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HARDY-LITTLEWOOD-SOBOLEV THEOREMS OF FRACTIONAL INTEGRATION ON HERZ-TYPE SPACES AND ITS APPLICATIONS

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ABSTRACT. In this paper, the authors first establish the Hardy-Littlewood-Sobolev theorems of fractional integration on the Herz spaces and Herz-type Hardy spaces. Then the authors give some applications of these theorems to the Laplacian and wave equations.

1. **Introduction.** It is well-known that Baernstein and Sawyer in [1] have shown the Herz spaces are very useful in studying the sharpness of multiplier theorems on $H^p(\mathbb{R}^N)$ spaces. This paper will involve some other applications. First, let us introduce some notation. For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^N : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{C_k}$, where χ_{C_k} is the characteristic function of set C_k. Recently, the authors in [7] introduce the following weighted Herz spaces and give its decomposition characterization.

DEFINITION 1.1 ([7]). Assume $0 < \alpha < \infty$, $0 < p < \infty$, $1 \leq q < \infty$ and ω_i $(i = 1, 2)$ are non-negative weight functions.

(a) The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega_1, \omega_2) := \left\{f \in L^q_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}, \omega_2) : \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} < \infty\right\}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} := \left\{\sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/N} \|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^N)}\right\}^{1/p}$$

and

$$\|g\|_{L^q_{\omega_2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |g(x)|^q \omega_2(x)\, dx\right)^{1/q}.$$

(b) The non-homogeneous weighted Herz space $K_q^{\alpha,p}(\omega_1, \omega_2)$ is defined by

$$K_q^{\alpha,p}(\omega_1, \omega_2) := L^q_{\omega_2}(\mathbb{R}^N) \cap \dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$$

and

$$\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)} := \|f\|_{L^q_{\omega_2}(\mathbb{R}^N)} + \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

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And the authors [7] have pointed out that if $\omega_1 \in A_1$ (Muckenhoupt weight), then

$$\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)} \sim \left\{ \|f\chi_{B_0}\|^p_{L^q_{\omega_2}(\mathbb{R}^N)} + \sum_{k=1}^{\infty} [\omega_1(B_k)]^{\alpha p/N} \|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^N)} \right\}^{1/p}.$$

Obviously, if $\omega_1 \equiv \omega_2 \equiv 1$, then $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ and $K_q^{\alpha,p}(\omega_1,\omega_2)$ are the standard Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^N)$ and $K_q^{\alpha,p}(\mathbb{R}^N)$ respectively, see [1]. Also in [7], the authors establish a boundedness theorem of operators on the weighted Herz space with $0 < \alpha < N(1-1/q)$ for a large class of sublinear operators. But, this theorem is not true when $N(1-1/q) \le \alpha < \infty$. However, the authors in [8] find out a substitute result by a proper substitute space instead of the weighted Herz space when $N(1-1/q) \le \alpha < \infty$. This substitute space is just the following Hardy spaces $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ and $HK_q^{\alpha,p}(\omega_1,\omega_2)$. And in [8], the authors also give their atom decompositional theory.

DEFINITION 1.2 ([8]). Let $\omega_1, \omega_2 \in A_1$, $0 < p < \infty$, $1 < q < \infty$, $N(1-1/q) \le \alpha < \infty$ and $G(f)$ be the grand maximal function of f (see [4]).

(a) The Hardy space $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ associated with $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) := \{f \in \mathcal{S}'(\mathbb{R}^N) : G(f) \in \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)\}$$

and

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} := \|G(f)\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

(b) The Hardy space $HK_q^{\alpha,p}(\omega_1,\omega_2)$ associated with $K_q^{\alpha,p}(\omega_1,\omega_2)$ is defined by

$$HK_q^{\alpha,p}(\omega_1,\omega_2) := \{f \in \mathcal{S}'(\mathbb{R}^N) : G(f) \in K_q^{\alpha,p}(\omega_1,\omega_2)\}$$

and

$$\|f\|_{HK_q^{\alpha,p}(\omega_1,\omega_2)} := \|G(f)\|_{K_q^{\alpha,p}(\omega_1,\omega_2)}.$$

If $\omega_1 \equiv \omega_2 \equiv 1$, we denote $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ by $H\dot{K}_q^{\alpha,p}(\mathbb{R}^N)$, and $HK_q^{\alpha,p}(\omega_1,\omega_2)$ by $HK_q^{\alpha,p}(\mathbb{R}^N)$. Clearly, $HK_q^{N(1-1/q),1}(\mathbb{R}^N)$ is just the space introduced by Chen-Lau [2] and García-Cuerva [5].

On the other hand, it is also well-known that the Hardy-Littlewood-Sobolev theorems of fractional integration on $H^p(\mathbb{R}^N)$ spaces play a profound and extensive role in harmonic analysis and partial differential equations, see [3, 10, 12]. The main purpose of Section 2 in this paper is to establish the Hardy-Littlewood-Sobolev theorems of fractional integration on the Herz space and the Herz-type Hardy space by means of their decompositional characters in [7] and [8]. Using the boundedness theorem of fractional integration on $L^p_\omega(\mathbb{R}^N)$ (ω: power weight) established by Lu and Soria in [6] (also see [11]), in Section 3, we investigate the boundedness on the non-homogeneous Herz-type space with the power weight of fractional integration. These results are the generalization and supplement of the results of Lu-Soria [6], which generalize the results of Stein-Weiss [11]. In addition, many applications of the Hardy space theory to partial differential equations have been found, see [3] and its references. In Section 4 of this paper, by means of some ideas comming from [3] and the results in Section 2, we give some simple applications of the Herz-type space to the Laplacian equations and the wave equations. More interesting applications of Herz-type space refer to the authors' other papers.

2. **Hardy-Littlewood-Sobolev theorems.** In this section, we shall establish the Hardy-Littlewood-Sobolev theorems on the Herz spaces and Herz-type Hardy spaces by means of their decompositional characterizations in [7] and [8].

THEOREM 2.1. *Let* $0 < \ell < N$ *and*

$$I_\ell(f)(x) = C_{\ell,N}\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\ell}}\,dy.$$

Suppose $1 < q_1 < \infty$, $0 < p_1 \le \min\{q_1,p_2\}$, $0 < \alpha_1 < N(1-1/q_1)$, $1/q_2 = 1/q_1(1-\ell p_1/N)$ *and* $\alpha_2 = \alpha_1 + \ell(p_1/q_1 - 1)$. *Then*

$$I_\ell\colon \dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)\big(\,or\, K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)\big) \longrightarrow \dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^N)\big(\,or\, K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^N)\big).$$

PROOF. We only prove the theorem for the homogeneous case. Let $f \in \dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$, then $f(x) = \sum_{k=-\infty}^{\infty}\lambda_k b_k(x)$, where $\|f\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)} \sim \inf(\sum_k |\lambda_k|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of f), and b_k is a dyadic central (α_1,q_1)-unit with the support B_k, that is, $\operatorname{supp} b_k \subset \{x : |x| \le 2^k\}$ and $\|b_k\|_{L^{q_1}(\mathbb{R}^N)} \le |B_k|^{-\alpha_1/N}$, see [7] for the details. We write

$$\begin{aligned}
\|I_\ell(f)\|^{p_2}_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^N)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_2}\|I_\ell(f)\chi_k\|^{p_2}_{L^{q_2}(\mathbb{R}^N)}\\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_2}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|\,\|I_\ell(b_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)}\Big)^{p_2}\\
&\quad + C\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_2}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|\,\|I_\ell(b_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)}\Big)^{p_2}\\
&:= CI_1 + CI_2.
\end{aligned}$$

Let us first estimate I_2. Set $1/q_0 = 1/q_1 - \ell/N$. Using $I_\ell\colon L^{q_1}(\mathbb{R}^N) \longrightarrow L^{q_0}(\mathbb{R}^N)$ and Hölder's inequality, we get

$$\begin{aligned}
\|I_\ell(b_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} &\le C\|I_\ell(b_j)\chi_k\|_{L^{q_0}(\mathbb{R}^N)}2^{kN(1/q_2-1/q_0)}\\
&\le C\|b_j\|_{L^{q_1}(\mathbb{R}^N)}2^{kN(1/q_2-1/q_0)}\\
&\le C2^{-j\alpha_1+kN(1/q_2-1/q_0)}.
\end{aligned}$$

Thus, if set $1/p_1 + 1/p_1' = 1$, we have

$$\begin{aligned}
I_2^{p_1/p_2} &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_1} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{-j\alpha_1 + kN(1/q_2 - 1/q_0)} \Big)^{p_1} \\
&\le C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha_1} \Big)^{p_1} \\
&\le \begin{cases} C \sum_{k=-\infty}^{\infty} (\sum_{j=k-1}^{\infty} |\lambda_j|^{p_1} 2^{(k-j)\alpha_1 p_1}), & 0 < p_1 \le 1; \\ C \sum_{k=-\infty}^{\infty} (\sum_{j=k-1}^{\infty} |\lambda_j|^{p_1} 2^{(k-j)\alpha_1 p_1/2}) & \\ \quad \times (\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_1 p_1'/2})^{p_1/p_1'}, & 1 < p_1 < \infty \end{cases} \\
&\le \begin{cases} C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} (\sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha_1 p_1}), & 0 < p_1 \le 1; \\ C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} (\sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha_1 p_1/2}), & 1 < p_1 < \infty \end{cases} \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{aligned}$$

By the way, this computational technique will be used throughout this paper. We shall not go into details in the following. That is,

$$I_2 \le C \|f\|^{p_2}_{\dot{K}_{q_1}^{\alpha_1, p_1}(\mathbb{R}^N)}.$$

For I_1, note that $j \le k-2$, we then have

$$\begin{aligned}
\|I_\ell(b_j)\chi_k\|^{q_2}_{L^{q_2}(\mathbb{R}^N)} &\le C \int_{C_k} \frac{1}{|x|^{(N-\ell)q_2}} \Big(\int |b_j(y)|\, dy \Big)^{q_2} dx \\
&\le C 2^{-j\alpha_1 q_2 + jN(1-1/q_1)q_2 + kN - k(N-\ell)q_2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_1^{p_1/p_2} &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_1} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{-j\alpha_1 + jN(1-1/q_1) + kN/q_2 - k(N-\ell)} \Big)^{p_1} \\
&= C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(k-j)(\alpha_1 + N(1/q_1 - 1))} \Big)^{p_1} \le C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{aligned}$$

That is,

$$I_1 \le C \|f\|^{p_2}_{\dot{K}_{q_1}^{\alpha_1, p_1}(\mathbb{R}^N)}.$$

This finishes the proof of Theorem 2.1.

In Theorem 2.1, if we restrict $1 < q_1 < N/\ell$, we shall get the following more refined theorem.

THEOREM 2.2. *Let* $0 < \ell < N$ *and* $I_\ell(f)$ *be as in Theorem 2.1. Assume that* $0 < \alpha_1 < N(1 - 1/q_1)$, $1 < q_1 < N/\ell$, $1/q_2 = 1/q_1 - \ell/N$ *and* $0 < p_1 \le p_2 < \infty$. *Then* I_ℓ *maps* $\dot{K}_{q_1}^{\alpha_1, p_1}(\mathbb{R}^N)$ *(or* $K_{q_1}^{\alpha_1, p_1}(\mathbb{R}^N)$*) into* $\dot{K}_{q_2}^{\alpha_1, p_2}(\mathbb{R}^N)$ *(or* $K_{q_2}^{\alpha_1, p_2}(\mathbb{R}^N)$*).*

PROOF. We only prove it for the homogeneous case. Let $f \in \dot{K}_{q_1}^{\alpha_1, p_1}(\mathbb{R}^N)$; then $f(x) = \sum_{k=-\infty}^{\infty} \lambda_k b_k(x)$, where b_k is a dyadic central (α_1, q_1)-unit with the support B_k and

$\|f\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)} \sim \inf(\sum_k |\lambda_k|^{p_1})^{1/p_1}$. We write

$$\begin{aligned}\|I_\ell(f)\|^{p_2}_{\dot{K}_{q_2}^{\alpha_1,p_2}} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_2} \|I_\ell(f)\chi_k\|^{p_2}_{L^{q_2}(\mathbb{R}^N)} \\ &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_2} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| \, \|I_\ell(b_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} \Big)^{p_2} \\ &\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_2} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| \, \|I_\ell(b_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} \Big)^{p_2} \\ &:= CI_1 + CI_2\end{aligned}$$

For I_2, using $I_\ell\colon L^{q_1}(\mathbb{R}^N) \longrightarrow L^{q_2}(\mathbb{R}^N)$, we get

$$\begin{aligned}I_2^{p_1/p_2} &\leq \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| \, \|I_\ell(b_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} \Big)^{p_1} \\ &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| \, \|b_j\|_{L^{q_1}(\mathbb{R}^N)} \Big)^{p_1} \\ &\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha_1} \Big)^{p_1} \leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.\end{aligned}$$

That is,

$$I_2 \leq C\|f\|^{p_2}_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)}.$$

For I_1, note that if $j \leq k-2$, then

$$\begin{aligned}\|I_\ell(b_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} &\leq C\Big\{ \int_{C_k} \frac{1}{|x|^{(N-\ell)q_2}} \Big(\int |b_j(y)| \, dy \Big)^{q_2} dx \Big\}^{1/q_2} \\ &\leq C2^{-j\alpha_1 + (j-k)N(1-1/q_1)}.\end{aligned}$$

Thus,

$$\begin{aligned}I_1^{p_1/p_2} &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{-j\alpha_1 + (j-k)N(1-1/q_1)} \Big)^{p_1} \\ &\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(k-j)(\alpha_1 - N(1-1/q_1))} \Big)^{p_1} \leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.\end{aligned}$$

Therefore,

$$I_1 \leq C\|f\|^{p_2}_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)}.$$

This finishes the proof of Theorem 2.2.

Note that if $\alpha_1 = 0$ and $p_1 = q_1$, then Theorem 2.1 and 2.2 are the standard Hardy-Littlewood-Sobolev theorem, see [10]. Thus, Theorem 2.1 and 2.2 are the generalization and the supplement of the standard Hardy-Littlewood-Sobolev theorem. On the other hand, the above two theorems both require the restriction of $\alpha_1 < N(1 - 1/q_1)$. If we want to get rid of this restriction, similar to the $L^p(\mathbb{R}^N)$ case (see [12]), we must replace the Herz space by the Herz-type Hardy space, also see[8]. We have the following three cases.

THEOREM 2.3. *Let ℓ and $I_\ell(f)$ be as in Theorem 2.1. Suppose* $1 < q_1 < \infty$, $1/q_2 = 1/q_1(1-\ell p_1/N)$, $0 < p_1 \le \min\{q_1, p_2\} < \infty$, $N(1-1/q_1) \le \alpha_1 < \infty$ *and* $\alpha_2 = \alpha_1 + \ell(p_1/q_1 - 1)$. *Then I_ℓ maps* $H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$ *(or* $HK_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$*) into* $\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^N)$ *(or* $K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^N)$*).*

PROOF. Similar to Theorem 2.1, it suffices to consider homogeneous case. Let $f \in H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$, then $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)} \sim \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ and a_j is a dyadic central (α_1, q_1)-atom, that is

i) $\operatorname{supp} a_j \subset B_j$;

ii) $\|a_j\|_{L^{q_1}(\mathbb{R}^N)} \le |B_j|^{-\alpha_1/N}$;

iii) $\int a_j(x)x^\beta\,dx = 0$, $|\beta| \le s_1$, $s_1 \ge [\alpha_1 + N(1/q_1 - 1)]$,

see [8] for the details. Note that $p_1 \le p_2$, we write

$$\begin{aligned}
\|I_\ell(f)\|^{p_1}_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^N)} &:= \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_2}\|I_\ell(f)\chi_k\|^{p_2}_{L^{q_2}(\mathbb{R}^N)}\Big)^{p_1/p_2} \\
&\le \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_1}\|I_\ell(f)\chi_k\|^{p_1}_{L^{q_2}(\mathbb{R}^N)} \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_1}\Big(\sum_{j=-\infty}^{k-2} |\lambda_j|\,\|I_\ell(a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)}\Big)^{p_1} \\
&\quad + C\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_1}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|\,\|I_\ell(a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)}\Big)^{p-1} \\
&:= CI_1 + CI_2.
\end{aligned}$$

For I_2, note that $1/q_2 = 1/q_1 - (p_1/q_1)(\ell/N) \ge 1/q_1 - \ell/N := 1/q_0$, we then have

$$\begin{aligned}
\|I_\ell(a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} &\le \|I_\ell(a_j)\chi_k\|_{L^{q_0}(\mathbb{R}^N)}|C_k|^{1/q_2 - 1/q_0} \\
&\le C\|a_j\|_{L^{q_1}(\mathbb{R}^N)}2^{kN(1/q_2-1/q_0)} \\
&\le C2^{-j\alpha_1 + kN(1/q_2-1/q_0)}.
\end{aligned}$$

Thus

$$\begin{aligned}
I_2 &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_1}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|2^{-j\alpha_1 + kN(1/q_2-1/q_0)}\Big)^{p_1} \\
&\le C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_1}|\lambda_j|\Big)^{p_1} \\
&\le C\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{aligned}$$

For I_1, we first make $|x-y|^{-N+\ell}$ into the Taylor expansion at x and use the vanishing-moment condition of a_j, we get

$$\begin{aligned}
\|I_\ell(a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} &\le C\Big\{\int_{C_k}\Big(\int_{B_j}\frac{|a(y)||y|^{s_1+1}}{|x|^{N-\ell+s_1+1}}\,dy\Big)^{q_2}dx\Big\}^{1/q_2} \\
&\le C2^{k\{N/q_2 - (N-\ell+s_1+1)\} + j\{s_1+1-\alpha_1+N(1-1/q_1)\}}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
I_1 &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_1} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{k\{N/q_2-(N-\ell+s_1+1)\}+j\{s_1+1-\alpha_1+N(1-1/q_1)\}} \Big)^{p_1} \\
&\le C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(k-j)(\alpha_1+N/q_1-N-s_1-1)} \Big)^{p_1} \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{aligned}
$$

This finishes the proof of the theorem.

Similar to the case of Herz space, if we restrict $1 < q_1 < N/\ell$, we can get the following more exact theorem.

THEOREM 2.4. *Let ℓ and $I_\ell(f)$ be as in Theorem 2.1. Assume that* $1 < q_1 < N/\ell$, $1/q_2 = 1/q_1 - N/\ell$, $0 < p_1 \le p_2 < \infty$ *and* $N(1-1/q_1) \le \alpha_1 < \infty$. *Then I_ℓ maps* $H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$ *(or* $HK_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$*) into* $\dot{K}_{q_2}^{\alpha_1,p_2}(\mathbb{R}^N)$ *(or* $K_{q_2}^{\alpha_1,p_2}(\mathbb{R}^N)$*).*

PROOF. It suffices to study the homogeneous case. Suppose $f \in H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$, as in Theorem 2.3, we set $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where a_j is a dyadic central (α_1, q_1)-atom with the support B_j and s_1-order of vanishing moments, $s_1 \ge [\alpha_1 + N(1/q_1 - 1)]$. Note that $p_1 \le p_2$, we have

$$
\begin{aligned}
\|I_\ell(f)\|^{p_1}_{\dot{K}_{q_2}^{\alpha_1,p_2}(\mathbb{R}^N)} &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| \, \|I_\ell(a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} \Big)^{p_1} \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| \, \|I_\ell(a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} \Big)^{p_1} \\
&:= CI_1 + CI_2.
\end{aligned}
$$

For I_2, using $I_\ell \colon L^{q_1}(\mathbb{R}^N) \longrightarrow L^{q_2(\mathbb{R}^N)}$, we get

$$
\begin{aligned}
I_2 &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| \, \|a_j\|_{L^{q_1}(\mathbb{R}^N)} \Big)^{p_1} \\
&\le C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha_1} \Big)^{p_1} \le C \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{aligned}
$$

For I_1, similar to the proof of Theorem 2.3, using the Taylor expansion of $|x-y|^{-N+\ell}$ at $|x|$ and the s_1-order vanishing moments of a_j with $s_1 \ge [\alpha_1 + N(1/q_1 - 1)]$, we first get

$$
\begin{aligned}
\|I_\ell(a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^N)} &\le C \Big\{ \int_{C_k} \Big(\int_{B_j} \frac{|a_j(y)| \, |y|^{s_1+1}}{|x|^{N-\ell+s_1+1}} \, dy \Big)^{q_2} dx \Big\}^{1/q_2} \\
&\le C 2^{j(s_1+1-\alpha_1+N(1-1/q_1))+k\left(N/q_2-(N-\ell+s_1+1)\right)}.
\end{aligned}
$$

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Thus,

$$\begin{aligned} I_1 &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1}\Big(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{j(s_1+1-\alpha_1+N(1-1/q_1))+k(N/q_2-(N-\ell+s_1+1))}\Big)^{p_1} \\ &\le C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)(s_1+1-\alpha_1+N(1-1/q_1))}\Big)^{p_1} \\ &\le \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}, \end{aligned}$$

and we finish the proof of Theorem 2.4.

In Theorem 2.4, if we also restrict $\alpha_1 \ge N(1-1/q_2)$, then we can get more refined results. Before doing that, we first come to establish the molecular decomposition of the space $\dot{HK}_q^{\alpha,p}(1,|x|^{-\beta})$ with $0 \le \beta < N$.

DEFINITION 2.1. Let $\omega(x) = |x|^{-\beta}$, $0 \le \beta < N$, $1 < q < \infty$, $N(1-1/q) \le \alpha < \infty$, non-negative integer $s \ge [\alpha + N(1/q-1)]$, $\varepsilon > \max\{s/N + \beta/(Nq), \alpha/N + 1/q - 1\}$, $a = 1 - 1/q - \alpha/N + \varepsilon$ and $b = 1 - 1/q + \varepsilon$. A function $M_\ell \in L^q_\omega(\mathbb{R}^N)$ with $\ell \in \mathbb{Z}$ is called a dyadic central $(\alpha, q, s, \varepsilon)_{\ell,\omega}$-molecule, if it satisfies

i) $\|M_\ell\|_{L^q_\omega(\mathbb{R}^N)} \le 2^{-\ell\alpha}$;

ii) $\mathfrak{R}_{q,\ell,\omega}(M_\ell) := \|M_\ell\|^{a/b}_{L^q_\omega(\mathbb{R}^N)} \| |x|^{Nb} M_\ell(x)\|^{1-a/b}_{L^q_\omega(\mathbb{R}^N)} < \infty$;

iii) $\int M(x)x^\beta\,dx = 0$, $|\beta| \le s$.

DEFINITION 2.2. Let ω, q, α, s, ε, a and b be as in Definition 2.1.

(a) A function $M \in L^q_\omega(\mathbb{R}^N)$ is called a central $(\alpha, q, s, \varepsilon)_\omega$-molecule, if it satisfies

i) $\mathfrak{R}_{q,\omega}(M) = \|M\|^{a/b}_{L^q_\omega(\mathbb{R}^N)} \| |x|^{Nb} M(x)\|^{1-a/b}_{L^q_\omega(\mathbb{R}^N)} < \infty$;

ii) $\int M(x)x^\beta\,dx = 0$, $|\beta| \le s$.

(b) A function $M \in L^q_\omega(\mathbb{R}^N)$ is called a central $(\alpha, q, s, \varepsilon)_\omega$-molecule of restrict type, if it satisfies i), ii) and

iii) $\|M\|_{L^q_\omega(\mathbb{R}^N)} \le 1$.

THEOREM 2.5. *Let ω, q, α, s, ε be as in Definition 2.1, and* $0 < p < \infty$. *Then* $f \in \dot{HK}_q^{\alpha,p}(1,\omega)$ *(or $HK_q^{\alpha,p}(1,\omega)$) if and only if* $f \overset{S'}{=} \sum_{\ell=-\infty}^{\infty} \lambda_\ell M_\ell$ *(or $\sum_{\ell=0}^{\infty} \lambda_\ell M_\ell$), where each M_ℓ is a dyadic central $(\alpha, q, s, \varepsilon)_{\ell,\omega}$-molecule, $\mathfrak{R}_{q,\ell,\omega}(M_\ell) \le C < \infty$, C is independent of M_ℓ, and $\sum_{\ell=-\infty}^{\infty} |\lambda_\ell|^p < \infty$ (or $\sum_{\ell=0}^{\infty} |\lambda_\ell|^p < \infty$).*

For the proof of Theorem 2.5, we refer to [12]. And, similar to the atom-decomposition case, if $0 < p \le 1$, then we can replace the dyadic central $(\alpha, q, s, \varepsilon)_{\ell,\omega}$-molecule by the central $(\alpha, q, s, \varepsilon)_\omega$-molecule or the central $(\alpha, q, s, \varepsilon)_\omega$-molecule of restrict type, respectively.

Now, we give an application of this theorem.

THEOREM 2.6. *Let ℓ and $I_\ell(f)$ be as in Theorem 2.4,* $1 < q_1 < N/\ell$, $1/q_2 = 1/q_1 - \ell/N$, $0 < p_1 \le p_2 < \infty$ *and* $N(1-1/q_1) < N(1-1/q_2) \le \alpha_1 < \infty$. *Then I_ℓ maps $\dot{HK}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$ (or $HK_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$) into $\dot{HK}_{q_2}^{\alpha_1,p_2}(\mathbb{R}^N)$ (or $HK_{q_2}^{\alpha_1,p_2}(\mathbb{R}^N)$).*

PROOF. We only prove the theorem for the homogeneous case and shall use the atom-molecule theory of $H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^N)$ and $H\dot{K}_{q_2}^{\alpha_1,p_2}(\mathbb{R}^N)$. Let f be a central dyadic (α_1, q_1)-atom with the support B_j and the s_1-order of vanishing moments, $s_1 \geq [\alpha_1 + N(1/q_1 - 1)]$, see the proof of Theorem 2.3. We must prove that $I_\ell(f)$ is a central dyadic $(\alpha_1, q_2, s_2, \varepsilon)$-molecule by Theorem 2.5, that is,

i) $\|I_\ell(f)\|_{L^{q_2}(\mathbb{R}^N)} \leq C2^{-j\alpha_1}$;

ii) $\Re_{q_2}\big(I_\ell(f)\big) := \|I_\ell(f)\|_{L^{q_2}(\mathbb{R}^N)}^{a/b} \| \, |x|^{Nb} I_\ell(f)\|_{L^{q_2}(\mathbb{R}^N)}^{1-a/b} \leq C < \infty$;

iii) $\int I_\ell(f)(x)x^\beta \, dx = 0$, $|\beta| \leq s_2$, $s_2 \geq [\alpha_1 + N(1/q_2 - 1)]$,

where $\varepsilon > \max\{s_2/N, \alpha_2/N + 1/q_2 - 1\}$, $a = 1 - 1/q_2 - \alpha_1/N + \varepsilon$, $b = 1 - 1/q_2 - \varepsilon$ and C is a constant independent of f.

Since I_ℓ maps $L^{q_1}(\mathbb{R}^N)$ into $L^{q_2}(\mathbb{R}^N)$, i) is obvious. We now verify ii). Using I_ℓ: $L^{q_1}(\mathbb{R}^N) \longrightarrow L^{q_2}(\mathbb{R}^N)$, we first get

$$\Big(\int_{B_{j+2}} |I_\ell(f)|^{q_2} |x|^{Nbq_2} \, dx\Big)^{1/q_2} \leq C2^{jNb} \|I_\ell(f)\|_{L^{q_2}(\mathbb{R}^N)} \leq C2^{jNb} \|f\|_{L^{q_1}(\mathbb{R}^N)} \leq C2^{j(Nb-\alpha_1)}.$$

Next, using the Taylor expansion of $|x-y|^{-N+\ell}$ at $|x|$ and the s_1-order vanishing moments of a_j, we have

$$\int_{|x|>2^{j+2}} |I_\ell(f)|^{q_2} |x|^{Nbq_2} \, dx \leq C \int_{|x|\geq 2^{j+2}} |x|^{Nbq_2} \Big(\int_{B_j} \frac{|f(y)| \, |y|^{s_1+1}}{|x|^{N-\ell+s_1+1}} \, dy\Big)^{q_2} dx \leq C2^{j(Nb-\alpha_1)q_2},$$

where we choose s_1 such that $(s_1 + 1 - \ell)/N > \varepsilon$. Therefore,

$$\Re_{q_2}\big(I_\ell(f)\big) = \|I_\ell(f)\|_{L^{q_2}(\mathbb{R}^N)}^{a/b} \| \, |x|^{Nb} I_\ell(f)\|_{L^{q_2}(\mathbb{R}^N)}^{1-a/b} \leq C2^{-j\alpha_1 a/b + j(Nb-\alpha_1)(1-a/b)} = c < \infty.$$

This proves ii). Take $s_2 = [\alpha_1 + N(1/q_2 - 1)]$, it remains to verify iii). In fact, by the inequality

$$\int_{|x|>1} |I_\ell(f)(x)x^\beta| \, dx \leq \|I_\ell(f)(x)|x|^{Nb}\|_{L^{q_2}(\mathbb{R}^N)} \Big(\int_{|x|>1} |x|^{(|\beta|-Nb)q_2'} \, dx\Big)^{1/q_2'} < \infty,$$

we see that $I_\ell(f)(x)x^\beta \in L^1(\mathbb{R}^N)$, where $1/q_2 + 1/q_2' = 1$. From this, it follows that

$$\big(I_\ell(f)(t)t^\beta\big)^\wedge(x) = D^\beta\big\{\big(I_\ell(f)\big)^\wedge(x)\big\} \in C(\mathbb{R}^N).$$

Thus, in order to prove

$$\big(I_\ell(f)(t)t^\beta\big)^\wedge(0) = \int I_\ell(f)(t)t^\beta \, dt = 0,$$

it suffices to show

$$\lim_{|x|\to 0} D^\beta\{I_\ell(f)^\wedge(x)\} = 0.$$

Let β_1 and β_2 satisfy $|\beta_1| + |\beta_2| = |\beta|$. Note that

$$D^{\beta_1}(|x|^{-\ell}) = O(|x|^{-\ell-|\beta_1|})$$

and

$$\begin{aligned} D^{\beta_2}\hat{f}(x) &= \int f(\xi)(-2\pi i\xi)^{\beta_2} e^{-2\pi i\xi\cdot x}\,dx \\ &= \int f(\xi)(-2\pi i\xi)^{\beta_2}[e^{-2\pi i\xi\cdot x} - P(\xi)]\,d\xi, \end{aligned}$$

where $P(\xi)$ is the $(s_1 - |\beta_2|)$-order Taylor expansion of $e^{-2\pi i\xi\cdot x}$ at the origin, we get that $|D^{\beta_2}\hat{f}(x)| \le C|x|^{s_1-|\beta_2|+1}$. From this, it deduces that $\left|D^{\beta_1}(|x|^{-\ell})D^{\beta_2}\hat{f}(x)\right| \le C|x|^{s_1-|\beta|-\ell+1}$. Note that $s_1 - \ell + 1 > s_2 \ge |\beta|$, we get

$$\lim_{|x|\to 0} D^{\beta_1}(|x|^{-\ell})D^{\beta_2}\hat{f}(x) = 0.$$

And therefore, $\lim\limits_{|x|\to 0} D^{\beta}\left\{\left(I_\ell(f)\right)^{\wedge}(x)\right\} = 0$. This finishes the proof of Theorem 2.6.

3. **Hardy-Littlewood-Sobolev theorems with power weights.** In this section, we shall generalize Theorems 2.1–2.4 and 2.6 of the non-homogeneous case into the power weight case. First, we quote the theorem of Lu-Soria [6] as follows, which is the generalization of the theorem of Stein-Weiss [11].

THEOREM 3.0 ([6] OR [11]). *Let $1 < p < \infty$, $0 \le \ell < N$, $1/p_1 = 1/p + (\alpha_1 + \beta_1)/N$, $0 \le \alpha_1 + \beta_1 \le \ell$, $\alpha_1 \le 0$ and $1/q = 1/p_1 - \ell/N$. If a sublinear operator I_ℓ satisfies*

$$|I_\ell f(x)| \le C\int \frac{|f(y)|}{|x-y|^{N-\ell}}\,dy$$

and I_ℓ maps $L^{p_1}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$, then I_ℓ also maps $L^p(\mathbb{R}^N, |x|^{-\alpha}dx)$ into $L^q(\mathbb{R}^N, |x|^{-\beta}dx)$, where $\alpha = -p\alpha_1$, $\beta = q\beta_1$ and $\beta_1 < N/q$.

In this section, we redefine that $\overline{\chi}_0 = \chi_{B_0}$, $\overline{\chi}_k = \chi_k$ for $k \in \mathbb{N}$. Corresponding to Theorem 2.1 of non-homogeneous case, we have

THEOREM 3.1. *Let $0 < \ell < N$ and $I_\ell(f)(x) = C_{\ell,N}\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\ell}}\,dy$, $1 < q_1 < \infty$, $0 < p_1 \le \min\{q_1, p_2\}$, $0 < \alpha_1 < N(1-1/q_1) + \alpha/q_1$, $0 \le \alpha < N - \ell q_1$, $\beta = \alpha N/(N-\ell q_1)$, $1/q_2 = 1/q_1(1 - \ell p_1/N)$ and $\alpha_2 = \alpha_1 + \ell(p_1/q_1 - 1) + \ell\alpha/(N-\ell q_1)(1 - p_1/q_1)$. Then I_ℓ maps $K_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$, into $K_{q_2}^{\alpha_2,p_2}(1,\omega_\beta)$, where $\omega_\alpha = |x|^{-\alpha}$.*

PROOF. Suppose $f \in K_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$, then $f(x) = \sum_{k=0}^{\infty}\lambda_k a_k(x)$, where $\inf\{\sum_{k=0}^{\infty}|\lambda_k|^{p_1}\}^{1/p_1} \sim \|f\|_{K_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)}$ and a_k is a weighted dyadic $(\alpha_1, q_1; 1, \omega_\alpha)$-unit, that is, $\operatorname{supp} a_k \subset B_k$ and $\|a_k\|_{L^{q_1}_{\omega_\alpha}(\mathbb{R}^N)} \le |B_k|^{-\alpha_1/N}$, see [7] for the details. Note that $p_1 \le p_2$, we get

$$
\begin{aligned}
\|I_\ell(f)\|^{p_1}_{K^{\alpha_2,p_2}_{q_2}(1,\omega_\beta)} &\le \sum_{k=0}^{\infty} 2^{k\alpha_2 p_1} \|I_\ell(f)\overline{\chi}_k\|^{p_1}_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} \\
&\le C\sum_{k=0}^{1} \|I_\ell(f)\overline{\chi}_k\|^{p_1}_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} \\
&\quad + C\sum_{k=2}^{\infty} 2^{k\alpha_2 p_1}\Big(\sum_{j=0}^{k-2} |\lambda_j|\, \|I_\ell(a_j)\overline{\chi}_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)}\Big)^{p_1} \\
&\quad + C\sum_{k=2}^{\infty} 2^{k\alpha_2 p_1}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|\, \|I_\ell(a_j)\overline{\chi}_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)}\Big)^{p_1} \\
&:= C(I_1 + I_2 + I_3).
\end{aligned}
$$

For $k = 1$ and 2, note that $1/q_2 = 1/q_1 - (p_1/q_1)(\ell/N) \ge 1/q_1 - \ell/N := 1/q_0$, by using the Hölder inequality and Theorem 3.0, we have

$$
\begin{aligned}
\|I_\ell(f)\overline{\chi}_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} &\le \|I_\ell(f)\|_{L^{q_0}_{\omega_\beta}(\mathbf{R}^N)} \omega_\beta(B_1)^{(1-q_2/q_0)(1/q_2)} \\
&\le C\|f\|_{L^{q_1}_{\omega_\alpha}(\mathbf{R}^N)} \le C\sum_{j=0}^{\infty} |\lambda_j|^{p_1}.
\end{aligned}
$$

Thus, $I_1 \le C\|f\|^{p_1}_{K^{\alpha_1,p_1}_{q_1}(1,\omega_\alpha)}$.

Now, we come to estimate I_3. Using Theorem 3.0, we get

$$
\begin{aligned}
\|I_\ell(a_j)\overline{\chi}_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} &\le \|I_\ell(a_j)\overline{\chi}_k\|_{L^{q_0}_{\omega_\beta}(\mathbf{R}^N)} \omega_\beta(B_k)^{1/q_2 - 1/q_0} \\
&\le C2^{-j\alpha_1 + k(N-\beta)(1/q_2 - 1/q_0)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_3 &\le C\sum_{k=2}^{\infty} 2^{k\alpha_2 p_1}\Big(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{-j\alpha_1 + k(N-\beta)(1/q_2 - 1/q_0)}\Big)^{p_1} \\
&\le C\sum_{k=2}^{\infty}\Big(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha_1}\Big)^{p_1} \le C\sum_{j=1}^{\infty} |\lambda_j|^{p_1}.
\end{aligned}
$$

For I_2, note that $j \le k-2$, we first have

$$
\begin{aligned}
\|I_\ell(a_j)\overline{\chi}_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} &\le C\Big\{\int_{C_k} |x|^{-\beta}\Big(\int_{B_j} \frac{|a_j(y)|}{|x|^{N-\ell}}\,dy\Big)^{q_2} dx\Big\}^{1/q_2} \\
&\le C2^{-k\{\beta/q_2 - N/q_2 + N - \ell\} + j\{\alpha/q_1 + N(1-1/q_1) - \alpha_1\}}.
\end{aligned}
$$

From this, it follows that

$$
\begin{aligned}
I_2 &\le C\sum_{k=2}^{\infty}\Big(\sum_{j=0}^{k-2} |\lambda_j| 2^{-k\{\beta/q_2 - N/q_2 + N - \ell - \alpha_2\} + j\{\alpha/q_1 + N(1-1/q_1) - \alpha_1\}}\Big)^{p_1} \\
&\le C\sum_{k=2}^{\infty}\Big(\sum_{j=0}^{k-2} |\lambda_j| 2^{(j-k)(N(1-1/q_1) - \alpha_1 + \alpha/q_1)}\Big)^{p_1} \\
&\le C\sum_{j=0}^{\infty} |\lambda_j|^{p_1},
\end{aligned}
$$

where we use $\alpha_1 < N(1 - 1/q_1) + \alpha/q_1$. And we finish the proof of Theorem 3.1.

THEOREM 3.2. *Let ℓ and $I_\ell(f)$ be as in Theorem 3.0. Assume that* $0 < \alpha_1 < N - (N-\alpha)/q_1$, $1 < q_1 < \infty$, $0 < p_1 \le p_2 < \infty$, $1/q_0 = 1/q_1 + (\alpha_0+\beta_0)/N$, $0 \le \alpha_0 + \beta_0 \le \ell$, $\alpha_0 \le 0$, $1/q_2 = 1/q_0 - \ell/N$ *and* I_ℓ *maps* $L^{q_0}(\mathbb{R}^N)$ *into* $L^{q_2}(\mathbb{R}^N)$. *Then* I_ℓ *also maps* $K_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$ *into* $K_{q_2}^{\alpha_1,p_2}(1,\omega_\beta)$, *where* $\alpha = -q_1\alpha_0$, $\beta = q_2\beta_0$ *and* $\beta_0 < N/q_2$.

PROOF. Similar to the proof of Theorem 3.1, let $f \in K_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$, then $f = \sum_{j=0}^\infty \lambda_j a_j$, where a_j is a weighted dyadic central $(\alpha_1, q_1; 1, \omega_\alpha)$-unit with the support B_j and $\|f\|_{K_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)} \backsim \inf\{\ \sum_{j=0}^\infty |\lambda_j|^{p_1}\}^{1/p_1}$. Note that $p_1 \le p_2$, we get

$$\begin{aligned}\|I_\ell(f)\|^{p_1}_{K_{q_2}^{\alpha_1,p_2}(1,\omega_\beta)} &\le C\sum_{k=0}^{1} \|I_\ell(f)\chi_k\|^{p_1}_{L^{q_2}_{\omega_\beta}(\mathbb{R}^N)} \\ &\quad + C\sum_{k=2}^\infty 2^{k\alpha_1 p_1}\Big(\sum_{j=0}^{k-2} |\lambda_j|\, \|I_\ell(a_j)\chi_k\|_{L^{q_2}_{\omega_\beta}(\mathbb{R}^N)}\Big)^{p_1} \\ &\quad + C\sum_{k=2}^\infty 2^{k\alpha_1 p_1}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|\, \|I_\ell(a_j)\chi_k\|_{L^{q_2}_{\omega_\beta}(\mathbb{R}^N)}\Big)^{p_1} \\ &:= C(I_1 + I_2 + I_3).\end{aligned}$$

Using Theorem 3.0, we directly obtain

$$I_1 \le C\|f\|^{p_1}_{L^{q_1}_{\omega_\alpha}(\mathbb{R}^N)} \le C\|f\|^{p_1}_{K_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)},$$

and

$$\begin{aligned}I_3 &\le C\sum_{k=2}^\infty 2^{k\alpha_1 p_1}\Big(\sum_{j=k-1}^\infty |\lambda_j|\, \|a_j\|_{L^{q_1}_{\omega_\alpha}(\mathbb{R}^N)}\Big)^{p_1} \\ &\le C\sum_{k=2}^\infty \Big(\sum_{j=k-1}^\infty |\lambda_j| 2^{(k-j)\alpha_1}\Big)^{p_1} \le C\sum_{j=1}^\infty |\lambda_j|^{p_1}.\end{aligned}$$

For I_2, similar to the proof of Theorem 3.1, note that $j \le k-2$, we have

$$\|I_\ell(a_j)\chi_k\|_{L^{q_2}_{\omega_\beta}(\mathbb{R}^N)} \le C2^{k((N-\beta)/q_2 - N + \ell) + j\{N(1-1/q_1) + \alpha/q_1 - \alpha_1\}}.$$

Therefore,

$$\begin{aligned}I_2 &\le C\sum_{k=2}^\infty \Big(\sum_{j=0}^{k-2} |\lambda_j| 2^{k((N-\beta)/q_2 + \alpha_1 - N + \ell) + j\{N(1-1/q_1) + \alpha/q_1 - \alpha_1\}}\Big)^{p_1} \\ &\le C\sum_{k=2}^\infty \Big(\sum_{j=0}^{k-2} |\lambda_j| 2^{(k-j)(\alpha_1 - N + N/q_1 - \alpha/q_1)}\Big)^{p_1} \\ &\le C\sum_{j=0}^\infty |\lambda_j|^{p_1}.\end{aligned}$$

This finishes the proof of Theorem 3.2.

Note that if $\alpha_1 = 0$ and $p_1 = q_1$, then Theorem 3.2 is just Theorem 3.0 and Theorem 3.1 is a special case of Theorem 3.0. Thus, Theorems 3.1–3.2 are the generalization and the supplement of Theorem 3.0. The following three theorems correspond to Theorem 2.3, 2.4 and 2.6 respectively.

THEOREM 3.3. *Let* ℓ *and* $I_\ell(f)$ *be as in Theorem 3.1,* $1 < q_1 < \infty$, $0 < p_1 \leq \min\{q_1, p_2\}$, $N(1-1/q_1) \leq \alpha_1 < \infty$, $0 \leq \alpha < N - \ell q_1$, $\beta = \alpha N/(N-\ell q_1)$, $\alpha_2 = \alpha_1 + \ell(p_1/q_1 - 1) + \ell\alpha/(N-\ell q_1)(1-p_1/q_1)$ *and* $1/q_2 = 1/q_1(1-\ell p_1/N)$. *Then* I_ℓ *maps* $HK_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$ *into* $K_{q_2}^{\alpha_2,p_2}(1,\omega_\beta)$.

PROOF. Let $f \in HK_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$. Then $f = \sum_{j=0}^\infty \lambda_j a_j$, where $\|f\|_{HK_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)} \sim \inf(\sum_{j=0}^\infty |\lambda_j|^{p_1})^{1/p_1}$ and a_j is a dyadic central $(\alpha_1, q_1; 1, \omega_\alpha)$-atom, that is, $\operatorname{supp} a_j \subset B_j$; $\|a_j\|_{L^{q_1}_{\omega_\alpha}(\mathbf{R}^N)} \leq |B_j|^{-\alpha_1/N}$ and $\int a_j(x)x^\beta\,dx = 0$, $|\beta| \leq s_1$ and $s_1 \geq [\alpha_1 + N(1/q_1 - 1)]$, see [8] for the details. Note that $p_1 \leq p_2$, write

$$
\begin{aligned}
\|I_\ell(f)\|^{p_1}_{K_{q_2}^{\alpha_2,p_2}(1,\omega_\beta)} &\leq C\sum_{k=0}^{1} \|I_\ell(f)\chi_k\|^{p_1}_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} \\
&\quad + C\sum_{k=2}^\infty 2^{k\alpha_2 p_1}\Big(\sum_{j=0}^{k-2} |\lambda_j|\,\|I_\ell(a_j)\chi_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)}\Big)^{p_1} \\
&\quad + C\sum_{k=2}^\infty 2^{k\alpha_2 p_1}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|\,\|I_\ell(a_j)\chi_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)}\Big)^{p_1} \\
&:= C(I_1 + I_2 + I_3).
\end{aligned}
$$

Note that $1/q_2 = 1/q_1 - p_1/q_1 \cdot (\ell/N) \geq 1/q_1 - \ell/N := 1/q_0$, using Hölder's inequality and Theorem 3.0, we get

$$
\begin{aligned}
I_1 &\leq C\|I_\ell(f)\|_{L^{q_0}_{\omega_\beta}(\mathbf{R}^N)}\omega_\beta(B_1)^{1/q_2-1/q_0} \\
&\leq C\|f\|_{L^{q_1}_{\omega_\alpha}(\mathbf{R}^N)} \leq C\Big(\sum_{j=0}^\infty |\lambda_j|^{p_1}\Big);
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &\leq C\sum_{k=2}^\infty 2^{k\alpha_2 p_1}\Big(\sum_{j=k-1}^\infty |\lambda_j|\,\|I_\ell(a_j)\chi_k\|_{L^{q_0}_{\omega_\beta}(\mathbf{R}^N)}\omega_\beta(B_k)^{1/q_2-1/q_0}\Big)^{p_1} \\
&\leq C\sum_{k=2}^\infty 2^{k\alpha_2 p_1}\Big(\sum_{j=k-1}^\infty |\lambda_j|\,\|a_j\|_{L^{q_1}_{\omega_\alpha}(\mathbf{R}^N)}2^{k(N-\beta)(1/q_2-1/q_0)}\Big)^{p_1} \\
&\leq C\sum_{k=2}^\infty\Big(\sum_{j=k-1}^\infty |\lambda_j|2^{(k-j)\alpha_1}\Big)^{p_1} \leq C\sum_{j=1}^\infty |\lambda_j|^{p_1}.
\end{aligned}
$$

Next, we come to estimate I_2. By the Taylor expansion of $|x-y|^{-N+\ell}$ at x and the s_1-order vanishing moments of a_j, we get

$$
\begin{aligned}
\|I_\ell(a_j)\chi_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} &\leq C\Big\{\int_{C_k} |x|^{-\beta}\Big(\int_{B_j}\frac{|a_j(y)||y|^{s_1+1}}{|x|^{N-\ell+s_1+1}}\,dy\Big)^{q_2}dx\Big\}^{1/q_2} \\
&\leq C2^{-k(\beta/q_2+N-\ell+s_1+1-N/q_2)+j(s_1+1)}\int |a_j(y)|\,dy \\
&\leq C2^{-k(\beta/q_2+N-\ell+s_1+1-N/q_2)+j\{s_1+1-\alpha_1+N(1-1/q_1)+\alpha/q_1\}}.
\end{aligned}
$$

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Therefore,

$$I_2 \le C\sum_{k=2}^{\infty}\Big(\sum_{j=0}^{k-2}|\lambda_j|2^{j(s_1+1-\alpha_1+N(1-1/q_1)+\alpha/q_1)+k(\alpha_2-\beta/q_2+N/q_2-(N-\ell+s_1+1))}\Big)^{p_1}$$
$$\le C\sum_{k=2}^{\infty}\Big(\sum_{j=0}^{k-2}|\lambda_j|2^{(j-k)(s_1+1-\alpha_1+N(1-1/q_1)+\alpha/q_1)}\Big)^{p_1}$$
$$\le C\sum_{j=0}^{\infty}|\lambda_j|^{p_1}.$$

This finishes the proof of Theorem 3.3.

THEOREM 3.4. *Let ℓ and $I_\ell(f)$ be as in Theorem 3.1,* $1 < q_1 < \infty$, $N(1-1/q_1) \le \alpha_1 < \infty$, $0 < p_1 \le p_2 < \infty$, $1/q_2 = 1/q_1 + (\alpha_0+\beta_0-\ell)/N$, $0 \le \alpha_0+\beta_0 \le \ell$ *and* $\alpha_1 \le 0$. *Then I_ℓ maps $HK_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$ into $K_{q_2}^{\alpha_1,p_2}(1,\omega_\beta)$, where $\alpha = -q_1\alpha_0$, $\beta = q_2\beta_0$ and $\beta_0 < N/q_2$.*

PROOF. Similar to the proof of Theorem 3.3, let $f \in HK_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$, then $f = \sum_{j=0}^{\infty}\lambda_j a_j$, where $\|f\|_{HK_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)} \sim \inf(\sum_{j=0}^{\infty}|\lambda_j|^{p_1})^{1/p_1}$ and a_j is a dyadic central $(\alpha_1, q_1; 1, \omega_\alpha)$-atom with the support B_j and the s_1-order vanishing moments, $s_1 \ge [\alpha_1 + N(1/q_1 - 1)]$. Note that $p_1 \le p_2$, we then have

$$\|I_\ell(f)\|^{p_1}_{K_{q_2}^{\alpha_1,p_2}(1,\omega_\beta)} \le C\sum_{k=0}^{1}\|I_\ell(f)\chi_k\|^{p_1}_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)}$$
$$+ C\sum_{k=2}^{\infty}2^{k\alpha_1p_1}\Big(\sum_{j=0}^{k-2}|\lambda_j|\,\|I_\ell(a_j)\chi_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)}\Big)^{p_1}$$
$$+ C\sum_{k=2}^{\infty}2^{k\alpha_1p_1}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|\,\|I_\ell(a_j)\chi_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)}\Big)^{p_1}$$
$$:= C(I_1+I_2+I_3).$$

Using Theorem 3.0, we get

$$I_1 \le C\|f\|^{p_1}_{L^{q_1}_{\omega_\alpha}(\mathbf{R}^N)} \le C\sum_{j=0}^{\infty}|\lambda_j|^{p_1},$$

and

$$I_3 \le C\sum_{k=2}^{\infty}2^{k\alpha_1p_1}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|\,\|a_j\|_{L^{q_1}_{\omega_\alpha}(\mathbf{R}^N)}\Big)^{p_1}$$
$$\le C\sum_{k=2}^{\infty}2^{k\alpha_1p_1}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|2^{-j\alpha_1}\Big)^{p_1} \le C\sum_{j=1}^{\infty}|\lambda_j|^{p_1}.$$

For I_2, by the Taylor expansion of $|x-y|^{-N+\ell}$ at x and the s_1-order vanishing moments of a_j, we get

$$\|I_\ell(a_j)\chi_k\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} \le C\Big\{\int_{C_k}|x|^{-\beta}\Big(\int_{B_j}\frac{|a_j(y)||y|^{s_1+1}}{|x|^{N-\ell+s_1+1}}\,dy\Big)^{q_2}dx\Big\}^{1/q_2}$$
$$\le C2^{-k\{(\beta-N)/q_2+N-\ell+s_1+1\}+j\{s_1+1+\alpha/q_1-\alpha_1+N(1-1/q_1)\}}.$$

Therefore,

$$I_2 \le C\sum_{k=2}^{\infty} 2^{k\alpha_1 p_1}\Big(\sum_{j=0}^{k-2}|\lambda_j|2^{-k\{(\beta-N)/q_2+N-\ell+s_1+1\}+j\{s_1+1+\alpha/q_1-\alpha_1+N(1-1/q_1)\}}\Big)^{p_1}$$
$$\le C\sum_{k=2}^{\infty}\Big(\sum_{j=0}^{k-2}|\lambda_j|2^{(j-k)(N(1-1/q_1)+\alpha/q_1+s_1+1-\alpha_1)}\Big)^{p_1}$$
$$\le C\sum_{j=0}^{\infty}|\lambda_j|^{p_1}.$$

And we finish the proof of Theorem 3.4.

THEOREM 3.5. *Set ℓ and $I_\ell(f)$ as in Theorem 3.4. Let $1 < q_1 < \infty$, $0 < p_1 \le p_2 < \infty$, $N(1-1/q_1) < N(1-1/q_2) \le \alpha_1 < \infty$, $1/q_2 = 1/q_1 + (\alpha_0+\beta_0-\ell)/N$, $0 \le \alpha_0+\beta_0 \le \ell$ and $\alpha_0 \le 0$. Then I_ℓ maps $HK_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$ into $HK_{q_2}^{\alpha_1,p_2}(1,\omega_\beta)$, where $\alpha = -q_1\alpha_0$, $\beta = q_2\beta_0$ and $\beta_0 < N/q_2$.*

PROOF. We shall use the atom-molecule theory of $HK_{q_1}^{\alpha_1,p_1}(1,\omega_\alpha)$ and $HK_{q_2}^{\alpha_1,p_2}(1,\omega_\beta)$ to prove this theorem. Let f be a dyadic central $(\alpha_1,q_1;1,\omega_\alpha)$-atom with the support B_j and the s_1-order vanishing moments, $s_1 \ge [\alpha_1+N(1/q_1-1)]$. We must prove that $I_\ell(f)$ is a dyadic central $(\alpha_1,q_1,s_2,\varepsilon)_{\omega_\beta}$-molecule by Theorem 2.5, that is

i) $\|I_\ell(f)\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} \le C2^{-j\alpha_1}$;

ii) $\Re_{q_2,\omega_\beta}\big(I_\ell(f)\big) := \|I_\ell(f)\|^{a/b}_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)}\|\,|x|^{Nb}I_\ell(f)\|^{1-a/b}_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)} \le C < \infty$;

iii) $\int I_\ell(f)(x)x^\nu\,dx = 0$, $|\nu| \le s_2$, $s_2 \ge [\alpha_1+N(1/q_2-1)]$,

where $\varepsilon > \max\{s_2/N+\beta/(Nq_2), \alpha_1/N+1/q_2-1\}$, $a = 1-1/q_2-\alpha_1/N+\varepsilon$, $b = 1-1/q_2-\varepsilon$ and C is a constant independent of f.

Using Theorem 3.0, we see that i) is obvious. iii) can be proved by a method similar to the proof of Theorem 2.6. We only need to verify ii). By Theorem 3.0, we first have

$$\Big(\int_{B_{j+2}}|I_\ell(f)|^{q_2}|x|^{Nbq_2}|x|^{-\beta}\,dx\Big)^{1/q_2} \le C2^{jNb}\|I_\ell(f)\|_{L^{q_2}_{\omega_\beta}(\mathbf{R}^N)}$$
$$\le C2^{jNb}\|f\|_{L^{q_1}_{\omega_\alpha}(\mathbf{R}^N)}$$
$$\le C2^{jNb-j\alpha_1}.$$

Next, using the s_1-order Taylor expansion of $|x-y|^{-N+\ell}$ at x and the s_1-order vanishing moments of f, we get

$$\int_{|x|\ge 2^{j+2}}|I_\ell(f)|^{q_2}|x|^{Nbq_2}|x|^{-\beta}\,dx$$
$$\le C\int_{|x|\ge 2^{j+2}}|x|^{Nbq_2-\beta}\Big(\int_{B_j}\frac{|f(y)|\,|y|^{s_1+1}}{|x|^{N-\ell+s_1+1}}\,dy\Big)^{q_2}dx$$
$$\le C2^{j(-\alpha_1+Nb)}.$$

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Therefore,

$$\begin{aligned}\Re_{q_2,\omega_\beta}\big(I_\ell(f)\big) &= \|I_\ell(f)\|^{a/b}_{L^{q_2}_{\omega_\beta}(\mathbb{R}^N)} \, \| \, |x|^{Nb} I_\ell(f)\|^{1-a/b}_{L^{q_2}_{\omega_\beta}(\mathbb{R}^N)} \\ &\le C 2^{-j\alpha_1 a/b + j(Nb-\alpha_1)(1-a/b)} \\ &= C < \infty.\end{aligned}$$

This finishes the proof of Theorem 3.5.

4. **Some applications.** In this section, we shall give some applications of the theorems in Section 2. For more interesting applications, we refer to the authors' other paper [9].

Let $N \ge 3$ and $f \in K^{(1-1/q)(N+2)/2, 2N/(N+2)}_{2qN/(N+2)}(\mathbb{R}^N)$, where $1 \le q < \infty$. If $-\Delta u = f$, then $u \in K^{(1-1/q)(N-2)/2, 2N/(N+2)}_{2qN/(N-2)}(\mathbb{R}^N)$ by Theorem 2.1. Moreover, if let $R = (R_1, \cdots, R_N)$ and $\{R_j\}_{j=1}^N$ be the Riesz transforms on $\mathbb{R}^N$, noting that $\nabla u = R\big(I_1(f)\big)$, we get that $\nabla u \in K^{N(1-1/q)/2, 2}_{2q}(\mathbb{R}^N)$ by Theorem 2.1 in Section 2 of this paper and Theorem 2.3 in the authors' paper [7]. We claim that $|\nabla u|^2 - fu \in HK^{N(1-1/q), 1}_q(\mathbb{R}^N)$. In order to prove this claim, we follow the idea in the proof of Theorem II.1 in [3]. Take $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \ge 0$, $\operatorname{supp}\phi \subset B(0,1)$, $\int \phi(x)\,dx = 1$ and $\phi_t(x) = t^{-N}\phi(x/t)$ for $t > 0$, we get

$$\begin{aligned}\{\phi_t * (|\nabla u|^2 - fu)\}(x) &= \int \nabla u(y) \frac{1}{t}\Big[u(y) - \frac{1}{B(x,t)} \int_{B(x,t)} u\Big] \nabla\phi\Big(\frac{x-y}{t}\Big)\frac{1}{t^N}\,dy \\ &\quad - \Big(\frac{1}{|B(x,t)|}\int_{B(x,t)} u\Big) \int_{B(x,t)} f(y)\phi\Big(\frac{x-y}{t}\Big)\frac{1}{t^N}\,dy,\end{aligned}$$

where $B(x,t) = \{y \in \mathbb{R}^N : |x-y| \le t\}$. Then,

$$\begin{aligned}&(|\nabla u|^2 - fu)^*(x) \\ &\quad := \sup_{t>0}\big|\{\phi_t * (|\nabla u|^2 - fu)\}(x)\big| \\ &\quad \le \sup_{t>0} \frac{C}{t|B(x,t)|}\int_{B(x,t)} |\nabla u(y)| \Big| u(y) - \frac{1}{|B(x,t)|}\int_{B(x,t)} u \Big|\,dy \\ &\qquad + C\sup_{t>0}\Big(\frac{1}{|B(x,t)|}\int_{B(x,t)}|u|\Big)\Big(\frac{1}{|B(x,t)|}\int_{B(x,t)}|f|\Big) \\ &\quad \le C\sup_{t>0} \frac{1}{t|B(x,t)|}\int_{B(x,t)} |\nabla u(y)| \Big| u(y) - \frac{1}{|B(x,t)|}\int_{B(x,t)} u \Big|\,dy \\ &\qquad + CM(|u|)M(|f|) \\ &\quad \le C\sup_{t>0} \frac{1}{t}\Big(\frac{1}{|B(x,t)|}\int_{B(x,t)} |\nabla u|^{2N/(N+1)}\Big)^{(N+1)/(2N)} \\ &\qquad\quad \times \Big(\frac{1}{|B(x,t)|}\int_{B(x,t)} \Big| u(y) - \frac{1}{|B(x,t)|}\int_{B(x,t)} u\Big|^{2N/(N-1)}\,dy\Big)^{(N-1)/(2N)} \\ &\qquad + CM(|u|)M(|f|)\end{aligned}$$

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$$\leq C\sup_{t>0}\Big(\frac{1}{|B(x,t)|}\int_{B(x,t)}|\nabla u|^{2N/(N+1)}\Big)^{(N+1)/N}$$
$$+ CM(|u|)^{2N/(N-2)} + CM(|f|)^{2N/(N+2)}$$
$$\leq CM\big(|\nabla u|^{2N/(N+1)}\big)^{(N+1)/N}$$
$$+ CM(|u|)^{2N/(N-2)} + CM(|f|)^{2N/(N+2)},$$

where M denotes the Hardy-Littlewood maximal function and we have used the Sobolev-Poincaré inequality in the inverse second inequality. Using the equivalent characterization of $HK_q^{N(1-1/q),1}(\mathbb{R}^N)$ (see [2,5] or [8]), we obtain

$$\begin{aligned}\|\,|\nabla u|^2 - fu\|_{HK_q^{N(1-1/q),1}(\mathbb{R}^N)} &:= \|(|\nabla u|^2 - fu)^*\|_{K_q^{N(1-1/q),1}(\mathbb{R}^N)}\\
&\leq C\|M(|\nabla u|^{2N/(N+1)})^{(N+1)/N}\|_{K_q^{N(1-1/q),1}(\mathbb{R}^N)}\\
&\quad + C\|M(|u|)^{2N/(N-2)}\|_{K_q^{N(1-1/q),1}(\mathbb{R}^N)}\\
&\quad + C\|M(|f|)^{2N/(N+2)}\|_{K_q^{N(1-1/q),1}(\mathbb{R}^N)}\\
&= C\|M(|\nabla u|^{2N/(N+1)})\|^{(N+1)/N}_{K_{q(N+1)/N}^{N(1-1/q)\cdot N/(N+1),(N+1)/N}(\mathbb{R}^N)}\\
&\quad + C\|M(|u|)\|^{2N/(N-2)}_{K_{2qN/(N-2)}^{N(1-1/q)(N-2)/(2N),2N/(N-2)}(\mathbb{R}^N)}\\
&\quad + C\|M(|f|)\|^{2N/(N+2)}_{K_{2qN/(N+2)}^{N(1-1/q)(N+2)/(2N),2N/(N+2)}(\mathbb{R}^N)}\\
&\leq C\|\nabla u\|^2_{K_{2q}^{N(1-1/q)/2,2}(\mathbb{R}^N)}\\
&\quad + C\|u\|^{2N/(N-2)}_{K_{2qN/(N-2)}^{(1-1/q)(N-2)/2,2N/(N+2)}(\mathbb{R}^N)}\\
&\quad + C\|f\|^{2N/(N+2)}_{K_{2qN/(N+2)}^{N(1-1/q)(N+2)/(2N),2N/(N+2)}(\mathbb{R}^N)} < \infty,\end{aligned}$$

where we use Theorem 2.3 of the authors' paper [7] in the inverse second inequality. That is, $|\nabla u|^2 - fu \in HK_q^{N(1-1/q),1}(\mathbb{R}^N)$. More generally, by a similar method, we can prove the following proposition.

PROPOSITION 4.1. *Let $N \geq 3$ and $1 \leq q < \infty$. If $\nabla u \in K_{2q}^{N(1-1/q)/2,2}(\mathbb{R}^N)$, $u \in K_{pq}^{N(1-1/q)/p,p}(\mathbb{R}^N)$, $2N/(N-2) \leq p < \infty$ and $\Delta u \in K_{p'q}^{N(1-1/q)/p',p'}(\mathbb{R}^N)$, where $1/p + 1/p' = 1$, then $\Delta u \cdot u + |\nabla u|^2 \in HK_q^{N(1-1/q),1}(\mathbb{R}^N)$.*

For the wave equations $\Box u := \big(\frac{\partial^2}{\partial t^2} - \Delta\big)u = f$, we have a similar result.

PROPOSITION 4.2. *Let $N \geq 2$, $1 \leq q < \infty$. If $\Box u = \big(\frac{\partial^2}{\partial t^2} - \Delta\big)u = f$ in $\mathbb{R}_t \times \mathbb{R}_x^N$, $\frac{\partial u}{\partial t}$, $\nabla u \in K_{2q}^{N(1-1/q)/2,2}(\mathbb{R}^{1+N})$, $u \in K_{pq}^{N(1-1/q)/p,p}(\mathbb{R}^{1+N})$ with $2(N+1)/(N-1) \leq p < \infty$ and $f \in K_{p'q}^{N(1-1/q)/p',p'}(\mathbb{R}^N)$ where $1/p + 1/p' = 1$, then $\frac{1}{2}\Box(u^2) = fu + \big|\frac{\partial u}{\partial t}\big|^2 - |\nabla u|^2 \in HK_q^{N(1-1/q),1}(\mathbb{R}^{1+N})$.*

REMARK 4.1. The Proposition 4.1 and 4.2 are also true for the homogeneous Herz and Herz-type Hardy spaces.

REMARK 4.2. If $q = 1$, then Proposition 4.1 and 4.2 are just the results of [3]. Thus, Proposition 4.1 and 4.2 are the generalization of the corresponding results of [3]. However, Proposition 4.1 and 4.2 are the bases of our following work [9].

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