrite the magnetohydrodynamics (MHD) equations with these Q-variables. We consider also the linearised version of the resulting MHD equations in a uniform plasma, and recover the classical Alfvén waves, but also separate the fast and slow magnetosonic waves into upward- and downward-propagating waves. Moreover, we show that the Q-variables may also track the upward- and downward-propagating surface Alfvén waves in a non-uniform plasma, displaying the power of our generalisation. In the end, we lay the mathematical framework for driving solar wind models with a multitude of wave drivers.

Key words: plasma waves, plasma nonlinear phenomena, plasma instabilities

1. Introduction

The Elsässer variables (Elsasser 1950) are expressed as

$$\boldsymbol{Z}^{\pm} = \boldsymbol{V} \pm \boldsymbol{V}_A, \tag{1.1}$$

where $\boldsymbol{V}$ is the speed of the plasma and $\boldsymbol{V}_A = \boldsymbol{B}/\sqrt{\mu\rho}$ is the vectorial Alfvén speed expressed in terms of the magnetic field $\boldsymbol{B}$, density ρ and magnetic permeability μ. In magnetohydrodynamics (MHD), these Elsässer variables play a unique role, which conveniently corresponds to Alfvén waves. A single Alfvén wave may be expressed through a single Elsässer variable.

Because of this convenient property and the prevalence of Alfvén wave turbulence in the solar wind, the Elsässer variables have been used numerous times in the description of the plasma in the solar wind (e.g. Dobrowolny, Mangeney & Veltri 1980; Marsch & Tu 1989; Tu, Marsch & Thieme 1989; Velli, Grappin & Mangeney 1989; Zhou & Matthaeus 1989; Grappin, Mangeney & Marsch 1990; Bruno & Carbone 2013). With the Elsässer variables, it is straightforward to show that Alfvén wave turbulence exists because of the interaction of counterpropagating Alfvén waves (Bruno & Carbone 2013) in incompressible MHD.

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Given the great success of the Elsässer variables, Marsch & Mangeney (1987) have even gone so far as to rewrite the entire set of MHD equations in terms of the independent variables, comprising the Elsässer variables and the density. In that paper, it is clear that the entire machinery of MHD waves can be recovered for this set of equations in terms of Elsässer variables and density. This set of equations offers the possibility to study the evolution of MHD waves through the Elsässer variables. The caveat is that the Elsässer variables are really only well suited to model Alfvén waves.

However, for other waves, the Elsässer variables are less well suited, because other MHD waves necessarily consist of a combination of both Elsässer variables. For example, Magyar, Van Doorsselaere & Goossens (2019*a*) show that this is particularly true for slow and fast magnetosonic waves in a homogeneous plasma. But this statement also holds for most waves in a non-uniform plasma. Ismayilli *et al.* (2022) calculated the Elsässer variables for surface Alfvén waves on a discontinuous interface between two homogeneous plasmas, and clearly show that both Elsässer components are non-zero for this surface Alfvén wave. Moreover, the Elsässer variables are no longer uniquely associated with upward or downward propagation. For instance, an upward-propagating kink wave in a cylindrical plasma has both Elsässer variables co-propagating along the magnetic field (Van Doorsselaere *et al.* 2020). Their continuous interaction would lead to an efficient formation of turbulence, and this turbulence from a unidirectional transverse wave is called uniturbulence (Magyar, Van Doorsselaere & Goossens 2017). To study the nonlinear evolution of such waves in inhomogeneous plasmas, a more general approach than Elsässer variables is needed.

In direct measurements in the solar wind, it has been found many times that the magnetic field fluctuations and the velocity fluctuations are highly correlated, showing that they are highly Alfvénic (Bavassano & Bruno 2000). This is expressed through the Alfvén ratio r_A, which is the ratio of the kinetic energy and the magnetic energy, which is found to be close to 1 close to the Sun. However, it has also been found in solar wind data that the slope of the correlation between the magnetic field fluctuations and the velocity fluctuations is not always 1 (Marsch & Tu 1993). This is potentially because of the presence of other wave modes than Alfvén waves. Thus, also observationally, there is a need for a generalisation of the Elsässer variables.

Here, we consider a generalisation of the Elsässer variables by considering them as co-moving with the wave, using the phase speed as a parameter. We call these the Q-variables. However, the push for a generalisation of Elsässer variables is embraced in the wider community. For example, Galtier (2023) has considered so-called canonical variables. With these canonical variables, he described successfully the interaction and cascade of fast mode waves. Thus, it seems that more general Elsässer variables are possible, and this should be a research question that is actively pursued, given the tremendous impact of the Elsässer variables.

2. Results

2.1. *The MHD equations written in terms of Q-variables*

In what follows, we introduce a new parameter α, which describes the wave phase speed, for a general wave. We then introduce the Q-variables by

$$\boldsymbol{Q}^{\pm} = \boldsymbol{V} \pm \alpha \boldsymbol{B}, \tag{2.1}$$

where it is clear that the limit $\alpha = 1/\sqrt{\mu\rho}$ recovers the special case of Elsässer variables. Taking this limit thus always allows us to check our equations against the relevant equations in Marsch & Mangeney (1987).

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We start from the same set of ideal MHD equations as Marsch & Mangeney (1987) do. They read

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla\boldsymbol{V} = -\frac{1}{\rho}\nabla P_T + \frac{1}{\mu\rho}\boldsymbol{B}\cdot\nabla\boldsymbol{B}, \tag{2.2}$$

$$\frac{\partial \ln\rho}{\partial t} + \boldsymbol{V}\cdot\nabla\ln\rho = -\nabla\cdot\boldsymbol{V}, \tag{2.3}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{B}\nabla\cdot\boldsymbol{V} - \boldsymbol{V}\cdot\nabla\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\boldsymbol{V}, \tag{2.4}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{2.5}$$

where the total pressure is defined as $P_T = p + \frac{1}{2}\rho V_A^2$, using the gas pressure p and vectorial Alfvén speed $\boldsymbol{V}_A = \boldsymbol{B}/\sqrt{\mu\rho}$. They are complimented with an adiabatic assumption for the energy equation

$$p = p(\rho) = p_0(\rho/\rho_0)^\gamma, \tag{2.6}$$

where γ is the adiabatic exponent.

2.1.1. *Solenoidal constraint*

Let us first consider the solenoidal constraint (2.5). We rewrite it in terms of Q-variables, through the expression of $\boldsymbol{B}$ in terms of $\boldsymbol{Q}^\pm$

$$\boldsymbol{B} = \frac{1}{2\alpha}(\boldsymbol{Q}^+ - \boldsymbol{Q}^-). \tag{2.7}$$

Inserting that into (2.5) allows us to write

$$0 = \frac{1}{2\alpha}\nabla\cdot(\boldsymbol{Q}^+ - \boldsymbol{Q}^-) - \frac{1}{2\alpha}(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)\cdot\nabla\ln\alpha, \tag{2.8}$$

or, after simplification,

$$0 = \nabla\cdot(\boldsymbol{Q}^+ - \boldsymbol{Q}^-) - (\boldsymbol{Q}^+ - \boldsymbol{Q}^-)\cdot\nabla\ln\alpha. \tag{2.9}$$

Considering the limiting case of $\alpha^2 = 1/\mu\rho, \boldsymbol{Q}^\pm = \boldsymbol{Z}^\pm, \boldsymbol{Z}^+ - \boldsymbol{Z}^- = \boldsymbol{V}_A$, we recover equation (10) of Marsch & Mangeney (1987).

2.1.2. *Conservation of mass*

Next, we rewrite the conservation of mass (2.3). We use it for finding an expression for $(\mathrm{D}^\pm/\mathrm{D}t)(\ln\rho)$, where $\mathrm{D}^\pm/\mathrm{D}t = \partial/\partial t + \boldsymbol{Q}^\pm\cdot\nabla$ is the derivative co-moving with the wave, in the so-called waveframe. We find

$$\frac{\mathrm{D}^\pm}{\mathrm{D}t}(\ln\rho) = \frac{\partial\ln\rho}{\partial t} + \boldsymbol{Q}^\pm\cdot\nabla\ln\rho, \tag{2.10}$$

$$= \frac{\partial\ln\rho}{\partial t} + \boldsymbol{V}\cdot\nabla\ln\rho \pm \alpha\boldsymbol{B}\cdot\nabla\ln\rho, \tag{2.11}$$

$$= -\nabla\cdot\boldsymbol{V} \pm \alpha\boldsymbol{B}\cdot\nabla\ln\rho, \tag{2.12}$$

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where the continuity equation (2.3) was used in the last equation. To this last equation, we add on the right-hand side ($\mp\times$ (2.9)) to find

$$\frac{\mathrm{D}^{\pm}}{\mathrm{D}t}(\ln\rho) = -\frac{1}{2}\nabla\cdot(\boldsymbol{Q}^{+}+\boldsymbol{Q}^{-}) \mp \nabla\cdot(\boldsymbol{Q}^{+}-\boldsymbol{Q}^{-}) \pm \frac{\boldsymbol{Q}^{+}-\boldsymbol{Q}^{-}}{2}\cdot\nabla\ln\rho\alpha^{2}, \tag{2.13}$$

$$= -\frac{1}{2}\nabla\cdot(3\boldsymbol{Q}^{\pm}-\boldsymbol{Q}^{\mp}) \pm \frac{\boldsymbol{Q}^{+}-\boldsymbol{Q}^{-}}{2}\cdot\nabla\ln\rho\alpha^{2}, \tag{2.14}$$

where we have used the expressions for $\boldsymbol{V}$ in terms of $\boldsymbol{Q}^{\pm}$ in the equations

$$\boldsymbol{V} = \tfrac{1}{2}(\boldsymbol{Q}^{+}+\boldsymbol{Q}^{-}). \tag{2.15}$$

When the limit of $\alpha^2 \to 1/\sqrt{\mu\rho}$ is considered, the last term of (2.14) cancels out and (16) of Marsch & Mangeney (1987) is readily recovered.

2.1.3. *Momentum equation*

Now we turn to the momentum equation and the induction equation, (2.2) and (2.4), which form the key equation (17) of Marsch & Mangeney (1987). Following their lead, we add (2.2)$\pm\alpha$(2.4). In the first step, we use the expansion of $\boldsymbol{Q}^{\mp}\cdot\nabla\boldsymbol{Q}^{\pm}$ as

$$\boldsymbol{Q}^{\mp}\cdot\nabla\boldsymbol{Q}^{\pm} = \boldsymbol{V}\cdot\nabla\boldsymbol{V} \mp \alpha\boldsymbol{B}\cdot\nabla\boldsymbol{V} \pm \alpha\boldsymbol{V}\cdot\nabla\boldsymbol{B} - \alpha^{2}\boldsymbol{B}\cdot\nabla\boldsymbol{B} \pm \boldsymbol{B}\boldsymbol{Q}^{\mp}\cdot\nabla\alpha, \tag{2.16}$$

where we have used the vector identity $\boldsymbol{C}\cdot\nabla(f\boldsymbol{D}) = f\boldsymbol{C}\cdot\nabla\boldsymbol{D} + \boldsymbol{D}(\boldsymbol{C}\cdot\nabla f)$ for any vector fields $\boldsymbol{C}$ and $\boldsymbol{D}$ and scalar field f. We also define the parameter

$$\Delta\alpha^{2} = \alpha^{2} - \frac{1}{\mu\rho}. \tag{2.17}$$

The $\Delta\alpha^2$ parameter expresses how far a wave's phase speed is from the Alfvén speed. Since a wave can be slower or faster than the Alfvén speed, the $\Delta\alpha^2$ parameter may be positive or negative, despite the square! The square in the notation is kept for dimensional purposes to keep $\Delta\alpha$ in the same units as α. In the limit of $\alpha = 1/\sqrt{\mu\rho}$, the parameter $\Delta\alpha^2$ will turn to 0: $\Delta\alpha^2 = 0$ and $\boldsymbol{Q}^{\pm} = \boldsymbol{Z}^{\pm}$ turns into the classical Elsässer variable. Here, it is also useful to point out that it will be convenient to use expressions with $\rho\alpha^2$, which are constant in this limit.

With the above expressions, we obtain from (2.2)$\pm\alpha$(2.4) the result

$$\frac{\partial\boldsymbol{Q}^{\pm}}{\partial t} \mp \boldsymbol{B}\frac{\partial\alpha}{\partial t} = -\boldsymbol{Q}^{\mp}\cdot\nabla\boldsymbol{Q}^{\pm} \pm \boldsymbol{B}\boldsymbol{Q}^{\mp}\cdot\nabla\alpha - \Delta\alpha^{2}\boldsymbol{B}\cdot\nabla\boldsymbol{B} - \frac{1}{\rho}\nabla P_{T} \mp \alpha\boldsymbol{B}\nabla\cdot\boldsymbol{V}. \tag{2.18}$$

The first two terms on the right-hand side group with the left-hand side to form the co-moving derivative

$$\frac{\mathrm{D}^{\mp}}{\mathrm{D}t}\boldsymbol{Q}^{\pm} \mp \boldsymbol{B}\frac{\mathrm{D}^{\mp}}{\mathrm{D}t}\alpha = -\frac{1}{\rho}\nabla P_{T} - \Delta\alpha^{2}\boldsymbol{B}\cdot\nabla\boldsymbol{B} \mp \alpha\boldsymbol{B}\nabla\cdot\boldsymbol{V}. \tag{2.19}$$

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We now find an expression for the terms on the right-hand side. For the total pressure term, we find

$$\frac{1}{\rho}\nabla P_T = \frac{1}{\rho}\nabla\left(p + \frac{B^2}{2\mu}\right), \tag{2.20}$$

$$\begin{aligned} &= v_s^2\nabla\ln\rho + \frac{1}{8\alpha^2}(\alpha^2 - \Delta\alpha^2)\nabla(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)^2 \\ &\quad + \frac{1}{8}(\alpha^2 - \Delta\alpha^2)(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)^2\nabla\left(\frac{1}{\alpha^2}\right), \end{aligned} \tag{2.21}$$

$$\begin{aligned} &= v_s^2\nabla\ln\rho + \frac{1}{8}\left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)\nabla(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)^2 \\ &\quad - \frac{1}{4}\left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)^2\nabla\ln\alpha, \end{aligned} \tag{2.22}$$

where we have used the adiabatic relationship of $p(\rho)$ which introduces the expression for the sound speed $v_s = \sqrt{\gamma p/\rho}$.

The second term on the right-hand side of (2.19) can be rewritten with the expression for $\boldsymbol{B}$ in terms of $\boldsymbol{Q}^\pm$ as

$$\begin{aligned} -\Delta\alpha^2\boldsymbol{B}\cdot\nabla\boldsymbol{B} = &-\frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)\cdot\nabla(\boldsymbol{Q}^+ - \boldsymbol{Q}^-) \\ &+ \frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)((\boldsymbol{Q}^+ - \boldsymbol{Q}^-)\cdot\nabla\ln\alpha). \end{aligned} \tag{2.23}$$

The third term on the right-hand side of (2.19) should be handled through the modified version of the continuity relation (2.13). From that equation, we have that

$$\begin{aligned} \mp\alpha\boldsymbol{B}\nabla\cdot\boldsymbol{V} = &\pm\left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\frac{\mathrm{D}^\mp}{\mathrm{D}t}(\ln\rho) - \left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\nabla\cdot(\boldsymbol{Q}^+ - \boldsymbol{Q}^-) \\ &+ \left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\left(\left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\cdot\nabla\ln\rho\alpha^2\right). \end{aligned} \tag{2.24}$$

Substituting everything in (2.19), we now have

$$\begin{aligned} \frac{\mathrm{D}^\mp}{\mathrm{D}t}\boldsymbol{Q}^\pm \mp \left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\frac{\mathrm{D}^\mp}{\mathrm{D}t}\ln\alpha = &-v_s^2\nabla\ln\rho - \frac{1}{8}\left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)\nabla(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)^2 \\ &+ \frac{1}{4}\left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)^2\nabla\ln\alpha - \frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)\cdot\nabla(\boldsymbol{Q}^+ - \boldsymbol{Q}^-) \\ &+ \frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)((\boldsymbol{Q}^+ - \boldsymbol{Q}^-)\cdot\nabla\ln\alpha) \pm \left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\frac{\mathrm{D}^\mp}{\mathrm{D}t}(\ln\rho) \\ &- \left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\nabla\cdot(\boldsymbol{Q}^+ - \boldsymbol{Q}^-) + \left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\left(\left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\cdot\nabla\ln\rho\alpha^2\right). \end{aligned} \tag{2.25}$$

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After moving the right-hand side convective derivative to the left-hand side and subsequently adding ($\pm((\boldsymbol{Q}^+ - \boldsymbol{Q}^-)/4)\times$ (2.14)) and using (2.9), we obtain the final result

$$\begin{aligned}\frac{\mathrm{D}^{\mp}}{\mathrm{D}t}\boldsymbol{Q}^{\pm} \mp \left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{4}\right)\frac{\mathrm{D}^{\mp}}{\mathrm{D}t}\ln\rho\alpha^2 &= -v_s^2\nabla\ln\rho - \frac{1}{8}\left(1-\frac{\Delta\alpha^2}{\alpha^2}\right)\nabla(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)^2\\
&\quad + \frac{1}{4}\left(1-\frac{\Delta\alpha^2}{\alpha^2}\right)(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)^2\nabla\ln\alpha\\
&\quad - \frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)\boldsymbol{\cdot}\nabla(\boldsymbol{Q}^+ - \boldsymbol{Q}^-) + \frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)\nabla\boldsymbol{\cdot}(\boldsymbol{Q}^+ - \boldsymbol{Q}^-)\\
&\quad \mp \left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{8}\right)\nabla\boldsymbol{\cdot}(3\boldsymbol{Q}^{\pm} - \boldsymbol{Q}^{\mp}) + \left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{4}\right)\left(\left(\frac{\boldsymbol{Q}^+ - \boldsymbol{Q}^-}{2}\right)\boldsymbol{\cdot}\nabla\ln\rho\alpha^2\right).\end{aligned} \tag{2.26}$$

Taking the limit of $\alpha = \sqrt{1/\mu\rho}$ allows us to confirm that this equation converges in that case to equation (17) of Marsch & Mangeney (1987). The last term on the left-hand side and the last three terms on the right-hand side are terms parallel to the magnetic field. We remind the reader that these equations are valid for any choice of α (satisfying basic dimensional arguments).

2.2. *Linearised Q-equations*

In a first attempt to better understand the Q-variables and the role that α plays in the MHD equations, we shall linearise the MHD equations (2.9), (2.14), (2.26) around a uniform equilibrium. We take $\rho = \rho_0 + \delta\rho$, $\boldsymbol{B} = B_0\boldsymbol{e}_z + \boldsymbol{\delta B}$, $\boldsymbol{V} = \boldsymbol{V_0} + \boldsymbol{\delta V}$, $\boldsymbol{Q}^{\pm} = \boldsymbol{Q}_0^{\pm} + \boldsymbol{\delta Q}^{\pm}$, where quantities with subscript 0 are constant equilibrium quantities, and δ indicates Eulerian perturbations (where we have used the Chandrasekhar notation for such). The Cartesian coordinate system (x, y, z) is aligned with the magnetic field in the z-direction. We have not linearised α, because we shall show later that it is proportional to the phase speed of the wave. Moreover, a linearisation of α would result in terms rewritten from $\delta\rho$ and other physical parameters, and consequently the equation for the linearised α would be linearly dependent on the previous equations.

Adopting a similar notation as Marsch & Mangeney (1987), we have

$$\nabla\ln\rho = \nabla\ln\left(\rho_0\left(1+\frac{\delta\rho}{\rho_0}\right)\right) = \nabla\frac{\delta\rho}{\rho_0} \equiv \nabla\delta R, \tag{2.27}$$

$$\nabla\ln\rho\alpha^2 = \nabla\ln\rho + \nabla\ln\alpha^2 = \nabla\delta R. \tag{2.28}$$

We have utilised that the background variables are uniform, and that α does not need to be linearised. We have also rejected any terms higher than the first order in perturbations and defined the quantity δR. Additionally we linearise the co-moving advective derivative $D^{\pm}/Dt$ as

$$\frac{\mathrm{D}^{\pm}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{Q}_0^{\pm}\boldsymbol{\cdot}\nabla + \boldsymbol{\delta Q}^{\pm}\boldsymbol{\cdot}\nabla \equiv \frac{\mathrm{d}^{\pm}}{\mathrm{d}t} + \boldsymbol{\delta Q}^{\pm}\boldsymbol{\cdot}\nabla, \tag{2.29}$$

where the notation of Marsch & Mangeney (1987) was once again used to define $\mathrm{d}^{\pm}/\mathrm{d}t$. Note also that the last term always results in 0 when operating on equilibrium quantities, given their assumed homogeneity. Action of the last term on linear quantities results in a second-order contribution, which is neglected. With this notation, the MHD equations are

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rewritten as

$$\begin{aligned}\frac{\mathrm{d}^{\mp}}{\mathrm{d}t}\delta\boldsymbol{Q}^{\pm} \mp \left(\frac{\boldsymbol{Q}_0^{+}-\boldsymbol{Q}_0^{-}}{4}\right)\frac{\mathrm{d}^{\mp}}{\mathrm{d}t}\delta R = {}& -v_{s0}^2\nabla\delta R \\ & -\frac{1}{4}\left(1-\frac{\Delta\alpha^2}{\alpha^2}\right)\nabla((\delta\boldsymbol{Q}^{+}-\delta\boldsymbol{Q}^{-})\boldsymbol{\cdot}(\boldsymbol{Q}_0^{+}-\boldsymbol{Q}_0^{-})) \\ & -\frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(\boldsymbol{Q}_0^{+}-\boldsymbol{Q}_0^{-})\boldsymbol{\cdot}\nabla(\delta\boldsymbol{Q}^{+}-\delta\boldsymbol{Q}^{-}) \\ & +\frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(\boldsymbol{Q}_0^{+}-\boldsymbol{Q}_0^{-})\nabla\boldsymbol{\cdot}(\delta\boldsymbol{Q}^{+}-\delta\boldsymbol{Q}^{-}) \\ & \mp\left(\frac{\boldsymbol{Q}_0^{+}-\boldsymbol{Q}_0^{-}}{8}\right)\nabla\boldsymbol{\cdot}(3\delta\boldsymbol{Q}^{\pm}-\delta\boldsymbol{Q}^{\mp}) + \left(\frac{\boldsymbol{Q}_0^{+}-\boldsymbol{Q}_0^{-}}{4}\right)\left(\left(\frac{\boldsymbol{Q}_0^{+}-\boldsymbol{Q}_0^{-}}{2}\right)\boldsymbol{\cdot}\nabla\delta R\right),\end{aligned} \tag{2.30}$$

$$\frac{\mathrm{d}^{\pm}}{\mathrm{d}t}\delta R = -\frac{1}{2}\nabla\boldsymbol{\cdot}(3\delta\boldsymbol{Q}^{\pm}-\delta\boldsymbol{Q}^{\mp}) \pm \left(\frac{\boldsymbol{Q}_0^{+}-\boldsymbol{Q}_0^{-}}{2}\right)\boldsymbol{\cdot}\nabla\delta R, \tag{2.31}$$

$$0 = \nabla\boldsymbol{\cdot}(\delta\boldsymbol{Q}^{+}-\delta\boldsymbol{Q}^{-}). \tag{2.32}$$

Given the homogeneity, the linear wave solutions may be written with the plane wave notation $\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\mathrm{i}\omega t)$, where we choose the x-axis to be in the $\boldsymbol{k}-\boldsymbol{B}_0$-plane resulting in $k_y \equiv 0$. For the plane waves, the co-moving derivative is rewritten as $\mathrm{d}^{\pm}/\mathrm{d}t = -\mathrm{i}(\omega - \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{Q}_0^{\pm}) \equiv -\mathrm{i}\omega^{\pm}$, where we have yet again used the notation of Marsch & Mangeney (1987). With these notations, we can split the Q-equations (2.30)–(2.31) into its components

$$-\omega^{\mp}\delta R = -\frac{1}{2}k_x(3\delta Q_x^{\mp}-\delta Q_x^{\pm}) - \frac{1}{2}k_z(3\delta Q_z^{\mp}-\delta Q_z^{\pm}) \mp \alpha B_0 k_z\delta R, \tag{2.33}$$

$$\begin{aligned}-\omega^{\mp}\delta Q_x^{\pm} = {}& -v_{s0}^2 k_x\delta R - \frac{1}{2}\left(1-\frac{\Delta\alpha^2}{\alpha^2}\right)\alpha B_0 k_x(\delta Q_z^{+}-\delta Q_z^{-}) \\ & -\frac{1}{2}\frac{\Delta\alpha^2}{\alpha^2}\alpha B_0 k_z(\delta Q_x^{+}-\delta Q_x^{-}),\end{aligned} \tag{2.34}$$

$$-\omega^{\mp}\delta Q_y^{\pm} = -\frac{1}{2}\frac{\Delta\alpha^2}{\alpha^2}\alpha B_0 k_z(\delta Q_y^{+}-\delta Q_y^{-}), \tag{2.35}$$

$$\begin{aligned}-\omega^{\mp}\delta Q_z^{\pm} \pm \frac{1}{2}\alpha B_0\omega^{\mp}\delta R = {}& -v_{s0}^2 k_z\delta R - \frac{1}{2}\left(1-\frac{\Delta\alpha^2}{\alpha^2}\right)\alpha B_0 k_z(\delta Q_z^{+}-\delta Q_z^{-}) \\ & +\frac{1}{2}\frac{\Delta\alpha^2}{\alpha^2}\alpha B_0 k_x(\delta Q_x^{+}-\delta Q_x^{-}) \mp \frac{1}{4}\alpha B_0(k_x(3\delta Q_x^{\pm}-\delta Q_x^{\mp}) \\ & + k_z(3\delta Q_z^{\pm}-\delta Q_z^{\mp})) + \frac{1}{2}\alpha^2 B_0^2 k_z\delta R,\end{aligned} \tag{2.36}$$

which form a system of 7 equations for 7 unknowns. It has eigenvalue ω. Remember, in these equations, α can still be chosen freely!

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2.2.1. *Alfvén waves*

As expected, the y-component (2.35) is separated from the other equations. This equation is rewritten in the following system:

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{Q}_0^-)\delta Q_y^+ = \frac{1}{2}\frac{\Delta\alpha^2}{\alpha^2}\alpha B_0 k_z(\delta Q_y^+ - \delta Q_y^-), \tag{2.37}$$

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{Q}_0^+)\delta Q_y^- = \frac{1}{2}\frac{\Delta\alpha^2}{\alpha^2}\alpha B_0 k_z(\delta Q_y^+ - \delta Q_y^-), \tag{2.38}$$

resulting in a dispersion relation

$$\omega^2 - \boldsymbol{k}\cdot(\boldsymbol{Q}_0^+ + \boldsymbol{Q}_0^-)\omega + (\boldsymbol{k}\cdot\boldsymbol{Q}_0^+)(\boldsymbol{k}\cdot\boldsymbol{Q}_0^-) + \frac{1}{2}\boldsymbol{k}\cdot(\boldsymbol{Q}_0^+ - \boldsymbol{Q}_0^-)\frac{\Delta\alpha^2}{\alpha}B_0 k_z = 0, \tag{2.39}$$

with solutions

$$\omega = \boldsymbol{k}\cdot\boldsymbol{V}_0 \pm \sqrt{(\boldsymbol{k}\cdot\boldsymbol{V}_0)^2 - (\boldsymbol{k}\cdot\boldsymbol{Q}_0^+)(\boldsymbol{k}\cdot\boldsymbol{Q}_0^-) - \Delta\alpha^2 k_z^2 B_0^2} \tag{2.40}$$

$$= \boldsymbol{k}\cdot\boldsymbol{V}_0 \pm \sqrt{\left(\frac{\boldsymbol{k}}{2}\cdot(\boldsymbol{Q}_0^+ - \boldsymbol{Q}_0^-)\right)^2 - \Delta\alpha^2 k_z^2 B_0^2} \tag{2.41}$$

$$= \boldsymbol{k}\cdot\boldsymbol{V}_0 \pm k_z B_0\sqrt{\alpha^2 - \Delta\alpha^2} \tag{2.42}$$

$$= \boldsymbol{k}\cdot\boldsymbol{V}_0 \pm \frac{k_z B_0}{\sqrt{\mu\rho_0}}, \tag{2.43}$$

which nicely converges to the well-known Alfvén wave solution $\omega = \boldsymbol{k}\cdot(\boldsymbol{V}_0 \pm \boldsymbol{V}_A) = \boldsymbol{k}\cdot\boldsymbol{Z}_0^{\pm}$.

This subsection also points us in the direction of the meaning and importance of the α parameter. If we would change variables to the co-moving frame (co-moving with $\boldsymbol{Q}_0^{\pm}$), then that frame would require that either $\omega^{\pm} = 0$ separately. Implementing these conditions in (2.37) and (2.38), leads to the (single) condition

$$\frac{1}{2}\frac{\Delta\alpha^2}{\alpha^2}\alpha B_0 k_z(\delta Q_y^+ - \delta Q_y^-) = 0. \tag{2.44}$$

From this condition, we obtain that $(\delta Q_y^+ - \delta Q_y^-) \equiv 0$ or that $\Delta\alpha^2 \equiv 0$. The former condition would lead to $\delta Q_y^{\pm} \equiv 0$ through the companion equation (e.g. (2.38) for $\omega^- = 0$), which tells us that there is no physical solution with non-zero amplitude. The latter condition $\Delta\alpha^2 = 0$ leads to the well-known solution $\alpha^2 = 1/\mu\rho_0$, which is equivalent to the limit where the Q-variables coincide with the Elsässer variables. This thus shows that the Elsässer variables are the only co-propagating waveframe variables in which the Alfvén waves have a non-zero amplitude. It shows that α should be chosen according to the phase speed, through the solution of $\omega^{\pm} = 0$

$$0 = \omega^{\pm} = \omega - \boldsymbol{k}\cdot\boldsymbol{Q}_0^{\pm} = \omega - \boldsymbol{k}\cdot\boldsymbol{V}_0 \mp \alpha\boldsymbol{k}\cdot\boldsymbol{B}_0, \tag{2.45}$$

resulting in an expression for α

$$\alpha = \pm\frac{\omega - \boldsymbol{k}\cdot\boldsymbol{V}_0}{\boldsymbol{k}\cdot\boldsymbol{B}_0}. \tag{2.46}$$

The reader is cautioned to be careful with this expression, given that the expression diverges if $k \to 0$ or perpendicular $\boldsymbol{k}$ and $\boldsymbol{B}_0$.

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2.2.2. *Magnetoacoustic waves*

Let us now investigate magnetoacoustic waves as they appear in terms of Q-variables. For the specific geometry chosen without loss of generality in § 2.2, linear magnetoacoustic modes perturb the Q-variables in the x–z plane, and density. The system of equations to be solved for magnetoacoustic modes is composed of (2.33)–(2.36), except 2.35, which were treated in the previous subsection, yielding Alfvén waves. The dispersion relation is given by the determinant of this system of 5 equations for 5 unknowns. However, it turns out that, in this system, there are only 4 independent equations, (2.33) being linearly dependent on the other equations. Instead, we use the linearised solenoidal constraint (2.32) as a fifth equation

$$k_x(\delta Q_x^+ - \delta Q_x^-) + k_z(\delta Q_z^+ - \delta Q_z^-) = 0. \tag{2.47}$$

Next, we use the standard dispersion relation of magnetosonic waves. Assuming that $|k| = 1$, so that $k_z = \cos(\theta)$ and $k_x = \sin(\theta)$, with θ being the angle between the background magnetic field $B_0\boldsymbol{e}_z$ and the wavevector $\boldsymbol{k}$, and that there are no background flows $V_0 = 0$, the dispersion relation is

$$\alpha B_0 \cos(\theta)(V_{A0}^2 v_{s0}^2 \cos^2(\theta) - \omega^2(V_{A0}^2 + v_{s0}^2) + \omega^4) = 0. \tag{2.48}$$

Note that we have not yet assumed any form for α, which is not needed for isolating the magnetoacoustic solutions. If we assume a form for α like in (2.46), we recover the fifth, trivial, solution of the dispersion relation, $\omega = 0$, the entropy wave, which represents non-propagating perturbations of plasma density and temperature. The other four solutions are the up- and downward-propagating (with respect to $\boldsymbol{e}_z$) fast and slow magnetoacoustic modes, as found also elsewhere through e.g. the velocity representation of MHD (Goedbloed & Poedts 2004)

$$\omega_{s,f} = \pm_{\mathrm{ud}} \sqrt{\frac{1}{2}(V_{A0}^2 + v_{s0}^2)} \sqrt{1 \pm_{\mathrm{sf}} \sqrt{1 - \frac{\cos^2(\theta)(4V_{A0}^2 v_{s0}^2)}{(V_{A0}^2 + v_{s0}^2)^2}}}. \tag{2.49}$$

Here, we have 4 solutions with the symbol $\pm_{\mathrm{ud}}$ differentiating between upward- and downward-propagating waves, and the symbol $\pm_{\mathrm{sf}}$ is the usual differentiation between the slow and fast magnetoacoustic waves. Recovering the magnetoacoustic solutions demonstrates the validity of the formulation of compressible MHD equations in terms of the Q-variables.

Using the eigenvalues in terms of ω (2.49), the eigenfunctions for $\boldsymbol{Q}^{\pm}$ can be determined for fast and slow waves from (2.30)–(2.32). The Q-variables can also be computed directly from the velocity and magnetic field eigenfunctions, if we assume a form for α. Note that the definition of α from (2.46) diverges for purely perpendicularly propagating ($k_z = 0$) fast waves, thus this definition is not suitable for fast waves. This uncovers a curious property of (2.33)–(2.36), in that advection (in the form of the co-moving advective derivative) is only explicitly present along the magnetic field, leaving the definition of the phase speed in α only in terms of k_z. A straightforward remedy is then to use the full magnitude of the wavevector instead of only the k_z component in the definition of $\alpha = \omega k^{-1} B_0^{-1}$.

In figure 1 we represent the parallel and perpendicular eigenfunctions of the Q-variables for fast and slow waves. From this figure, it is clear that only the perpendicular components are separated as a function of propagation direction with respect to the background magnetic field. In other words, $Q_{s,f\perp}^+$ is non-zero only when $\boldsymbol{k} \cdot \boldsymbol{B_0} > 0$, and $Q_{s,f\perp}^-$ is

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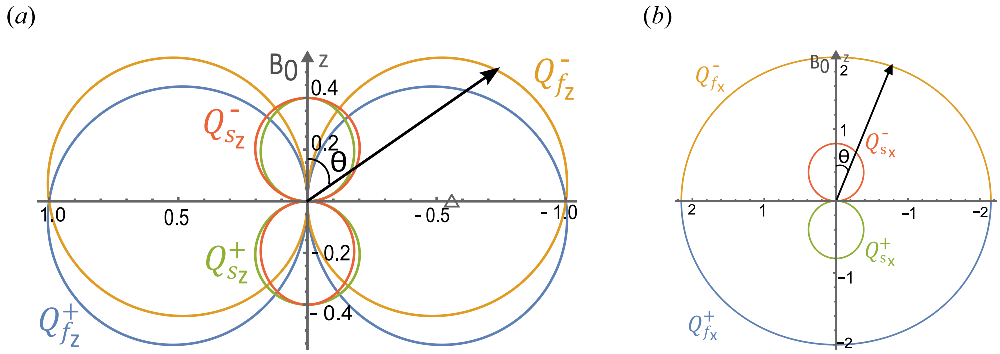

FIGURE 1. Polar plots representing the θ-dependence of the magnitude of the parallel (a) and perpendicular (b) $Q^{\pm}$-variables, for both fast and slow waves, indicated with the subscripts f and s, respectively. The magnitudes are normalised by multiplying with the phase speed $\omega_{s,f}/|k|$ where $|k| = 1$. Here, the plasma-β is set to 0.2.

non-zero for $\boldsymbol{k}\cdot\boldsymbol{B_0} < 0$. The parallel components $Q^{\pm}_{s,f\|}$ are generally both perturbed, thus, based on the present form of the Q-variables, parallel perturbations cannot be separated into parallel- and anti-parallel-propagating components. The parallel components $Q^{\pm}_{s,f\|}$ vanish only for purely parallel-propagating fast waves, which are just Alfvén waves polarised in plane. In the next section (§ 2.2.3), we show that this is because of the connection of $Q_{\|}$ to the magnetic pressure.

We conjecture that the full separation of waves, including the component parallel to the background field, is possible by constructing a waveframe variable which includes the total pressure or density perturbation as well in its formulation, but this will solely work in a homogeneous plasma where such a neat separation is possible.

2.2.3. *Kink waves*

In order to model kink waves, we start from (2.30)–(2.31), written out in components. Once again, we use the same frame of reference: the magnetic field $\boldsymbol{B}_0$ is pointing in the z-direction, and we also take the flow in the z-direction $\boldsymbol{V}_0 = V_0\boldsymbol{e}_z$. Additionally, we take the assumption of a pressureless plasma $v_s = 0$ and we take a density step function at $x = 0$, with a constant density ρ_L (ρ_R) on the left (right) side of the interface. In each half-space, the waves may be Fourier analysed in y, z and t, putting every quantity proportional to $\exp(\mathrm{i}k_z z - \mathrm{i}\omega t)$, where we have once again considered $k_y \equiv 0$ as in § 2.2. The resulting equations will be just like (2.33)–(2.36), except that the terms with k_x will be replaced by a derivative $\mathrm{d}/\mathrm{d}x$. In what follows, we ignore (2.35), because we will not concentrate on the Alfvén waves, but rather on the kink waves, which are solely polarised in the x, z-directions for $k_y = 0$.

Following the earlier strategy, we take (e.g.) $\omega^{+}_{L,R} = 0$ to find the upward-propagating kink waves. This immediately implies a connection

$$V_L + \alpha_L B_L = V_R + \alpha_R B_R, \tag{2.50}$$

between $\alpha_{R,L}$. Each quantity in this equation is the corresponding background quantity in the left half-space or right half-space, respectively, for subscripts L and R. With this

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assumption, we then have the following set of equations:

$$0 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(3\delta Q_x^+ - \delta Q_x^-) - \frac{1}{2}k_z(3\delta Q_z^+ - \delta Q_z^-) + \alpha B_0 k_z \delta R, \tag{2.51}$$

$$\begin{aligned} 0 = &-\frac{1}{4}\left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)(Q_0^+ - Q_0^-)\frac{\mathrm{d}}{\mathrm{d}x}(\delta Q_z^+ - \delta Q_z^-) \\ &- \frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(Q_0^+ - Q_0^-)\mathrm{i}k_z(\delta Q_x^+ - \delta Q_x^-), \end{aligned} \tag{2.52}$$

$$\begin{aligned} -\mathrm{i}k_z(Q_0^+ - Q_0^-)\delta Q_x^+ = &-\frac{1}{4}\left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)(Q_0^+ - Q_0^-)\frac{\mathrm{d}}{\mathrm{d}x}(\delta Q_z^+ - \delta Q_z^-) \\ &- \frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(Q_0^+ - Q_0^-)\mathrm{i}k_z(\delta Q_x^+ - \delta Q_x^-), \end{aligned} \tag{2.53}$$

$$\begin{aligned} &-\frac{Q_0^+ - Q_0^-}{4}\left[\frac{\mathrm{d}}{\mathrm{d}x}(\delta Q_x^+ + \delta Q_x^-) + \mathrm{i}k_z(\delta Q_z^+ + \delta Q_z^-)\right] \\ &= -\frac{1}{4}\left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)(Q_0^+ - Q_0^-)\mathrm{i}k_z(\delta Q_z^+ - \delta Q_z^-) \\ &\quad + \frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(Q_0^+ - Q_0^-)\frac{\mathrm{d}}{\mathrm{d}x}(\delta Q_x^+ - \delta Q_x^-), \end{aligned} \tag{2.54}$$

$$\begin{aligned} &-\mathrm{i}k_z(Q_0^+ - Q_0^-)\delta Q_z^+ + \frac{Q_0^+ - Q_0^-}{4}\left[\frac{\mathrm{d}}{\mathrm{d}x}(\delta Q_x^+ + \delta Q_x^-) + \mathrm{i}k_z(\delta Q_z^+ + \delta Q_z^-)\right] \\ &= -\frac{1}{4}\left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)(Q_0^+ - Q_0^-)\mathrm{i}k_z(\delta Q_z^+ - \delta Q_z^-) \\ &\quad + \frac{1}{4}\frac{\Delta\alpha^2}{\alpha^2}(Q_0^+ - Q_0^-)\frac{\mathrm{d}}{\mathrm{d}x}(\delta Q_x^+ - \delta Q_x^-), \end{aligned} \tag{2.55}$$

in which all quantities are subscripted with R and L, respectively, for each half-space. Combining (2.53) and (2.52) isolates δQ_x^+ as

$$\mathrm{i}k_z(Q_0^+ - Q_0^-)\delta Q_x^+ = 0, \tag{2.56}$$

showing that the kink wave is uniquely described by δQ_x^- only, because δQ_x^+ is 0 if $k_z \neq 0$ and $B_0 \neq 0$. If we find a value for $\alpha_{R,L}$ and ω, then the kink wave is written with only one of $\delta Q_x^{\pm}$, as was the intention of the Q-variables for separating upward- and downward-propagating waves. Similarly, from the combination of (2.54) and (2.55) (and using $\delta Q_x^+ = 0$), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta Q_x^- - \mathrm{i}k_z(\delta Q_z^+ - \delta Q_z^-) = 0. \tag{2.57}$$

Thus, we obtain a set of equations describing the kink waves (or any other wave under these assumptions) from (2.52) and (2.57)

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\delta Q_x^- - \mathrm{i}k_z(\delta Q_z^+ - \delta Q_z^-), \tag{2.58}$$

$$0 = \left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}(\delta Q_z^+ - \delta Q_z^-) - \frac{\Delta\alpha^2}{\alpha^2}\mathrm{i}k_z\delta Q_x^-. \tag{2.59}$$

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Introducing a new variable $\varPi = \delta Q_z^+ - \delta Q_z^-$, we obtain the set

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\delta Q_x^- - \mathrm{i}k_z\varPi, \tag{2.60}$$

$$0 = \left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right)\frac{\mathrm{d}\varPi}{\mathrm{d}x} - \frac{\Delta\alpha^2}{\alpha^2}\mathrm{i}k_z\delta Q_x^-. \tag{2.61}$$

This set is reminiscent of the coupled differential equations between the perturbed total pressure and displacement that other works have found for the description of kink waves (Appert, Gruber & Vaclavik 1974; Goossens, Hollweg & Sakurai 1992; Ismayilli *et al.* 2022), which have a strong correspondence to the currently modelled surface Alfvén waves (Goossens *et al.* 2012).

Since each quantity is constant in each half-space, we can substitute one of the equations in the other. Then, we obtain a single second-order differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\delta Q_x^- + k_z^2\left(\frac{\Delta\alpha^2}{\alpha^2 - \Delta\alpha^2}\right)\delta Q_x^- = 0. \tag{2.62}$$

In the left and right half-spaces, we consider respectively the solution

$$\delta Q_{x,L}^- = A_L \exp(\kappa_L x), \quad \delta Q_{x,R}^- = A_R \exp(-\kappa_R x), \tag{2.63}$$

where

$$\kappa^2 = k_z^2\left|\frac{\Delta\alpha^2}{\Delta\alpha^2 - \alpha^2}\right|. \tag{2.64}$$

The solution for $\varPi$ can be calculated from (2.60).

Next, we need to apply boundary conditions at $x = 0$. Namely, we take (as usual)

$$0 = [v_x], \tag{2.65}$$

$$0 = [P'], \tag{2.66}$$

where $P' = B_0 b_z/\mu$ is the perturbed total pressure and the square brackets are differences between the left and the right of the interface. Note that the extra term in the Lagrangian pressure perturbation is 0 in the linear regime, since the magnetic pressure is uniform in each half-space. Translated to our variables, these boundary conditions are

$$0 = [Q_x^-], \tag{2.67}$$

$$0 = \left[\frac{B_0\varPi}{\alpha}\right]. \tag{2.68}$$

The first condition states that $A_L = A_R$, while the second condition results in the dispersion relation

$$-\frac{\kappa_L B_L}{\alpha_L} = \frac{\kappa_R B_R}{\alpha_R}. \tag{2.69}$$

Squaring this relation, and inserting the expression for $\kappa^2 = k_z^2|\mu\rho\alpha^2 - 1|$, we obtain

$$\left(\mu\rho_L - \frac{1}{\alpha_L^2}\right)B_L^2 = -\left(\mu\rho_R - \frac{1}{\alpha_R^2}\right)B_R^2, \tag{2.70}$$

where we have used the fact that the absolute values in κ^2 take a different sign on either side of the interface. Solving this equation in conjunction with (2.50) (and considering

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$V_0 = 0$ for simplicity), we obtain finally the allowed values for α

$$\alpha B_0 = \alpha_L B_L = \alpha_R B_R = \sqrt{\frac{B_R^4 + B_L^4}{\mu(\rho_R B_R^2 + \rho_L B_L^2)}}, \quad \omega = k_z \sqrt{\frac{B_R^4 + B_L^4}{\mu(\rho_R B_R^2 + \rho_L B_L^2)}}. \tag{2.71}$$

Given that $B_L = B_R = B_0$ for a pressureless plasma, these equations reduce to

$$\alpha = \alpha_L = \alpha_R = \sqrt{\frac{2}{\mu(\rho_R + \rho_L)}}, \quad \omega = k_z \sqrt{\frac{2B_0^2}{\mu(\rho_R + \rho_L)}}, \tag{2.72}$$

as is well known from other works.

2.2.4. *General waves in field-aligned flows*

Now we will prove explicitly that the proper choice of α splits the Q-variable between wave modes of propagation directions. We follow the derivation of Magyar, Van Doorsselaere & Goossens (2019*b*) and their equation (19). In this subsection, we consider the general configuration with a magnetic field pointing in the z-direction, but still dependent on x and y. Moreover, we also take the background flow along the magnetic field

$$\boldsymbol{B}_0 = B_0(x, y)\boldsymbol{e}_z, \quad \boldsymbol{V}_0 = V_0(x, y)\boldsymbol{e}_z. \tag{2.73}$$

Let us now consider the linearised induction equation

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times ((\boldsymbol{V_0} + \boldsymbol{v}) \times \boldsymbol{B}_0), \tag{2.74}$$

of which we will only consider the perpendicular component. We can reduce this induction equation with vector identities to

$$\frac{\partial \boldsymbol{b}_\perp}{\partial t} = \boldsymbol{B}_0 \cdot \nabla \boldsymbol{v}_\perp - \boldsymbol{V}_0 \cdot \nabla \boldsymbol{b}_\perp. \tag{2.75}$$

Here, we have naturally used that $\nabla \cdot \boldsymbol{B}_0 = \nabla \cdot \boldsymbol{b} = 0$, but we have also used $\nabla \cdot \boldsymbol{V}_0 = 0$ because $\boldsymbol{V}_0$ only has a z-component that does not depend on z. Using Fourier analysis for the ignorable coordinates z and t, we then have

$$-\mathrm{i}\omega \boldsymbol{b}_\perp = \mathrm{i}k_z(B_0 \boldsymbol{v}_\perp - V_0 \boldsymbol{b}_\perp). \tag{2.76}$$

Using (2.7) and (2.15), we then have

$$(\omega - k_z V_0 + k_z \alpha B_0)\delta \boldsymbol{Q}_\perp^+ = (\omega - k_z V_0 - k_z \alpha B_0)\delta \boldsymbol{Q}_\perp^-. \tag{2.77}$$

This equation shows that the correct choice of α indeed splits a wave mode with a specific ω and k_z between different $\delta \boldsymbol{Q}_\perp$ components. Using the Q-variable terminology, the equation is more elegantly written as

$$(\omega - \boldsymbol{k} \cdot \boldsymbol{Q}_0^-)\delta \boldsymbol{Q}_\perp^+ = (\omega - \boldsymbol{k} \cdot \boldsymbol{Q}_0^+)\delta \boldsymbol{Q}_\perp^-. \tag{2.78}$$

This equation states that a wave with phase speed $\boldsymbol{Q}_0^\pm$ has the associated wave only present in $\delta \boldsymbol{Q}_\perp^\mp$, with the other Q-variable $\delta \boldsymbol{Q}_\perp^\pm = 0$.

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2.3. *Splitting the equations for different wave modes*

The linearised Q-equations (2.30)–(2.31) and their component versions (2.33)–(2.36) show that the operator on the right-hand side of these equations is a linear operator, and yields a vector proportional to its input plane wave solution with dependence $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\omega t)$. Moreover, we understand that the wave vector $\boldsymbol{k}$ and ω must satisfy the dispersion relation.

In a future work, we want to construct models for the solar atmosphere, which are driven by different wave modes. In our upcoming models, we want to take a step back from the linear approach, and once again use the full operator. The plan is to use a WKB approach (after Wentzel–Kramers–Brillouin) as detailed in Marsch & Tu (1989), Tu & Marsch (1993) and van der Holst *et al.* (2014). Such a model with only Alfvén wave drivers is called an AWSOM (Alfvén wave driven solar model) (van der Holst *et al.* 2014). We want to extend this model by also including the kink waves, their self-interaction and damping, leading to a model named UAWSOM (uniturbulence and Alfvén wave driven solar model). Thus we are looking forward to taking

$$\boldsymbol{Q}^{\pm}=\boldsymbol{Q}_0^{\pm}+\delta\boldsymbol{Q}_k^{\pm}+\delta\boldsymbol{Q}_A^{\pm}, \tag{2.79}$$

where $\boldsymbol{Q}_0^{\pm}$ stands for the slowly varying background, $\delta\boldsymbol{Q}_k^{\pm}$ is the contribution of the (respectively up- and downward-propagating) kink waves and $\delta\boldsymbol{Q}_A^{\pm}$ the contribution from the (up- and downward) Alfvén waves, as classically used in AWSOM type models (Evans *et al.* 2012; van der Holst *et al.* 2014; Réville *et al.* 2020). In the employed WKB approximation, we consider a background $\boldsymbol{Q}_0^{\pm}$ slowly varying along $\boldsymbol{B}_0$ and in time. We thus consider the dominant Fourier components of $\boldsymbol{Q}_0^{\pm}$ to be with wavelengths (or scale heights, if you wish) much larger than the wavelengths of the kink and Alfvén waves, and periods (or time scales of variation, if you like) much larger than the periods of the kink and Alfvén waves.

Let us first only consider the linearised version of (2.26) and (2.14), and call its associated operator $\mathcal{L}_\alpha$ acting on the eigenvector to be $\boldsymbol{\mathcal{U}}$ (consisting of $\boldsymbol{Q}^{\pm}$ and ρ)

$$\mathcal{L}_\alpha\boldsymbol{\mathcal{U}}=0. \tag{2.80}$$

We realise that the linear operators on the left-hand side and right-hand side will just split out over the different contributions $\delta\boldsymbol{Q}_k^{\pm}$ and $\delta\boldsymbol{Q}_A^{\pm}$, because of the linear character of the operators. Each term for the kink wave in the equation will have a dependence $\exp(\mathrm{i}k_{z,k}z-\mathrm{i}\omega_k t)$, and likewise the terms for the Alfvén waves will have a dependence of $\exp(\mathrm{i}k_{z,A}z-\mathrm{i}\omega_A t)$, where the pairs $(\omega_k, k_{z,k})$ and $(\omega_A, k_{z,A})$ satisfy their respective dispersion relation for a (different!) driving frequency ω_k or ω_A that finds its origin in the photospheric convective motions or p-modes (Morton, Weberg & McLaughlin 2019). By using a Fourier transform of the linearised Q-equations, we then obtain a separated set of equations for each contribution

$$\mathcal{L}_\alpha\boldsymbol{\mathcal{U}}_k=0, \tag{2.81}$$

$$\mathcal{L}_\alpha\boldsymbol{\mathcal{U}}_A=0, \tag{2.82}$$

$$\mathcal{L}_\alpha\boldsymbol{\mathcal{U}}_0=0. \tag{2.83}$$

Here, the last equation for the equilibrium is in the WKB approximation an integration of the Fourier components ω smaller than the smallest wave frequency

$$\omega<\min\{\omega_A,\omega_k\}, \tag{2.84}$$

which thus represents the slow evolution of the background. It is irrelevant for this last equation for the equilibrium which α value is chosen or used, because the equations are more conveniently written in terms of the classical MHD variables.

The key point to realise in (2.81)–(2.82) is that they are still valid for any possible α that you prefer. Moreover, they are clearly independent, and thus α may be chosen freely for both separately! Thus, for (2.82), we use the choice of $\alpha = 1/\sqrt{\mu\rho}$ reverting to the classical equation of van der Holst *et al.* (2014). However, for the kink waves (2.81), we make the choice of the appropriate α, as found in (2.72). That then allows us to formulate the appropriate equations for upward- and downward-propagating kink waves, separating out their contributions.

If we assume that the nonlinearity and field-aligned inhomogeneity are sufficiently weak, we can consider the re-inclusion of the nonlinear terms in (2.26) and (2.14). They will be of the form $\boldsymbol{\delta Q}_k^{\pm} \cdot \nabla \boldsymbol{\delta Q}_k^{\pm}$ and $\boldsymbol{\delta Q}_A^{\pm} \cdot \nabla \boldsymbol{\delta Q}_A^{\mp}$, and also include cross-terms between $\boldsymbol{\delta Q}_A$ and $\boldsymbol{\delta Q}_k$. Using the same Fourier argument as before, we should realise that the cross-terms will make no net contribution to the equations (2.81)–(2.82) when integrated over a longer time (this seems, however, in contradiction with the numerical experiments of Guo *et al.* 2019). The other terms will contain the classical interaction of counterpropagating waves in Alfvén wave turbulence (Iroshnikov 1964; Kraichnan 1967), acting as a net sink in the equations (2.81)–(2.82), but added as a source term in the equilibrium equations as in Marsch & Tu (1989), Tu & Marsch (1993), Evans *et al.* (2012), van der Holst *et al.* (2014) and Réville *et al.* (2020). The terms in $\boldsymbol{\delta Q}_k^{\pm} \cdot \nabla \boldsymbol{\delta Q}_k^{\pm}$ model the damping of the kink wave due to uniturbulence (Magyar *et al.* 2017, 2019*b*) due to its self-deformation. In Van Doorsselaere *et al.* (2020) it was found that this term also leads to a net contribution when averaged over longer times, similar to the Alfvén wave cascade. This extra contribution also acts as a sink in the kink wave evolution equation (2.81), and is added as an extra heating and pressure term in background MHD equations, just like the Alfvén wave cascade in the AWSOM model.

3. Conclusions

In this paper, we have started from the success of the Elsässer variables in describing and separating upward- and downward-propagating Alfvén waves. With the earlier realisation that any other wave than an Alfvén wave necessarily has both Elsässer components (Magyar *et al.* 2019*b*), we have realised that the Elsässer variables need generalisation to other waves as well.

To fill this need, we have proposed the Q-variables given by

$$\boldsymbol{Q}^{\pm} = \boldsymbol{V} \pm \alpha \boldsymbol{B}, \tag{3.1}$$

with a parameter α that we have proven to be proportional to the phase speed of the wave. The value of α is dependent on the type of wave and equilibrium parameters through the dispersion relation. We have rewritten the MHD equations in these Q-variables, following the lead of Marsch & Mangeney (1987).

In the next section of the paper, we have shown that (i) the modelling of Alfvén waves reverts back to the classical Elsässer variables, (ii) that slow and fast waves have also the perpendicular component of $\boldsymbol{Q}^{\pm}$ split between upward- and downward-propagating waves and (iii) that surface Alfvén waves in a non-uniform plasma can also be described by the Q-variables, separating out upward- and downward-propagating waves. This shows that the generalisation of the Elsässer variables, as we set out to do, has been successful. Indeed, going beyond the Elsässer description, with the current Q-variables, we can separate upward- and downward-propagating waves of many different types, including waves in inhomogeneous plasmas.

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The significance of these Q-variables is in enabling a more general approach to the Alfvén wave driven solar wind models (e.g. van der Holst *et al.* 2014). These models encapsulate in a one-dimensional way the additional heating by Alfvén waves (see Cranmer *et al.* (2015), for a review). Thanks to this new development of the Q-variables, it will be possible to construct new solar wind models that also include wave driving by other wave modes. In particular, we have laid the mathematical groundwork for the creation of the UAWSOM model, which also incorporates the propagation of kink waves on inhomogeneous structures, such as plumes. Kink waves have been ubiquitously observed in the solar corona (Tomczyk *et al.* 2007; Nechaeva *et al.* 2019) and possibly deliver significant energy input in coronal loops (Lim *et al.* 2023) and plumes (Thurgood, Morton & McLaughlin 2014). These kink waves self-interact nonlinearly and show uniturbulence (Magyar *et al.* 2019*b*). This potentially leads to extra heating in the solar wind model, possibly resolving current shortcomings of the AWSOM model which underperforms in open field regions (Verdini *et al.* 2010; van der Holst *et al.* 2014; van Ballegooijen & Asgari-Targhi 2016, 2017; Verdini, Grappin & Montagud-Camps 2019). This potential extra heating by kink waves will be the subject of a future publication, in which we will derive the governing equations for the UAWSOM model, based on the current Q-variables. These will incorporate the evolution equations of the wave energy density. Moreover, but more speculatively, this formalism could be useful in deriving the effect of the parametric instability on the solar wind driving with Alfvén waves (Shoda *et al.* 2019).

Furthermore, the adoption of these new Q-variables allows the exploration of Solar Orbiter or Parker Solar Probe data, in regimes which are not highly Alfvénic. In particular, data series of low Alfvénicity could be re-analysed with the Q-variables to expose other wave modes in these regimes.

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Declaration of interests

The authors report no conflict of interest.

Author contributions

TVD derived the theory, NM made the numerical solutions, all contributed to discussions during the research, all contributed to writing and editing the manuscript.

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