NEW APPROACH TO THE EARTH'S ROTATION PROBLEM CONSISTENT WITH THE GENERAL PLANETARY THEORY

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Abstract. The equations of the translatory motion of the major planets and the Moon and the Poisson equations of the Earth's rotation in Euler parameters are reduced to the secular system describing the evolution of the planetary and lunar orbits (independent of the Earth's rotation) and the evolution of the Earth's rotation (depending on the planetary and lunar evolution).

1. Introduction

According to the general planetary theory (GPT) the coordinates of the major planets and the Moon may be represented by means of the power series in eccentric and oblique variables with trigonometric coefficients in mean longitudes of the planets (Brumberg, 1995 and references therein). The behaviour of the evolutionary eccentric and oblique variables is governed by an autonomous secular system. The aim of the present paper is to present in the same form the theory of the Earth's rotation around its centre of mass.

2. Earth's Rotation Equations in Euler Parameters

In the Earth's rotation problem one has to deal with two geocentric coordinate systems, an inertial system $\mathbf{x} = (x_i)$ with the fixed ecliptic as a

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main reference plane and a rotating, Earth–fixed system $\mathbf{x}' = (x_i')$ with the equator of date as a main reference plane. These two systems are related by the transformation (Tisserand, 1891; Smart, 1953)

$$\mathbf{x} = \Lambda \mathbf{x}', \qquad \mathbf{x}' = \Lambda^T \mathbf{x}, \qquad \Lambda = D_3(\psi) D_1(\theta) D_3(-\varphi), \tag{2.1}$$

D_i being elementary rotation matrices. For computer manipulation it is reasonable to replace three Euler angles ψ, θ and φ by four Euler parameters regarded as the components of the unitary 4–vector $\mathbf{u} = (u_1, u_2, u_3, u_4)$

$$u_1 = -\sin\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}, \qquad u_3 = -\cos\frac{\theta}{2}\sin\frac{\psi-\varphi}{2},$$

$$u_2 = \sin\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}, \qquad u_4 = \cos\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}. \tag{2.2}$$

In terms of Euler parameters the kinematical and dynamical equations of the Earth's rotation are described in the form (Brumberg, 1995; Maciejewski, 1985)

$$2\dot{\mathbf{u}} = \Omega(\boldsymbol{\omega})\mathbf{u}, \qquad I_i\dot{\omega}_i - \sum_{j,k=1}^{3} \varepsilon_{ijk} I_j \omega_j \omega_k = M_i \qquad (i = 1, 2, 3) \tag{2.3}$$

with principal inertia moments I_i and vector $\boldsymbol{\omega} = (\omega_i)$ of the Earth's rotation angular velocity with components ω_i referred to x_i' axes. Ω is 4×4 matrix with components

$$\Omega_{ij} = \sum_{k=1}^{3} \varepsilon_{ijk}\omega_k, \quad \Omega_{i4} = -\Omega_{4i} = \omega_i, \quad \Omega_{44} = 0, \tag{2.4}$$

ε_{ijk} in being the Levi-Civita symbol. The torque vector $\mathbf{M} = (M_i)$ is determined by

$$2\mathbf{M} = Q(\mathbf{u})\mathrm{grad}_u U \tag{2.5}$$

with the force function U and 4×4 matrix Q with components

$$Q_{ij} = \sum_{k=1}^{3} \varepsilon_{ijk} u_k + \delta_{ij} u_4, \quad Q_{4i} = -Q_{i4} = u_i, \quad Q_{44} = u_4, \tag{2.6}$$

δ_{ij} being the Kronecker symbol. In dealing with (2.3) one may use the relationship $\boldsymbol{\omega} = 2Q(\mathbf{u})\dot{\mathbf{u}}$.

As initial step here we shall confine ourselves to the axially symmetrical Earth's model with $I_1 = I_2$ and $M_3 = \frac{\partial U}{\partial \varphi} = 0$ resulting to ω_3 =const and the equations

$$\dot{\psi} = -\frac{1}{I_3\omega_3 \sin\theta}\frac{\partial U}{\partial \theta} + L_1, \qquad \dot{\theta} = \frac{1}{I_3\omega_3 \sin\theta}\frac{\partial U}{\partial \psi} + L_2, \tag{2.7}$$

with small correction functions L_1 and L_2 depending on θ and first and second derivatives of ψ and θ. In neglecting L_1 and L_2 one gets from (2.7) the well-known Poisson equations which may be regarded as the first approximation to the advanced Earth's rotation problem solutions Introducing complex Euler parameters $u = u_1 + \mathrm{i}\, u_2$, $v = u_3 + \mathrm{i}\, u_4$, one may rewrite the equations corresponding to the Poisson case in the form

$$\dot{u} = \mathrm{i}\, n(u + R_5), \qquad \dot{v} = \mathrm{i}\, n(v + R_7), \tag{2.8}$$

with constant frequency $n = -\frac{1}{2}\omega_3$ and the right-hand members

$$R_5 = \frac{1}{I_3\omega_3^2}\left(u\frac{\partial U}{\partial \overline{v}} - v\frac{\partial U}{\partial \overline{u}}\right)\overline{v}, \qquad R_7 = -R_5\overline{v}^{-1}\overline{u}. \tag{2.9}$$

One should add to (2.9) two conjugate equations for $\overline{u}$ and $\overline{v}$ with the right-hand members $R_6 = -\overline{R}_5$ and $R_8 = -\overline{R}_7$, respectively.

3. Force Function

Retaining only the quadrupole terms, the force function due to the action of the Sun and the Moon can be presented in the form

$$U = \frac{1}{2}K I_3\omega_3^2\left[\left(\frac{A_3}{r_0}\right)^5\left(\frac{z_0'}{A_3}\right)^2 + \varepsilon\left(\frac{A_9}{r_9}\right)^5\left(\frac{z_9'}{A_9}\right)^2\right]. \tag{3.1}$$

Here and below we designate

$$K = -3\frac{I_3 - I_1}{I_3}\frac{GM_0}{A_3^3\omega_3^2}, \qquad \varepsilon = \frac{M_9}{M_0}\left(\frac{A_3}{A_9}\right)^3, \qquad \sigma = \frac{M_9}{M_3}\frac{A_9}{A_3}. \tag{3.2}$$

$M_0, \mathbf{r}_0$ and $M_9, \mathbf{r}_9$ are masses and geocentric position vectors of the Sun and the Moon, respectively. M_i and $\mathbf{r}_i$ are masses and heliocentric position vectors of the major planets for $i = 1, 2, \ldots, 8$ with the value $i = 3$ relating to the Earth-Moon barycentre. The expressions for $\mathbf{r}_i$ $(i = 1, 2, \ldots, 8)$ are given in GPT. The expression for $\mathbf{r}_9$ may be also given in the GPT form (Brumberg and Ivanova, 1985). More specifically, designating now the rectangular coordinates by x, y, z we have for $i = 1, 2, \ldots, 9$

$$x_i + \mathrm{i}\, y_i = A_i(1 - p_i)\zeta_i, \quad z_i = A_i w_i, \quad \zeta_i = \exp \mathrm{i}\,\lambda_i, \quad \lambda_i = n_i t + \varepsilon_i \tag{3.3}$$

with $n_i^2 A_i^3 = G(M_0 + M_i)$ $(i = 1, 2, \ldots, 8)$, $n_9^2 A_9^3 = GM_3$ and

$$x_0 + \mathrm{i}\, y_0 = A_3\left[-(1 - p_3)\zeta_3 + \sigma(1 - p_9)\zeta_9\right], \quad z_0 = A_3(-w_3 + \sigma w_9). \tag{3.4}$$

The third coordinate in the rotating system, z', may be found from

$$z' = \overline{u}v(x + \mathrm{i}\,y) + u\overline{v}(x - \mathrm{i}\,y) + (-u\overline{u} + v\overline{v})z\,, \tag{3.5}$$

enabling one to determine z'_0 and z'_9. Substituting all these expressions into (3.1), one gets the expansion

$$U = \frac{1}{2}KI_3\omega_3^2\big[k + u^2\overline{v}^2 f + \overline{u}^2 v^2 \overline{f} + 2u\overline{v}(u\overline{u} - v\overline{v})g + 2\overline{u}v(u\overline{u} - v\overline{v})\overline{g} + 2u\overline{u}v\overline{v}h\big] \tag{3.6}$$

with coefficients f, g, h (k is not needed to derive the equations) as Poisson series with power variables p_i, q_i, w_i ($q_i = \overline{p}_i$) and exponential variables ζ_i for $i = 3$ and $i = 9$. Substituting (3.6) into (2.9) one gets the expansion for the right–hand member

$$R_5 = K\big[u^3\overline{v}f - \overline{u}v^3\overline{f} + u^2(u\overline{u} - 3v\overline{v})g + v^2(v\overline{v} - 3u\overline{u})\overline{g} + uv(u\overline{u} - v\overline{v})h\big]\overline{v}. \tag{3.7}$$

4. Reduction to the Secular System

The right–hand members of (2.8) depend on planetary and lunar coordinates. Usually, they are regarded as known functions of time. Instead of this, we shall regard them as functions satisfying some definite differential equations. Combining (2.8) with these equations we obtain a complete system describing the planetary and lunar motions and Earth's rotation. In GPT extended for 8 major planets ($i = 1, 2, \ldots, 8$ with $i = 3$ for the Earth–Moon barycentre) and the Moon ($i = 9$) one takes, as a starting point, a quasi–periodic intermediary with arbitrary constants n_i and ε_i and then constructs a normalizing transformation from p_i, q_i, w_i to eccentric and oblique variables $a_i, \overline{a}_i, b_i, \overline{b}_i$ admitting the immediate reduction to the secular system. The resulting equations have the form

$$\dot{X} = \mathrm{i}\,\mathcal{N}\big[PX + R(X,t)\big]. \tag{4.1}$$

Here $X = (a, \overline{a}, b, \overline{b}, u, \overline{u}, v, \overline{v})$ and $R = (R_1, \ldots, R_8)$ are vectors with 40 components (a, b and R_i for $i = 1, 2, 3, 4$ are 9–vectors). $\mathcal{N}$ and P are 40×40 diagonal matrices of the structure

$$\mathcal{N} = \mathrm{diag}(N, N, N, N, n, n, n, n), \quad P = \mathrm{diag}(E, -E, E, -E, 1, -1, 1, -1), \tag{4.2}$$

N is 9×9 diagonal matrix of mean motions n_i and E is 9×9 unitary matrix. The GPT right–hand members are

$$R_1 = DA\overline{D}a + D\Phi\,, \quad R_2 = -\overline{R}_1\,, \quad R_3 = DB\overline{D}b + D\Psi\,, \quad R_4 = -\overline{R}_3 \tag{4.3}$$

with 9×9 diagonal matrix $D = \mathrm{diag}(\exp \mathrm{i}\, \lambda_k)$. In GPT the transformation

$$a = D\alpha\,, \qquad b = D\beta \tag{4.4}$$

results in the autonomous secular system for the major planets and the Moon

$$\dot{\alpha} = \mathrm{i}\, N[A\alpha + \Phi(\alpha, \overline{\alpha}, \beta, \overline{\beta})], \qquad \dot{\beta} = \mathrm{i}\, N[B\beta + \Psi(\alpha, \overline{\alpha}, \beta, \overline{\beta})]. \tag{4.5}$$

Equations (4.1) are treated as in GPT. The transformation to new variables Y

$$X = Y + \Gamma(Y, t) \tag{4.6}$$

results in a new system

$$\dot{Y} = \mathrm{i}\, \mathcal{N}\big[PY + F(Y, t)\big]. \tag{4.7}$$

Functions Γ and F are found by iterations as series in powers of Y with quasi–periodic coefficients of t (by means of the differences of the mean longitudes λ_i)

$$U = R - \mathcal{N}^{-1}\Gamma_Y \mathcal{N} U^*\,, \qquad \Gamma_t + \mathrm{i}(\Gamma_Y \mathcal{N} PY - \mathcal{N} P \Gamma) = \mathrm{i}\, \mathcal{N} U^+\,, \tag{4.8}$$

$$U = U^* + U^+\,, \qquad F = U^*\,. \tag{4.9}$$

The splitting of U is aimed to ensure the integration of (4.8) without t–secular terms. Since planetary and lunar equations are already written in the demanded form, we have for $\kappa = 1, \ldots, 4$

$$Y_\kappa = X_\kappa\,, \quad \Gamma_\kappa = U_\kappa^+ = 0\,, \quad F_\kappa = U_\kappa^* = R_\kappa \tag{4.10}$$

with vector Y of new variables $Y = (a, \overline{a}, b, \overline{b}, c, \overline{c}, d, \overline{d})$. The initial expansions of the right–hand members for $\kappa = 5,\ 7$ are of the form

$$R_\kappa = \sum_{p,q,r,s} R^{(\kappa)}_{pqrs} u^p \overline{u}^q v^r \overline{v}^s\,, \qquad R^{(\kappa)}_{pqrs} = \sum_{i,k,l,m} R^{(\kappa pqrs)}_{iklm} \prod_{j=1}^{9} a_j^{i_j} \overline{a}_j^{k_j} b_j^{l_j} \overline{b}_j^{m_j}\,, \tag{4.11}$$

$$R^{(\kappa pqrs)}_{iklm} = \sum_\sigma R^{(\kappa pqrs)}_{iklm\sigma} \exp \mathrm{i}(\sigma\lambda), \qquad \sigma\lambda = \sigma_1\lambda_1 + \ldots + \sigma_9\lambda_9 \tag{4.12}$$

with i, k, l, m being power 9–indices and σ being trigonometric 9–index. The resulting series for U_κ and Γ_κ ($\kappa = 5,\ 7$) have the same form with the coefficients related in virtue of equations (4.8) by means of

$$\Gamma^{(\kappa pqrs)}_{iklm\sigma} = \frac{n U^{(\kappa pqrs)}_{iklm\sigma}}{\sum_{j=1}^{9} n_j(i_j - k_j + l_j - m_j + \sigma_j) + n(p - q + r - s - 1)}\,. \tag{4.13}$$

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The critical values of indices

$$p - q + r - s = 1, \qquad i_j - k_j + l_j - m_j + \sigma_j = 0 \tag{4.14}$$

should be excluded from Γ–series.. These values marked by an asterisk correspond to the series of the secular system for the Earth's rotation

$$U_\kappa^* = \sum\nolimits^* U_{iklm\sigma}^{(\kappa pqrs)} c^p \overline{c}^q d^r \overline{d}^s \prod_{j=1}^{9} \alpha_j^{i_j} \overline{\alpha}_j^{k_j} \beta_j^{l_j} \overline{\beta}_j^{m_j} . \tag{4.15}$$

Thus, the transformation

$$u = c + \Gamma_5, \qquad v = d + \Gamma_7 \tag{4.16}$$

transforms Poisson equations (2.8) into the secular system

$$\dot{c} = \mathrm{i}\, n(c + F_5), \qquad \dot{d} = \mathrm{i}\, n(d + F_7) \tag{4.17}$$

with $F_\kappa = U_\kappa^*(c, \overline{c}, d, \overline{d}, \alpha, \overline{\alpha}, \beta, \overline{\beta})$. Expressions (4.16) describe the short–period nutation depending on mean longitudes of the Sun, the Moon and major planets. System (4.17) is responsible for precession and long–period nutation since it includes lunar evolutionary variables α_9 and β_9 related to the motions of the lunar perigee and node. It might be reasonable to separate precession and long–period nutation terms. Anyway, system (4.17) may be solved without any difficulties by different techniques.

Due to the lack of space, here we confine ourselves by the outline of the technique. All actual calculations were performed using a Poisson series processor described in Ivanova (1996). The work is now in progress and the practical results will be published elsewhere in the future.

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