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UNIQUENESS OF COMPLETE HYPERSURFACES WITH BOUNDED HIGHER ORDER MEAN CURVATURES IN SEMI-RIEMANNIAN WARPED PRODUCTS

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Abstract. In this paper, we deal with complete hypersurfaces immersed with bounded higher order mean curvatures in steady state-type spacetimes and in hyperbolic-type spaces. By applying a generalised maximum principle for the Yau's square operator [**11**], we obtain uniqueness results in each of these ambient spaces.

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1. Introduction. In this paper, we are interested in the study of complete non-compact Riemannian hypersurfaces with bounded r-th mean curvatures in a class of $(n+1)$-dimensional semi-Riemannian warped product spaces, which include the hyperbolic space $\mathbb{H}^{n+1}$ and the steady state space $\mathcal{H}^{n+1}$ (cf. Remark 4.3). Before giving details on our theorems, we present a brief outline of some recent results related to our results.

Alías et al. [**4**] extended the classical theorem of Bernstein for minimal graphs (that is, with zero mean curvature) in $\mathbb{R}^3$ to complete minimal surfaces in Riemannian ambient spaces of non-negative Ricci curvature and endowed with a Killing field. This was done under the assumption that the sign of the angle function between a global Gauss map and the Killing field remains unchanged along the surface.

In [**9**], the second author jointly with Caminha have studied complete vertical graphs of constant mean curvature in the hyperbolic and steady state spaces. They first derived suitable formulas for the Laplacians of the height function h and of a support-like function naturally attached to the graph; then, under appropriate restrictions on the values of the mean curvature and the growth of the height function, they obtained necessary conditions for the existence of such a graph. Further, in the 3-dimensional case, they proved the Bernstein-type results in each of these ambient spaces.

More recently, by applying a technique of Yau [**26**] and imposing suitable conditions on both the r-th mean curvatures and the norm of the gradient of the height function, the second author jointly with Camargo and Caminha [**8**] obtained another Bernstein-type results in the hyperbolic and steady state spaces.

Here, motivated by the works described above and using a generalised maximum principle developed in [**10**], we obtain in Sections 4 and 5 uniqueness theorems for the spacelike slices of the semi-Riemannian warped product spaces $\epsilon\mathbb{R} \times_{e^t} M^n$, under suitable conditions on both the r-th mean curvatures and the normal angle of the hypersurface (that is, angle between the Gauss map of the hypersurface and the unitary vector field, ∂_t). More precisely, we prove the following results (cf. Theorems 4.1 and 5.1; see also Corollaries 4.5 and 5.4).

Let M^n be a complete connected Riemmanian manifold of non-negative sectional curvature and $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a complete connected spacelike hypersurface with non-negative sectional curvature less than or equal to one, and bounded away from the future infinity of $-\mathbb{R} \times_{e^t} M^n$. Suppose that there exist positive constants α and β such that $\beta \le H_r \le H_{r+1} \le \alpha$, for some $0 \le r \le n-1$. If the normal hyperbolic angle θ of Σ^n satisfies $\cosh\theta \le \inf_\Sigma \frac{H_{r+1}}{H_r}$, *then Σ^n is a slice of $-\mathbb{R} \times_{e^t} M^n$.*

Let M^n be a complete connected Riemmanian manifold of zero sectional curvature and $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ be a complete connected hypersurface with non-negative sectional curvature and bounded away from the future infinity of $\mathbb{R} \times_{e^t} M^n$. Suppose that there exist positive constants α and β such that $\beta \le H_{r+1} \le H_r \le \alpha$, for some $0 \le r \le n-1$. If the normal angle θ of Σ^n satisfies $\cos\theta \ge \sup_\Sigma \frac{H_{r+1}}{H_r}$, *then Σ^n is a slice of $\mathbb{R} \times_{e^t} M^n$.*

Finally, we want to point out that our restrictions on the normal angle of the hypersurfaces are motivated by gradient estimates due to Montiel in [**20**], related to the steady state space, and López jointly with Montiel in [**16**], related to the hyperbolic space (cf. Remarks 4.6 and 5.5).

2. Preliminaries. Let $\overline{M}^{n+1}$ be a connected semi-Riemannian manifold with metric $\overline{g} = \langle\, ,\, \rangle$ of index $\nu \le 1$, and semi-Riemannian connection $\overline{\nabla}$. For a vector field $X \in \mathfrak{X}(\overline{M})$, let $\epsilon_X = \langle X, X\rangle$. We will say that X is a *unit* vector field if $\epsilon_X = \pm 1$, and *timelike* if $\epsilon_X = -1$.

In all that follows, we consider *Riemannian immersions* $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely immersions from a connected, n-dimensional orientable differentiable manifold Σ^n into $\overline{M}$ such that the induced metric $g = \psi^*(\overline{g})$ turns Σ into the Riemannian manifold (in the Lorentz case $\nu = 1$, we refer to (Σ^n, g) as a *spacelike* hypersurface of $\overline{M}$), with the Levi-Civita connection ∇. We orient Σ^n by the choice of a unit normal vector field N on it.

In this setting if we let A denote the corresponding shape operator, then at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$.

For $0 \le r \le n$, let $S_r(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p; this way one gets n smooth functions $S_r : \Sigma^n \to \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where S_0=1 by definition. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on indeterminates $X_1, \ldots, X_n$.

Also, we define the r-th mean curvature H_r of ψ, $0 \le r \le n$, by

$$\binom{n}{r} H_r = \epsilon_N^r S_r = \sigma_r(\epsilon_N \lambda_1, \ldots, \epsilon_N \lambda_n).$$

We observe that H_0=1 and H_1 is the usual mean curvature H of Σ^n.

For $0 \le r \le n$, one defines the r-th Newton transformation P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = \epsilon_N^r S_r I - \epsilon_N A P_{r-1}. \tag{2.1}$$

A trivial induction shows that

$$P_r = \epsilon_N^r (S_r I - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r),$$

so that the Cayley–Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of $T_p\Sigma$ diagonalising A at $p \in \Sigma^n$ also diagonalise all of the P_r at p. Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i \rangle^\perp \subset T_p\Sigma$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \le j_1 < \ldots < j_k \le n \\ j_1, \ldots, j_k \neq i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

It is also immediate to check that $P_r e_i = \epsilon_N^r S_r(A_i) e_i$ so that an easy computation (cf. Lemma 2.1 in **[6]**) gives the following.

LEMMA 2.1. *With the above notations, the following formulas hold*:

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$,
(b) $\mathrm{tr}(P_r) = \epsilon_N^r \sum_{i=1}^n S_r(A_i) = \epsilon_N^r (n-r) S_r = c_r H_r$,
(c) $\mathrm{tr}(AP_r) = \epsilon_N^r \sum_{i=1}^n \lambda_i S_r(A_i) = \epsilon_N^r (r+1) S_{r+1} = \epsilon_N c_r H_{r+1}$,
(d) $\mathrm{tr}(A^2 P_r) = \epsilon_N^r \sum_{i=1}^n \lambda_i^2 S_r(A_i) = \epsilon_N^r (S_1 S_{r+1} - (r+2) S_{r+2})$,

where $c_r = (n-r)\binom{n}{r}$.

Associated to each Newton transformation P_r one has the second-order linear differential operator, $L_r : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$, given by

$$L_r(h) = \mathrm{tr}(P_r \,\mathrm{Hess}\, h).$$

In particular, $L_0 = \Delta$ and if $\overline{M}$ has constant sectional curvature, Rosenberg proved in **[23]** that $L_r h = \mathrm{div}(P_r Dh)$, where div stands for the divergence on Σ^n and D denotes the field gradient of $h \in \mathcal{D}(\Sigma)$.

For a smooth $\varphi : \mathbb{R} \to \mathbb{R}$ and $h \in \mathcal{D}(\Sigma)$, it follows from the properties of the Hessian of functions that

$$L_r(\varphi \circ h) = \varphi'(h) L_r(h) + \varphi''(h) \langle P_r Dh, Dh \rangle. \tag{2.2}$$

3. Semi-Riemannian warped products. In order to study the semi-Riemannian warped products, we define conformal field vectors. A vector field V on $\overline{M}^{n+1}$ is said to be *conformal* if

$$\mathcal{L}_V\langle\ ,\ \rangle = 2\phi\langle\ ,\ \rangle \tag{3.1}$$

for some smooth function $\phi \in \mathcal{D}(\overline{M})$, where $\mathcal{L}$ stands for the Lie derivative of the metric of $\overline{M}$. The function ϕ is called the *conformal factor* of V.

Since $\mathcal{L}_V(X) = [V, X]$ for all $X \in \mathfrak{X}(\overline{M})$, it follows from the tensorial character of $\mathcal{L}_V$ that $V \in \mathfrak{X}(\overline{M})$ is conformal if and only if

$$\langle\overline{\nabla}_X V, Y\rangle + \langle X, \overline{\nabla}_Y V\rangle = 2\phi\langle X, Y\rangle \tag{3.2}$$

for all $X, Y \in \mathfrak{X}(\overline{M})$. In particular, V is the Killing vector field relatively to the metric $\langle\, ,\rangle$ if $\phi \equiv 0$.

Let M^n be a connected, n-dimensional-oriented Riemannian manifold, $I \subseteq \mathbb{R}$ an interval and $f : I \to \mathbb{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the I and M factors, respectively. A particular class of the semi-Riemannian manifolds having conformal fields is the one obtained by furnishing $\overline{M}$ with the metric

$$\langle v, w\rangle_p = \epsilon\langle(\pi_I)_* v, (\pi_I)_* w\rangle + f(p)^2\langle(\pi_M)_* v, (\pi_M)_* w\rangle,$$

for all $p \in \overline{M}$ and all $v, w \in T_p\overline{M}$, where $\epsilon = \epsilon_{\partial_t}$ and ∂_t is the standard unit vector field tangent to I. Moreover (cf. [**18**] and [**19**]), the vector field

$$V = (f \circ \pi_I)\partial_t$$

is conformal and closed (in the sense that its dual 1-form is closed) with conformal factor $\phi = f' \circ \pi_I$, where the prime denotes differentiation with respect to $t \in I$. Such a space is a particular instance of the semi-Riemannian *warped product*, and, from now on, we shall write $\overline{M}^{n+1} = \epsilon I \times_f M^n$ to denote it.

If $\psi : \Sigma^n \to \epsilon I \times_f M^n$ is the Riemannian immersion, with Σ oriented by the unit vector field N, one obviously has $\epsilon = \epsilon_{\partial_t} = \epsilon_N$. We let h denote the (vertical) height function naturally attached to Σ^n, namely $h = (\pi_I)|_\Sigma$.

Let $\overline{D}$ and D denote gradients with respect to the metrics of $\epsilon I \times_f M^n$ and Σ^n, respectively. A simple computation shows that the gradient of π_I on $\epsilon I \times_f M^n$ is given by

$$\overline{D}\pi_I = \epsilon\langle\overline{D}\pi_I, \partial_t\rangle = \epsilon\partial_t \tag{3.3}$$

so that the gradient of h on Σ^n is

$$Dh = (\overline{D}\pi_I)^\top = \epsilon\partial_t^\top = \epsilon\partial_t - \langle N, \partial_t\rangle N. \tag{3.4}$$

In particular, we get

$$|Dh|^2 = \epsilon(1 - \langle N, \partial_t\rangle^2), \tag{3.5}$$

where $|\ \ |$ denotes the norm of a vector field on Σ^n.

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In the Lorentz setting, the following result is a particular case of one obtained by Alías and Colares (cf. [**2**], Lemma 4.1).

LEMMA 3.1. *Let* $\psi : \Sigma^n \to \epsilon I \times_f M^n$ *be a Riemannian immersion. If* $h = (\pi_I)|_\Sigma : \Sigma^n \to I$ *is the height function of* Σ^n*, then*

$$L_r(h) = (\log f)'(\epsilon \operatorname{tr}(P_r) - \langle P_r Dh, Dh\rangle) + \langle N, \partial_t\rangle \operatorname{tr}(AP_r). \tag{3.6}$$

Proof. Fix $p \in M$, $v \in T_pM$ and write $v = w + \epsilon\langle v, \partial_t\rangle\partial_t$ so that $w \in T_p\overline{M}$ is tangent to the fibre of $\overline{M}$ passing through p. Therefore, by repeated use of the formulas of item (2) of Proposition 7.35 in [**22**], we get

$$\begin{aligned}\overline{\nabla}_v \partial_t &= \overline{\nabla}_w \partial_t + \epsilon\langle v, \partial_t\rangle \overline{\nabla}_{\partial_t}\partial_t = \overline{\nabla}_w \partial_t\\ &= (\log f)' w = (\log f)'(v - \epsilon\langle v, \partial_t\rangle\partial_t).\end{aligned}$$

Thus, from (3.4) we obtain that

$$\begin{aligned}\nabla_v Dh &= \overline{\nabla}_v Dh - \epsilon\langle Av, Dh\rangle N\\ &= \overline{\nabla}_v(\epsilon\,\partial_t - \langle N, \partial_t\rangle N) - \epsilon\langle Av, Dh\rangle N\\ &= \epsilon(\log f)' w - v(\langle N, \partial_t\rangle)N + \langle N, \partial_t\rangle Av - \epsilon\langle Av, Dh\rangle N\\ &= \epsilon(\log f)' w + (\langle Av, \partial_t\rangle - \langle N, \overline{\nabla}_v \partial_t\rangle)N + \langle N, \partial_t\rangle Av - \epsilon\langle Av, Dh\rangle N\\ &= \epsilon(\log f)' w + (\langle Av, \partial_t^\top\rangle - \langle N, (\log f)' w\rangle)N + \langle N, \partial_t\rangle Av - \epsilon\langle Av, Dh\rangle N\\ &= \epsilon(\log f)' w + \epsilon(\log f)'\langle v, \partial_t\rangle\langle N, \partial_t\rangle N + \langle N, \partial_t\rangle Av\\ &= \epsilon(\log f)'\{v - \langle v, \partial_t\rangle(\epsilon\,\partial_t - \langle N, \partial_t\rangle N)\} + \langle N, \partial_t\rangle Av\\ &= (\log f)'(\epsilon\, v - \epsilon\langle v, \partial_t^\top\rangle Dh) + \langle N, \partial_t\rangle Av\\ &= (\log f)'(\epsilon\, v - \langle v, Dh\rangle Dh) + \langle N, \partial_t\rangle Av.\end{aligned}$$

Now, by fixing $p \in \Sigma$ and an orthonormal frame $\{e_i\}$ at $T_p\Sigma$, one gets

$$\begin{aligned}L_r(h) &= \operatorname{tr}(P_r \operatorname{Hess}(h)) = \sum_{i=1}^n \langle \nabla_{e_i} Dh, P_r e_i\rangle\\ &= \sum_{i=1}^n \langle (\log f)'(\epsilon\, e_i - \langle e_i, Dh\rangle Dh) + \langle N, \partial_t\rangle Ae_i, P_r e_i\rangle\\ &= (\log f)'(\epsilon \operatorname{tr}(P_r) - \langle P_r Dh, Dh\rangle) + \langle N, \partial_t\rangle \operatorname{tr}(AP_r).\end{aligned}$$

□

REMARK 3.2. For $t_0 \in \mathbb{R}$, we orient the *slice* $\Sigma^n_{t_0} = \{t_0\} \times M^n$ by using the unit normal vector field ∂_t. According to [**5**], Σ_{t_0} has constant r-th mean curvature $H_r = -\epsilon(\frac{f'(t_0)}{f(t_0)})^r$ with respect to ∂_t. Since our applications in the next sections deal with the semi-Riemannian warped products with warping function $f(t) = e^t$, all slices will have r-th mean curvature, $H_r = -\epsilon$ with respect to ∂_t.

We will also need the well-known *Generalized Maximum Principle* due to Omori and Yau [**21, 25**].

LEMMA 3.3. *Let* Σ^n *be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and* $u \in \mathcal{D}(\Sigma)$ *be a smooth function which is bounded*

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from above on Σ^n. Then there exists a sequence $(p_k)_{k\geq 1}$ in Σ^n such that

$$u(p_k) > \sup_{\Sigma} u - \frac{1}{k}, \quad |Du(p_k)| < \frac{1}{k}, \quad \Delta u(p_k) < \frac{1}{k}.$$

In order to prove our uniqueness theorems related to the case of the higher order mean curvatures, we will need for Yau's square operator an analogue of Lemma 3.3.

Let Σ^n be a complete n-dimensional Riemannian manifold. Let also $\Phi : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denote a field of self-adjoint linear transformations on Σ^n. We consider the second-order linear differential operator $\Box : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$ defined by

$$\Box u = \operatorname{tr}(\Phi \operatorname{Hess} u). \tag{3.7}$$

In this setting, Caminha jointly with the second author have proved the following (cf. [**10**], Corollary 3.3).

LEMMA 3.4. *Let Σ^n be a complete Riemannian manifold with non-negative sectional curvature, and $u \in \mathcal{D}(\Sigma)$ be a function which is bounded from above on Σ^n. If Φ is positive semi-definite and* $\operatorname{tr}(\Phi)$ *is bounded from above on Σ^n, then there exists a sequence $(p_k)_{k\geq 1}$ in Σ^n such that*

$$u(p_k) > \sup_{\Sigma} u - \frac{1}{k}, \quad |Du(p_k)| < \frac{1}{k}, \quad \Box u(p_k) < \frac{1}{k}.$$

4. Uniqueness results in steady state-type spacetimes. In this section we consider (according to [**1**]; see also Remark 4.3) *steady state-type spacetimes*, i.e. the Lorentzian warped products,

$$-\mathbb{R} \times_{e^t} M^n, \tag{4.1}$$

where M^n is an n-dimensional complete, connected Riemannian manifold (see Remark 4.2).

As we have pointed out by the end of Section 3, each slice $\Sigma_{t_0} = \{t_0\} \times M^n$ is a complete, connected spacelike hypersurface with r-th mean curvature equal to 1 if we take the orientation given by the unit normal vector field $N = \partial_t$.

In what follows, we consider that our spacelike hypersurfaces $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ are oriented by the time-like unit vector field N such that $\langle N, \partial_t\rangle < 0$. The *normal hyperbolic angle* θ of ψ is defined as being the smooth function $\theta : \psi(\Sigma) \to [0, +\infty)$ such that

$$\cosh\theta = -\langle N, \partial_t\rangle \geq 1. \tag{4.2}$$

Following [**1**], we say that a spacelike hypersurface $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ is *bounded away from the future infinity* of $-\mathbb{R} \times_{e^t} M^n$ if there exists $\bar{t} \in \mathbb{R}$ such that

$$\psi(\Sigma) \subset \{(t, x) \in -\mathbb{R} \times_{e^t} M^n; t \leq \bar{t}\}.$$

Now we present our uniqueness theorem in the steady state-type space.

THEOREM 4.1. *Let M^n be a complete connected Riemmanian manifold of non-negative sectional curvature and $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a complete connected spacelike*

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hypersurface with non-negative sectional curvature less than or equal to one, and bounded away from the future infinity of $-\mathbb{R}\times_{e^t} M^n$*. Suppose that there exist positive constants* α *and* β *such that* $\beta \le H_r \le H_{r+1} \le \alpha$*, for some* $0 \le r \le n-1$*. If the normal hyperbolic angle* θ *of* Σ^n *satisfies* $\cosh\theta \le \inf_\Sigma \frac{H_{r+1}}{H_r}$*, then* Σ^n *is a slice of* $-\mathbb{R}\times_{e^t} M^n$*.*

Proof. We set

$$\Phi = H_r P_r.$$

Thus, since $\mathrm{tr}(\Phi) = c_r H_r^2 \ge 0$, Φ is positive semi-definite. In addition, since H_r is bounded on Σ^n, the same is true of $\mathrm{tr}(\Phi)$ and, hence, we can apply Lemma 3.4 to such a Φ.

On the other hand, from formulas (2.2) and (3.6), we get

$$L_r(e^h) = -c_r e^h (H_r + \langle N, \partial_t\rangle H_{r+1}).$$

Thus, we have

$$\begin{aligned}\Box e^h &= \mathrm{tr}(\Phi\,\mathrm{Hess}(e^h)) = H_r L_r(e^h)\\ &= -c_r e^h H_r (H_r + \langle N, \partial_t\rangle H_{r+1}).\end{aligned}$$

Consequently, since we are supposing that Σ^n is bounded away from the future infinity of $-\mathbb{R}\times_{e^t} M^n$, by applying Lemma 3.4 we obtain a sequence $(p_k)_{k\ge 1}$ in Σ^n such that

$$\lim_k (e^h)(p_k) = e^{\sup_\Sigma h}$$

and

$$0 \ge \lim_k \Box e^h(p_k) = c_r e^{\sup_\Sigma h} \lim_k \left(H_r(-\langle N, \partial_t\rangle H_{r+1} - H_r)\right).$$

Hence, our assumptions on H_r and H_{r+1} together with the reverse Cauchy–Schwarz inequality give

$$0 \ge \lim_k \Box e^h(p_k) \ge c_r e^{\sup_\Sigma h}\beta^2 \lim_k \left(\frac{H_{r+1}}{H_r} - 1\right) \ge 0.$$

Consequently,

$$\inf_\Sigma \frac{H_{r+1}}{H_r} = 1.$$

Therefore, from our hypothesis under the normal hyperbolic angle of Σ^n, we conclude that Σ^n is a slice of $-\mathbb{R}\times_{e^t} M^n$. $\square$

REMARK 4.2. According to Lemma 7 in **[1]** if $-\mathbb{R}\times_{e^t} M^n$ is to admit a complete hypersurface bounded away from the future infinity, then M^n must be necessarily complete.

REMARK 4.3. An interesting special case is that of the $(n+1)$-dimensional *steady state space*, i.e. the warped product $\mathcal{H}^{n+1} = -\mathbb{R}\times_{e^t}\mathbb{R}^n$, which is isometric to an open subset of the de Sitter space $\mathbb{S}_1^{n+1}$. In this case, the slice Σ_{t_0} is isometric to $\mathbb{R}^n$ and is called a *hyperplane* of $\mathcal{H}^{n+1}$.

The importance of considering $\mathcal{H}^{n+1}$ comes from the fact that in Cosmology $\mathcal{H}^4$ is the steady state model of the universe proposed by Bondi and Gold [**7**], and Hoyle [**15**] when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous) but also at all times (cf. Section 14.8 in [**24**], and Section 5.2 in [**14**]).

REMARK 4.4. Let M^n be of non-negative sectional curvature. As a consequence of the classical Bonnet–Myers theorem, if a complete spacelike hypersurface $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ has (not necessarily constant) mean curvature H satisfying

$$|H| \leq c < \frac{2\sqrt{n-1}}{n}$$

(c a positive real constant), then Σ^n has to be compact. In fact if we let Ric_Σ stand for the Ricci tensor of Σ^n, then inequality (16) of [**1**], together with the non-negativity of the sectional curvature of M and the above bound on H, gives

$$\mathrm{Ric}_\Sigma \geq (n-1) - \frac{n^2H^2}{4} > 0. \tag{4.3}$$

We observe that $\frac{2\sqrt{n-1}}{n} \leq 1$ for $n \geq 2$.

However, in case $M^n = \mathbb{R}^n$ (so that $-\mathbb{R} \times_{e^t} M^n = \mathcal{H}^{n+1}$) if Σ^n is bounded away from the future infinity, then Lemma 1 in [**1**] assures that Σ^n is diffeomorphic to $\mathbb{R}^n$; in particular, $\mathcal{H}^{n+1}$ does not possess any compact (without boundary) spacelike hypersurface.

On the other hand, it follows from the classification of totally umbilical spacelike hypersurfaces of the de Sitter space (cf. Example 1 in [**17**]) that there exists no totally umbilical complete immersed spacelike hypersurfaces with mean curvature $0 \leq H < 1$ in the steady state space.

It follows from all of the above that, in a certain sense, it is natural to restrict attention to mean curvature $H \geq 1$.

COROLLARY 4.5 *(Theorem 5.2 in [**12**]). Let M^n be a complete connected Riemmanian manifold of non-negative sectional curvature and $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a complete connected spacelike hypersurface bounded away from the future infinity of $-\mathbb{R} \times_{e^t} M^n$. Suppose that there exists a constant α such that $1 \leq H \leq \alpha$. If the normal hyperbolic angle θ of Σ^n satisfies* $\cosh\theta \leq \inf_\Sigma H$*, then Σ^n is a slice of $-\mathbb{R} \times_{e^t} M^n$.*

Proof. Following the same steps of the proof of Theorem 4.1 together with Lemma 3.3, and taking into account estimate (4.3) for the Ricci curvature of Σ^n, we obtain the result. □

REMARK 4.6. We observe that when Σ^n is a compact spacelike hypersurface immersed with constant mean curvature $H > 1$ in $\mathcal{H}^{n+1}$, and with its boundary $\partial\Sigma$ contained into a spacelike hyperplane of $\mathcal{H}^{n+1}$, a gradient estimate due to Montiel (cf. Theorem 7 in [**20**]) guarantees that the normal hyperbolic angle θ of Σ^n satisfies $\cosh\theta \leq H$.

REMARK 4.7. In [**1**], Albujer and Alías have proved that if a complete spacelike hypersurface with constant mean curvature is bounded away from the infinity of the steady state space $\mathcal{H}^{n+1}$, then its mean curvature must be identically 1. As a consequence

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of this result, they concluded that only complete Constant Mean Curvature (CMC) spacelike surfaces which lie between two planes of $\mathcal{H}^3$ are also the planes.

5. Uniqueness results in hyperbolic-type spaces. In analogy to the Lorentz case, we now turn our attention to *hyperbolic-type spaces*, i.e. warped products,

$$\mathbb{R} \times_{e^t} M^n,$$

where M^n is a complete, connected Riemannian manifold (see Remarks 5.2 and 4.2). According to Section 3, these hypersurfaces have constant mean curvature 1 if we take the orientation given by the unit normal vector field $N = -\partial_t$. For this reason, we will consider that our hypersurfaces $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ are such that their Gauss map satisfies $-1 \leq \langle N, \partial_t \rangle \leq 0$. In this setting, we define the *normal angle* θ of Σ^n as being the smooth function $\theta : \Sigma^n \to [0, \frac{\pi}{2}]$ given by

$$0 \leq \cos\theta = -\langle N, \partial_t \rangle \leq 1.$$

Similar to the Lorentz case, we say that a complete hypersurface $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ is *bounded away from the future infinity* of $\mathbb{R} \times_{e^t} M^n$ if there exists $\bar{t} \in \mathbb{R}$ such that $\psi(\Sigma)$ is contained below the slice $\Sigma_{\bar{t}}$.

We can finally state and prove, in the Riemannian setting, the analogue of Theorem 4.1.

THEOREM 5.1. *Let M^n be a complete connected Riemmanian manifold of zero sectional curvature and $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ be a complete connected hypersurface with non-negative sectional curvature and bounded away from the future infinity of $\mathbb{R} \times_{e^t} M^n$. Suppose that there exist positive constants α and β such that $\beta \leq H_{r+1} \leq H_r \leq \alpha$ for some $0 \leq r \leq n-1$. If the normal angle θ of Σ^n satisfies $\cos\theta \geq \sup_\Sigma \frac{H_{r+1}}{H_r}$, then Σ^n is a slice of $\mathbb{R} \times_{e^t} M^n$.*

Proof. As in the proof of Theorem 4.1, by setting $\Phi = H_r P_r$, we have that Φ is positive semi-definite with $\mathrm{tr}(\Phi)$ bounded on Σ^n. So, we can apply Lemma 3.4 to such a Φ.

On the other hand from formulas (2.2) and (3.6) we obtain

$$\begin{aligned} \Box e^h &= \mathrm{tr}(\Phi\,\mathrm{Hess}(e^h)) = H_r L_r(e^h) \\ &= c_r e^h H_r(H_r + \langle N, \partial_t \rangle H_{r+1}). \end{aligned}$$

Consequently, since we are supposing that Σ^n is bounded away from the future infinity of $\mathbb{R} \times_{e^t} M^n$, by applying Lemma 3.4 we obtain a sequence $(p_k)_{k\geq 1}$ in Σ^n such that

$$\lim_k (e^h)(p_k) = e^{\sup_\Sigma h}$$

and

$$0 \geq \lim_k \Box e^h(p_k) = c_r e^{\sup_\Sigma h} \lim_k \left(H_r(H_r + \langle N, \partial_t \rangle H_{r+1})\right).$$

Hence, our assumptions on H_r and H_{r+1} together with the Cauchy–Schwarz inequality give

$$0 \geq \lim_k \Box e^h(p_k) \geq c_r e^{\sup_\Sigma h} \beta^2 \lim_k \left(1 - \frac{H_{r+1}}{H_r}\right) \geq 0.$$

Consequently,

$$\sup_\Sigma \frac{H_{r+1}}{H_r} = 1.$$

Therefore, from our hypothesis under the normal angle of Σ^n, we conclude that Σ^n is a slice of $\mathbb{R} \times_{e^t} M^n$. □

REMARK 5.2. A motivation to consider the spaces $\mathbb{R} \times_{e^t} M^n$ comes from the fact that the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ is isometric to $\mathbb{R} \times_{e^t} \mathbb{R}^n$, an explicit isometry being found in **[3]**. It can easily be seen from such isometry that the slices $\Sigma_{t_0} = \{t_0\} \times \mathbb{R}^n$ of the warped product model of the hyperbolic space are precisely the horospheres.

REMARK 5.3. We note that when the ambient space $\epsilon\mathbb{R} \times_{e^t} M^n$ has constant sectional curvature, it follows from Proposition 7.42 in **[22]** that the sectional curvatures of the fibre M^n must vanish identically. Moreover, since our hypersurfaces are to be complete, Remark 4.2 shows that M^n must be also complete, i.e. M^n must be a space form of zero sectional curvature.

Taking into account Remark 5.3, we obtain the following extension of Theorem 3.3 in **[13]**.

COROLLARY 5.4. *Let M^n be a Riemmanian space form of zero sectional curvature and $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ be a complete connected hypersurface with bounded second fundamental form and bounded away from the future infinity of $\mathbb{R} \times_{e^t} M^n$. Suppose that there exists a positive constant β such that $\beta \leq H \leq 1$. If the normal angle θ of Σ^n satisfies $\cos\theta \geq \sup_\Sigma H$, then Σ^n is a slice of $\mathbb{R} \times_{e^t} M^n$.*

Proof. From the Gauss equation and with a straightforward computation, we obtain, for any unit tangent vector field X, that

$$\mathrm{Ric}_\Sigma(X) = -(n-1) + nH\langle AX, X\rangle - \langle AX, AX\rangle,$$

where Ric_Σ stands for the Ricci curvature of Σ^n. Hence,

$$\mathrm{Ric}_\Sigma \geq -(n-1) - nH|A| - |A|^2.$$

Thus, since we are supposing that Σ^n has bounded second fundamental form, following the same ideas of the proof of Theorem 5.1 together with Lemma 3.3, we get the result. □

REMARK 5.5. Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be an immersion from a compact manifold Σ^n with mean convex boundary $\partial\Sigma$ contained into a horosphere of $\mathbb{H}^{n+1}$. Suppose that ψ has constant mean curvature $0 \leq H \leq 1$. From the gradiente estimate (19) in **[16]**, taking into account our choice of the orientation N of Σ^n, we conclude that its normal angle θ satisfies $\cos\theta \geq H$.

Remark 5.6. In [**3**], Alías and Dajczer studied complete surfaces properly immersed in $\mathbb{H}^3$, which are contained between two horospheres, obtaining a Bernstein-type result for the case of constant mean curvature $-1 \leq H \leq 1$.

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