A REMARK BY PHILIP HALL

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The relationship between the representation theory of the full linear group GL(d) of all non-singular linear transformations of degree d over a field of characteristic zero and that of the symmetric group S_n goes back to Schur and has been expounded by Weyl in his <u>classical groups</u>, [4; cf also 2 and 3]. More and more, the significance of continuous groups for modern physics is being pressed on the attention of mathematicians, and it seems worth recording a remark made to the author by Philip Hall in Edmonton.

As is well known, the irreducible representations of S_n are obtainable from the Young diagrams $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_r]$ consisting of λ_1 nodes in the first row, λ_2 in the second row, etc., where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant {}_r$ and $\Sigma \lambda_i = n$. If we denote the j^{th} node in the i^{th} row of $[\lambda]$ by (i,j) then those nodes to the right of and below (i,j), constitute, along with the (i,j) node itself, the (i,j)-<u>hook</u> of length h_{ij}.

The notion of a hook arose first in the modular theory, but the significant formula (1) for the degree f^λ of $[\lambda]$:

$$(1) \qquad f^\lambda = n!/H^\lambda,$$

where $H^\lambda = \Pi h_{ij}$ for all (i,j) in $[\lambda]$, indicates its role in the ordinary representation theory of S_n. The proof of (1) is immediate when one recognizes that certain factors in Frobenius formula [3, 4.34] for f^λ may be cancelled, leaving just the h_{ij}'s in the denominator.

Professor Hall remarked that a similarly simple formula holds for the degree $\delta^\lambda(d)$ of the irreducible representation $\langle\lambda\rangle$ of GL(d) corresponding to $[\lambda]$. If we set

$$C_{ij} = d + j = i$$

and assume that $d \geqslant r$, then we may add d - r null rows to $[\lambda]$ and divide out [3, 4.41] as before to obtain the relation

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(2) $$\delta^{\lambda}(d) = C^{\lambda}(d)/H^{\lambda},$$

where $C^{\lambda}(d) = \prod C_{ij}$, for all (i,j) in $[\lambda]$.

Example. In order to calculate the degree of the irreducible representation $[3,2]$ of S_5 we write h_{ij} in place of the (i,j) node, yielding

$$\begin{matrix} 4 & 3 & 1 \\ 2 & 1 & \end{matrix}$$

so that $H^{3,2} = 24$ and $f^{3,2} = 5!/24 = 5$. The degree of the corresponding irreducible tensor representation $\langle 3,2 \rangle$ of GL(d) is obtained from the C_{ij} :

$$\begin{matrix} d & d+1 & d+2 \\ d-1 & d & \end{matrix}$$

so that $\delta^{3,2}(d) = d^2(d^2-1)(d+2)/24$. A more familiar example is the case n = 2, in which the symmetric tensor is of degree $\frac{1}{2}d(d+1)$ and the skew symmetric tensor of degree $\frac{1}{2}d(d-1)$.

The formula (2) suggests the possibility of expressing the character of the tensor representation $\langle\lambda\rangle$, i.e. the Schur function $\{\lambda\}$ directly in terms of the elementary symmetric functions $S_p = \sum \alpha_i^p$ The usual expression

(3) $$\{\lambda\} = \frac{1}{n!}\sum h_\rho \chi_\rho^\lambda S_\rho, \quad S_\rho = S_1^a S_2^b S_3^c \ldots,$$

where n = a+2b+3c+..., involves the characters χ_ρ^λ of S_n but if $[\lambda]$ is a hook representation of the form $[n-r, 1^r]$, then (2) can be made to yield (3) without involving character explicity. Using the Frobenius symbol $\binom{a_i}{b_i}$ for $[\lambda]$ and the formula [2, XII p. 112]:

$$\{\lambda\} = \left| \{ 1 + a_j, 1^{b_i} \} \right| ,$$

we have an alternative approach to (3).

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