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PROPERTIES OF QUOTIENT RINGS

S. PAGE

Introduction. In [**1**; **2**; **7**] Gabriel, Goldman, and Silver have introduced the notion of a localization of a ring which generalizes the usual notion of a localization of a commutative ring at a prime. These rings may not be local in the sense of having a unique maximal ideal. If we are to obtain information about a ring R from one of its localizations, $Q_\tau(R)$ say, it seems reasonable that $Q_\tau(R)$ be a tractable ring. This, of course, is what Goldie, Jans, and Vinsonhaler [**4**; **3**; **8**] did in the special case for $Q(R)$ the classical ring of quotients. In this note we extend the above results to arbitrary quotient rings and add to the list prime rings, uniserial and generalized uniserial rings and give a description of the prime and Jacobson radicals of $Q_\tau(R)$.

Preliminaries. Our basic references will be Goldman [**2**] and Lambek [**4**]. Throughout, all rings will have an identity and all modules will be left unitary.

Let R be a ring. By a *torsion theory*, τ, we will mean a family of R-modules closed under arbitrary extensions, homomorphic images, and submodules. We let $\mathscr{F}_\tau$ be the filter of left ideals of R which are annihilators of elements of members of τ. If M is a left R-module, set $\tau(M) = \{m \in M | Um = 0 \text{ for some } U \in \mathscr{F}_\tau\}$. The family τ is then the collection of left R-modules such that $\tau(M) = M$. We will refer to members of τ as torsion modules and $\tau(M)$ will be called the torsion part of M.

If τ is a torsion theory, set

$$Q_\tau(M) = \varinjlim_{U \in \mathscr{F}_\tau} \operatorname{Hom}_R(U, M/\tau(M))$$

for a left R-module M. $Q_\tau(M)$ is the quotient module of M relative to the torsion theory τ.

A left R-module M is called *τ-injective* if for every $U \in \mathscr{F}_\tau$ and every homomorphism $f : U \to M$, f lifts to a homomorphism $f : R \to M$.

The following theorem, in part due to Maranda and in part to Walker and Walker, and found in Goldman [**2**] is needed.

THEOREM 1. *Let $Q_\tau(R)$ be the quotient ring of R relative to the torsion theory τ. The following are equivalent:*

(a) *Every left $Q_\tau(R)$-module is τ-injective.*

(b) *Every left $Q_\tau(R)$-module is τ-torsion free.*

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(c) $Q_\tau(R)U = Q_\tau(R)$ *for every* $U \in \mathscr{F}_\tau$.
(d) $Q_\tau(R) \otimes_R M \cong Q_\tau(M)$ *for every left R-module* M.

Torsion theories which satisfy any or all of the conditions (a)–(d) are said to have property T.

Artinian quotient rings. We start with the problem of determining when $Q_\tau(R)$ is semi-simple Artinian.

LEMMA 1. *If* $Q_\tau(R)$ *is semi-simple Artinian, then* $\mathscr{F}_\tau$ *has property* T.

Proof. Since every left $Q_\tau(R)$-module is projective and hence a direct summand of a free $Q_\tau(R)$ module, all left $Q_\tau(R)$ modules are torsion free.

LEMMA 2. *If* $Q_\tau(R)$ *is semi-simple Artinian and* $0 \rightarrow R \rightarrow Q_\tau(R)$ *exact,* $\mathscr{F}_\tau$ *consists of all essential left ideals of* R.

Proof. If U is an essential left ideal of R, then $Q_\tau(R)U$ is an essential left ideal of $Q_\tau(R)$. But $Q_\tau(R)$ has no essential left ideals other than $Q_\tau(R)$ itself, so $Q_\tau(R)U = Q_\tau(R)$. This implies $U \in \mathscr{F}_\tau$; for if $x \in R$, $x = \sum_{i=1}^n q_i u_i$ where $u_i \in U$ and $q_i \in Q_\tau(R)$, $i = 1, \ldots, n$, and there exists a $U' \in \mathscr{F}_\tau$ such that $U'q_i \subset R$ for all $i = 1, 2, \ldots, n$, and so $U'x \subset U$, whence R/U is a τ-torsion module.

On the other hand, if U is in $\mathscr{F}_\tau$ and U is not essential, then there exists a left ideal A in R such that $A \neq 0$ and $A \cap U = 0$. But then A is isomorphic to a submodule of the τ-torsion module R/U, which means A is a τ-torsion module, which is a contradiction.

In [**5**] Morita shows that if a ring map $\alpha : R \rightarrow S$ is an epimorphism in the category of rings and S becomes a right flat R-module, then $S = Q_\tau(R)$ for some τ and if τ has property T, then $Q_\tau(R)$ is a flat epimorph of R.

The following generalizes the results of Sandomiersky [**6**].

THEOREM 2. *Let*

$$0 \rightarrow R \overset{\alpha}{\rightarrow} S$$

be an exact sequence of ring maps for which S becomes a right flat R-module and α *is a ring epimorphism. Then the following are equivalent:*

(a) S *is semi-simple Artinian.*

(b) S *is the complete ring of left quotients of* R, R *has finite Goldie dimension, and* $Z(R) = 0$.

(c) $S \cong Q_\tau(R)$, $\mathscr{F}_\tau = \{U | U$ *an essential left ideal of* $R\}$, *and for any sequence of left ideals* $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ *there exists a* k *such that* A_{k+i} *is essential in* A_k *for all* $i \geqq 1$.

Proof. Since S is a flat epimorph of R, $S \cong Q_\tau(R)$ for some torsion theory τ. If (a) holds, by Lemmas 1 and 2 $\tau(R)$ is the singular submodule of R, which must be zero because α is monic, so that dense left ideals and essential left ideals coincide making $Q_\tau(R)$ the complete ring of quotients. The finite dimension then follows easily since tensor products commute with direct sums. This shows that (a) $\Rightarrow$ (b).

For (b) $\Rightarrow$ (a) we need that S is semi-simple; but S is a regular ring which can be written as a sum of finitely many indecomposable left ideals which makes it semi-simple.

To prove (a) $\Leftrightarrow$ (c) we note that the condition on the left ideals given in (c) is equivalent to $Q_\tau(R)$ having D.C.C. on left ideals and that R is an essential submodule of $Q_\tau(R)$. With these two observations the proof follows easily.

Remark. This theorem generalizes Goldie's theorem on semi-simple classical quotient rings, for if we take $\tau(M) = \{m | bm = 0, b$ a regular element in $R\}$, then τ is a torsion theory provided R is a left Ore ring, and also every essential ideal contains a regular element if R is a semi-prime Goldie ring.

In case R is commutative we have:

COROLLARY. *If R is commutative, then $Q_\tau(R)$ is semi-simple Artinian if and only if $Q_T(R)$ is the classical ring of quotients and every essential ideal contains a regular element.*

Proof. If $Q_\tau(R)$ is semi-simple $\mathscr{F}_\tau$ is the set of essential ideals. If $U \in \mathscr{F}_\tau$, then $U \supset A_1 \oplus A_2 \oplus \ldots \oplus A_n$ where each $A_i \neq 0$, $Q_\tau(R)A_i$ is an indecomposable $Q_\tau(R)$ left ideal, and $Q_\tau(R)A_1 \oplus \ldots \oplus Q_\tau(R)A_n = Q_\tau(R)$. Now each $Q_\tau(R)A_i$ contains a primitive idempotent $q_i a_i$ such that $q_1 a_1 + q_2 a_2 + \ldots + q_n a_n = 1$ and $q_i a_i q_j a_j = 0$ if $i \neq j$. From this it follows that $a_i q_i a_i = a_i$ and $a_i q_j a_j = 0$ if $i \neq j$. Let $x \in R$. Then $(\sum_{i=1}^n a_i)x = 0 \Rightarrow \sum_{i,j} a_i a_j x = 0$ so that $\sum q_j a_j x = 0$. This shows that $\sum_{i=1}^n a_i$ is a regular element in U.

Conversely, if every essential ideal contains a regular element, then R can contain no infinite direct sums of left ideals. This implies $Q_\tau(R)$ has D.C.C. on left ideals for $Q_\tau(R)$ is already regular.

In case $Q_\tau(R)$ is quasi-Frobenius (Q.F.) we make the following observations:

LEMMA 3. *If $Q_\tau(R)$ is quasi-Frobenius, then τ has property T.*

Proof. Let M be any simple $Q_\tau(R)$ module. Then, since $Q_\tau(R)$ is Q.F., $M \cong L$, a simple left ideal. But $Q_\tau(R)$ is τ-torsion free, so M is τ-torsion free. It follows that every $Q_\tau(R)$ module is τ-torsion free and so τ has property T.

LEMMA 4. *If M is an injective $Q_\tau(R)$-module and $\tau(M) = 0$, then M is R-injective.*

Proof. Let I be a left ideal of R and $f : I \to M$. Then $0 \to QI \to Q$ is exact and $\tilde{f} : QI \to M$, given by $\tilde{f}(qi) = qf(i)$, is well defined because M is torsion free. Now there exists a map $f : Q \to M$ such that g restricted to QI is $\tilde{f}$. The restriction of g to the image of R yields the desired map because $\tau(R) \cap I \subset \ker f$ again because $\tau(M) = 0$.

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LEMMA 5. *If* $0 \to R \to Q_\tau(R)$ *is exact and* τ *has property* T, *then* $Q_\tau(R)$ *is neotherian if and only if for every sequence of left ideals* $A_1 \subset A_2 \subset \ldots$ *of* R *there exists* k *such that* $\tau(A_{k+i}/A_k) = A_{k+i}/A_k$ *for all* $i \geqq 0$.

Proof. The proof is essentially the same as that for the D.C.C. in Theorem 1.

THEOREM 3. *If* $0 \to R \to Q_\tau(R)$ *is exact, the following are equivalent:*

(a) $Q_\tau(R)$ *is a* Q.F. *quotient ring of* R.

(b) $Q_\tau(R)$ *is the injective hull of* R, τ *has property* T, *and for any sequence of left ideals* $A_1 \subset A_2 \subset \ldots \subset A_n \ldots$ *there exists* k *such that* $\tau(A_{k+j}/A_k) = A_{k+j}/A_k$ *for all* $j \geqq 0$.

(c) $Q_\tau(R)$ *is the injective hull of* R. *For every finitely generated* R*-module* M *which is torsion free there exists an integer* n *such that* $0 \to Q_\tau(M) \to \sum_{i=1}^n Q_\tau(R)$, τ *has property* T, *and* R *has the maximum condition on left annihilators of* $Q_\tau(R)$.

Proof. (a) $\Leftrightarrow$ (b) follows from the lemmas. (a) $\Leftrightarrow$ (c) follows by the argument of Jans in [**3**] with only minor modifications.

From another point of view we obtain:

THEOREM 4. *If* $0 \to R \to S$ *is an exact sequence of rings*, S *is* Q.F., $\tau(S \otimes_R S) = 0$, *and* $\tau(S/R) = S/R$, *then* $S \cong Q_\tau(R)$.

Proof. We first need that S is a quotient ring for some torsion theory. It suffices by [**5**] to show S is a flat epimorphism of R. $\tau(S \otimes_R S) = 0$ implies that $R \to S$ is an epimorphism, for the kernel of $S \otimes_R S \to S$ is $\tau(S \otimes_R S)$. To see that S is right flat over R we need to show that $\mathrm{Hom}_Z(S, Q/Z)$ is left injective over R (see [**4**]). But $\mathrm{Hom}_Z(S, Q/Z)$ is injective as an S module, hence projective as an S module, therefore τ-torsion free, and so injective by the same methods as used in Lemma 4. The fact that $\tau(S/R) = S/R$ completes the proof.

The class of Q.F. rings is contained in the class of Q.F. 3 rings. Namely, R is Q.F. 3 if R has a minimal faithful left R-module (minimal in the sense that it is a direct summand of every faithful R-module).

THEOREM 5. *If* $0 \to R \to Q_\tau(R)$ *exact and* R *is* Q.F. 3, *then* $Q_\tau(R)$ *is* Q.F. 3.

Proof. If V is a minimal faithful R module then V is both injective, for it is a direct summand of the hull of R, and projective for it is a direct summand of R. We then have $\tau(V) = 0$, so $Q_\tau(V) = V$, so that V is a $Q_\tau(R)$ module and is injective as such. Also, V is $Q_\tau(R)$ projective for V is a direct summand of $Q_\tau(R)$. V is $Q_\tau(R)$ faithful, for if $qV = 0$, $q \in Q_\tau(R)$, then for suitable $U \in \mathscr{F}_\tau$ we have $Uq \subseteq R$ and $UqV = 0$, giving $Uq = 0$ so that $q = 0$. Finally we need to show V is a direct summand of every faithful $Q_\tau(R)$ module. Suppose X is a faithful $Q_\tau(R)$-module, then X is a faithful R-module so V is a direct summand of X as an R-module. Let $X = V \oplus W$. Suppose W

is not a $Q_\tau(R)$ module. Then $qw \notin W$ for some $q \in Q_\tau(R)$. Let $qw = v_1 + w_1$. Then for some $U \in \mathscr{F}_\tau$, $Uq \subset R$ so that $Uqw = Uv_1 + Uw_1 \in W$ for all $u \in U$, i.e., $Uv_1 = 0$. But, V is τ-torsion free so $v_1 = 0$ and the sum is $Q_\tau(R)$-direct.

If we ask that $Q_\tau(R)$ be Q.F. 3 and Artinian we obtain:

THEOREM 6. *Let* $0 \to R \to Q_\tau(R)$ *be exact. Then the following are equivalent:*
(a) $Q_\tau(R)$ *is an Artinian* Q.F. 3 *ring.*
(b) (i) $Q_\tau(R)$ *is left Noetherian;*
(ii) $Q_\tau(R)$ *contains a faithful injective R-module X such that for every R-submodule Y of X, X/QY contains a simple Q module or is zero, and R imbeds a finite direct sum of copies of X.*

Proof (a) $\Rightarrow$ (b). If $Q_\tau(R)$ is an Artinian Q.F. 3 ring, then let X be the minimal faithful ideal of $Q_\tau(R)$. $Q_\tau(R)$ imbeds in a finite direct sum of copies of X, and therefore, so does R. By Lemma 4 X is R-injective. The last part of (ii) is clear so that X will serve for the faithful injecture of (ii).

Conversely, if (b) holds we can show the X of (ii) is a faithful injective projective $Q_\tau(R)$-module in much the same manner as in the proof of Theorem 4, so that it remains to show $Q_\tau(R)$ is Artinian. Note that $0 \to Q_\tau(R) \to \oplus \sum_{i=1}^{n} X$ is exact because R is essential in $Q_\tau(R)$. X is a Noetherian $Q_\tau(R)$-module such that every factor module has a non-zero socle and each of these socles is a finite direct sum of simple $Q_\tau(R)$-modules. A simple induction argument shows X has a composition series and hence has D.C.C., so $Q_\tau(R)$ has D.C.C. on left ideals which completes the proof.

Radicals. The Jacobson radical is the intersection of the maximal left ideals in any ring with unit. An alternative description is that it is the intersection of the annihilators of simple modules. This latter description provides a suitable method of calculating the Jacobson radical of $Q_\tau(R)$.

First we need a definition:

Definition. For a torsion theory τ, S is called a *support module* of τ if
(i) $S \neq 0$;
(ii) $\tau(S) = 0$; and
(iii) $\tau(S/S') = S/S'$ for all non-zero R-submodules S' of S.

We will call S *simply supporting* for τ if S is a support module and $Q_\tau(R)S' = Q_\tau(R)S$ for all non-zero R-submodules S'. Note that if τ has property T all supporting modules are simply supporting.

THEOREM 7. *The Jacobson radical of* $Q_\tau(R) = \bigcap$ {*annihilators of* $Q_\tau(S)$, *S a simply supporting module for* τ}.

Proof. We shall first show that a simple $Q_\tau(R)$-module is $Q_\tau(S)$ for some simply supporting R-module S and conversely. Let X be a simple $Q_\tau(R)$-module. Then $X \cong Q_\tau(R)/M$ where M is a maximal left ideal of $Q_\tau(R)$. Let Y be the inverse image of M in R under the canonical map of R to $Q_\tau(R)$.

Then R/Y is a simple supporting module for τ. On the other hand, if S is simply supporting for τ and Y is a $Q_\tau(R)$-submodule of $Q_\tau(S)$, then $Y \cap S \neq 0$ for S is essential in $Q_\tau(S)$. But then, $Q_\tau(R)(Y \cap S) \subset Y$, so $Q_\tau(S)$ is a simple $Q_\tau(R)$-module. The rest of the proof is clear.

In case τ has property T we obtain:

COROLLARY. *If τ has property T and $0 \to R \to Q_\tau(R)$ is exact, then the Jacobson radical of $Q_\tau(R) = Q_\tau(R)$ ($\bigcap$ {annihilators in R of supporting R-modules}).*

Proof. In this case each left ideal of $Q_\tau(R)$ is generated by a left ideal of R and each supporting module is simply supporting. A simple inclusion argument gives the desired result.

Connected with the Jacobson radical and Q.F. rings are the classes of uniserial and generalized uniserial rings.

THEOREM 8. *Let $0 \to R \to Q_\tau(R)$ be exact. $Q_\tau(R)$ is uniserial if and only if τ has property T and there exist a unique nilpotent left ideal X of R such that R/X is a supporting module and $Q_\tau(X^i/X^{i+1}) \cong Q_\tau(R/X)$ for all $i \geqq 0$.*

Proof. If $Q_\tau(R)$ is uniserial, then $Q_\tau(R)$ is a Q.F. ring; so τ has property T. Let N denote the Jacobson radial of $Q_\tau(R)$. Let $X = R \cap N$. Then R/X is supporting and X is nilpotent. Also, $QX^i = N^i$ for all i, for N is a two-sided ideal in $Q_\tau(R)$. It follows that $Q_\tau(X^i/X^{i+1}) = Q_\tau(R/X)$ for all i.

Conversely, if τ has property T, then $Q_\tau(R)X$ is the unique maximal left ideal of $Q_\tau(R)$; hence it must be a two-sided ideal of $Q_\tau(R)$ and also $Q_\tau(R)XQ_\tau(R)X = Q_\tau(R)X^2$. From this we see that $Q_\tau(R)X$ is nilpotent and the Jacobson radical of $Q_\tau(R)$. Moreover, since $Q_\tau(X^i/X^{i+1}) = Q_\tau(R/X)$ is simple, $Q_\tau(R)$ has a composition series, which ensures $Q_\tau(R)$ is a uniserial ring.

THEOREM 9. *If $0 \to R \to Q_\tau(R)$ is exact and τ has property T, then $Q_\tau(R)$ is generalized uniserial if and only if each indecomposable direct summand of R has a unique submodule X such that X is nilpotent and X^i/X^{i+1} is a support module for every τ.*

Proof. The proof follows from a similar argument to that used in Theorem 7.

The Baer lower radical is also the prime radical of a ring and is given by the intersection of the annihilators of prime modules. A module M is prime if the annihilator of any non-zero submodule N is the same as the annihilator of M. The next lemma tells where the prime modules over $Q_\tau(R)$ arise.

LEMMA 6. *If τ has property T and $\tau(M) = 0$, then $Q_\tau(M)$ is prime if and only if M is prime.*

Proof. Let $l_R(x) = \{r | rx = 0\}$ for $x \in M$, an R module. Suppose M is a prime R-module, τ has property T and $\tau(M) = 0$. Let N be a submodule of M.

Then we claim that $l_R(QN) = l_R(QM)$ if and only if $l_Q(QN) = l_Q(QM)$. To see this, let $r \in l_R(QN)$ and $r \notin l_R(QM)$. Then there exists $rqm \neq 0$ and $rqN = 0$. For some $U \in \mathscr{F}_\tau$, $Urq \subseteq R$ and $Urqm \cap M \neq 0$. But, $Urq \subseteq l_R(N)$ and so $Urq \subseteq l_R(M)$, a contradiction. This, of course, says that $QM = Q(M)$ is a prime Q-module.

Conversely, if τ has property T, M is a prime Q-module and N is an R-submodule of M, $l_R(N) = l_R(M)$, and $l_R(QN) \subset l_R(N)$. If the containment is proper, then for some $r \in R$, $rN = 0$ and $rqn \neq 0$ for some $q \in Q$ and $n \in N$. But there is a $U \in \mathscr{F}_\tau$ such that $Urq \subseteq R$ and $UrqN \cap N \neq 0$, which is a contradiction.

THEOREM 10. *If τ has property T, the Baer lower radical of $Q_\tau(R) = Q_\tau(R)$ ($\bigcap$ {annihilators in R of prime τ-torsion free R-modules})*.

Proof. A simple inclusion argument using Lemma 6 yields the desired proof.

Remark. The condition that $0 \to R \to Q_\tau(R)$ is exact in the theorems can be dropped with the obvious changes so that we work over $R/\tau(R)$ in effect.

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