THE $\sim$-REPRESENTATIONS OF SYMMETRIC HOMOGENEOUS ALGEBRAS

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Abstract

In 1947 I. E. Segal proved that to each non-degenerate $\sim$-representation $R$ of $L^1(= L^1(G)$ for a compact group $G$) with representation space $\mathcal{H}$, there corresponds a continuous unitary representation $W$ of $G$, also with representation space $\mathcal{H}$, which satisfies

$$\langle R(f) \mathcal{A}, \mathcal{A} \rangle = \int_G \langle W(x) \mathcal{A}, \mathcal{A} \rangle f(x) \, dx$$

for each $f \in L^1$ and $\mathcal{A}, \mathcal{A} \in \mathcal{H}$. This was extended to $L^p$, $1 \leq p < \infty$, in 1970 by E. Hewitt and K. A. Ross. We now generalize this result to any symmetric homogeneous convolution Banach algebra of pseudomeasures on $G$. Further we prove that the correspondence preserves irreducibility.


Throughout $G$ will denote a compact group with dual object $\Sigma$. As usual $\sigma$ will denote an arbitrary element of $\Sigma$, and $U_{\sigma}$ a representation in $\sigma$ acting on the Hilbert space $\mathcal{H}_{\sigma}$. For each $p$, $1 \leq p < \infty$, $L^p$ will denote the Banach space of $p$-integrable functions on $G$ with respect to Haar measure. It is well known that $L^p$ is a symmetric Banach algebra if multiplication is convolution and the involution $\sim$ is defined by $\tilde{f}(x) = f(x^{-1})$ for all $x \in G$ and $f \in L^p$.

For any Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ will denote the space of bounded linear operators on $\mathcal{H}$, and $\| \|$ the usual operator norm. If $\mathcal{B}(\mathcal{H})$ is involuted with the adjoint map, then it also is a symmetric Banach algebra.

We define a $\sim$-representation $R$ of $L^p$ on $\mathcal{H}$ to be any representation which preserves involution; it is non-degenerate if for each non-zero $\mathcal{A} \in \mathcal{H}$ there exists $f \in L^p$ such that $R_f \mathcal{A} \neq 0$, where $R_f$ denotes $R(f)$.
Segal [4] proved that to each non-degenerate \( R \) of \( L^1 \) there corresponds a unique continuous unitary representation \( W \) of \( G \), both \( R \) and \( W \) acting on a Hilbert space \( \mathcal{H} \), such that

\[
\langle Rf, \xi \rangle = \int_G \langle W_x f, \xi \rangle f(x) \, dx
\]

for all \( \xi, \xi' \in \mathcal{H} \) and \( f \in L^1 \). Hewitt and Ross [2] extend this result in (38.21) to \( L^p \) for \( 1 \leq p < \infty \).

We shall derive a corresponding result for all symmetric convolution Banach algebras of pseudomeasures on \( G \). Our results will include those of Segal, Hewitt and Ross since each \( L^p \), \( 1 \leq p < \infty \), is such a Banach algebra.

We will denote by \( A \) the space of continuous functions on \( G \) which possess absolutely summable Fourier transforms, and by \( P \) its continuous dual. The elements of \( P \) are called pseudomeasures.

Ward [5], [6] detail the properties of \( P \) and its elements. However, for convenience we provide a short summary. The Fourier transform \( \hat{S}(\sigma) \) of an element \( S \) of \( P \) at the point \( \sigma \) of \( \Sigma \) is defined to be the operator \( S_{\sigma} \) in \( \mathcal{B}(\mathcal{H}_{\sigma}) \) which satisfies

\[
S(f) = \sum_{\sigma \in \Sigma} d_{\sigma} \text{tr}(S_{\sigma} f(\sigma)^*)
\]

for all \( f \in A \). We note that \( \sup_{\sigma \in \Sigma} \|S_{\sigma}\| \) is finite and that \( \|S\|_P = \sup_{\sigma \in \Sigma} \|S_{\sigma}\| \), so that \( S \) can be identified with the sequence \((S_{\sigma})_{\sigma \in \Sigma}\). For each \( x \in G \), the left \( x \)-translate \( xS \) of \( S \) is defined by \( xS(f) = S(x^{-1} f) \) for all \( f \in A \). The involution \( \sim \) and convolution product \( \ast \) are defined on \( P \) by \((\hat{S})_{\sigma} = S_{\sigma}^*\) and \((S \ast T)_{\sigma} = S_{\sigma} T_{\sigma}\), respectively, for each \( \sigma \in \Sigma \).

Throughout \( B \) will denote a symmetric homogeneous convolution Banach algebra of pseudomeasures, and \( \| \|_B \) its norm. The homogeneity means that \( B \) is left translation invariant, that each left translation operator is continuous on \( B \), and that the map \( x \mapsto x.b \) is continuous on \( G \) for each \( b \in B \). It is further assumed that the injection of \( B \) into \( P \) is continuous.

As already noted each \( L^p \), \( 1 \leq p < \infty \), is homogeneous. Further examples are \( A, C \) (the space of continuous functions on \( G \)) and, for \( 1 \leq p < \infty \), \( U^p \) (the space of \( L^1 \)-functions with \( p \)-summable Fourier transforms). For a discussion of these and other examples see Section 5 of [5]. It is, in fact, shown in [5, (3.1)] that for any subset \( F \) of \( \Sigma \)

\[
B_F = \{ b \in B : \text{supp}(\hat{b}) \subseteq F \}
\]

is a closed symmetric subalgebra of \( B \), and so is also homogeneous. This provides us with a method of producing an abundance of non-trivial examples.
It is an important consequence of 2.4 and 2.6 of [5] that there exists a subset $F$ of $\Sigma$ such that $T_F$ (the set of trigonometric polynomials with Fourier transforms supported by $F$) is a dense subspace of $B$. Consequently, for each $\sigma \in F$, $B_{\{\sigma\}}$ is isomorphic to $B(\mathcal{H}_\sigma)$.

We now let $R$ denote a non-degenerate $\sim$ -representation of $B$ acting on some Hilbert space $\mathcal{H}$. Then, for each $\sigma \in F$, $R$ induces a $\sim$ -representation $Q_\sigma$ of $B(\mathcal{H}_\sigma)$ on $\mathcal{H}$ by

$$Q_\sigma(T) = R\left(x \rightarrow d_\sigma \operatorname{tr}(TU_\sigma(x)\ast)\right)$$

for each $T \in B(\mathcal{H}_\sigma)$.

Naimark [3] has completely determined the structure of $\sim$ -representations of finite dimensional operator algebras $B(\mathcal{H})$. It follows from [3, (22.2)] that, for each $\sigma \in F$, there exists a family of mutually orthogonal $Q_\sigma$-invariant closed subspaces $\mathcal{H}_\sigma^0, \mathcal{H}_\sigma^a, a \in \mathcal{A}$, of $\mathcal{H}$ satisfying

(i) $\mathcal{H} = \mathcal{H}_\sigma^0 \oplus \bigoplus_{a \in \mathcal{A}} \mathcal{H}_\sigma^a$,

(ii) dim $\mathcal{H}_\sigma^a = \dim \mathcal{H}_\sigma = d_\sigma$ for each $a \in \mathcal{A}$, and

(iii) $\mathcal{H}_\sigma^0 = \cap\{\ker Q_\sigma(T) : T \in B(\mathcal{H}_\sigma)\}$,

and a family, indexed by $\mathcal{A}$, of unitary operators $V_\sigma^a : \mathcal{H}_\sigma \rightarrow \mathcal{H}_\sigma^a$ such that, for each $T \in B(\mathcal{H}_\sigma), Q_\sigma(T)$ can be identified with the direct sum

$$\bigoplus_{a \in \mathcal{A}} V_\sigma^a TV_\sigma^a\ast.$$

That is, for each $\mathcal{H} \in \bigoplus_{a \in \mathcal{A}} \mathcal{H}_\sigma^a$, regarded as a subset of $\mathcal{H}$, we have

$$Q_\sigma(T)\mathcal{H} = \left(\bigoplus_{a \in \mathcal{A}} V_\sigma^a TV_\sigma^a\ast\right)\mathcal{H}.$$

It follows that if $P_\sigma^a$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\sigma^a$, then the series $\sum_{a \in \mathcal{A}} V_\sigma^a TV_\sigma^a P_\sigma^a$ is convergent in $B(\mathcal{H})$ in the strong operator topology.

We shall use this decomposition of the $Q_\sigma$ to obtain a decomposition of $R$ which will suggest an 'obvious' choice for $W$. First, however, we require a preliminary result concerning the orthogonality of the set of projections $\{P_\sigma^a : a \in \mathcal{A}, \sigma \in F\}$.

**Lemma 1.** $\{P_\sigma^a : a \in A, \sigma \in F\}$ is orthogonal.

**Proof.** Since $\mathcal{H}$ is the direct sum of $\mathcal{H}_\sigma^0$ and the $\mathcal{H}_\sigma^a$'s it is clear that for each $\sigma \in F, \{P_\sigma^a : a \in \mathcal{A}\}$ is orthogonal. So it is sufficient to show that, provided $\eta \neq \sigma$, we have $\bigoplus_{a \in \mathcal{A}} \mathcal{H}_\sigma^a \subseteq \mathcal{H}_\eta^0.$

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Assume now that $\eta \neq \sigma$. Observe that for each $b_1 \in B(\eta)$ and $b_2 \in B(\sigma)$, we have $R(b_1 \ast b_2) = R(0) = 0$, and so $Q_\eta(T_1)Q_\sigma(T_2) = 0$ for each $T_1 \in \mathcal{B}(\mathcal{H}_\eta)$ and $T_2 \in \mathcal{B}(\mathcal{H}_\sigma)$. We observe that if $T_2 = I_\sigma$ is the identity map on $\mathcal{H}_\sigma$, then $Q_\sigma(I_\sigma) = \sum_{a \in \mathcal{A}} P_a$, and so $Q_\eta(T_1)I_\sigma = 0$ for all $T_1 \in \mathcal{B}(\mathcal{H}_\eta)$ and $\sigma \in \mathcal{A}_\eta$. Hence $\bigoplus_{a \in \mathcal{A}} \mathcal{H}_\sigma^a \subset \mathcal{H}_\eta^0$.

We observe that, for each $\sigma \in \mathcal{F}$ and $a \in \mathcal{A}$, the operator $V_\sigma^a T V_\sigma^a$ is an operator in $\mathcal{B}(\mathcal{H}_\sigma)$ with $||V_\sigma^a T V_\sigma^a|| \leq ||T||$. It follows from Lemma 1 that if $(T_\sigma)_{\sigma \in \mathcal{F}}$ denotes a set of operators, $T_\sigma \in \mathcal{B}(\mathcal{H}_\sigma)$, with $\sup_{\sigma \in \mathcal{F}} ||T_\sigma|| < \infty$, then $\bigoplus_{\sigma \in \mathcal{F}} \bigoplus_{a \in \mathcal{A}} V_\sigma^a T_\sigma V_\sigma^a$ is a bounded operator in $\mathcal{B}(\mathcal{H})$. We can now prove that the representation $R$ can be reconstructed using the decomposition of the $Q_\sigma$'s.

**THEOREM 1.** For each $b \in B$, we have

$$R(b) = \bigoplus_{\sigma \in \mathcal{F}} \bigoplus_{a \in \mathcal{A}} V_\sigma^a \hat{b}(\sigma) V_\sigma^a.$$  

**PROOF.** If $b$ is a trigonometric polynomial, then the support of $\hat{b}$ is a finite subset $F_0$ of $F$, and so

$$R(b) = \sum_{\sigma \in F_0} R(x \to d_\sigma \text{ tr}(\hat{b}(\sigma) U_\sigma(x)))$$

$$= \sum_{\sigma \in F_0} Q_\sigma(\hat{b}(\sigma))$$

$$= \bigoplus_{\sigma \in F_0} \bigoplus_{a \in \mathcal{A}} V_\sigma^a \hat{b}(\sigma) V_\sigma^a$$

$$= \bigoplus_{\sigma \in \mathcal{F}} \bigoplus_{a \in \mathcal{A}} V_\sigma^a \hat{b}(\sigma) V_\sigma^a.$$

Now let $(k_\lambda)_{\lambda \in \Lambda}$ denote an approximate identity in $L^1$. It follows from [5, (2.5)] and from the continuity of $R$ that, for each $b \in B$,

$$\lim_{\lambda \in \Lambda} ||R(b) - R(k_\lambda \ast b)|| = 0.$$  

In particular, if each $k_\lambda$ is a trigonometric polynomial then $k_\lambda \ast b \in T_\mathcal{F}$ and so

$$R(k_\lambda \ast b) = \bigoplus_{\sigma \in \mathcal{F}} \bigoplus_{a \in \mathcal{A}} V_\sigma^a (k_\lambda \ast b)(\sigma) V_\sigma^a.$$  

We note that for each $b \in B$, we have

$$\sup_{\sigma \in \mathcal{F}} \|\hat{b}(\sigma)\| = \|b\|_p \leq C \|b\|_B$$

and so, as our discussion following Lemma 1 indicates, $\bigoplus_{\sigma \in \mathcal{F}} \bigoplus_{a \in \mathcal{A}} V_\sigma^a \hat{b}(\sigma) V_\sigma^a$ is an operator in $\mathcal{B}(\mathcal{H})$. Therefore, it follows from (3) that (2) holds, provided that

$$\lim_{\lambda \in \Lambda} \left\| \bigoplus_{\sigma \in \mathcal{F}} \bigoplus_{a \in \mathcal{A}} V_\sigma^a (\hat{b}(\sigma) - (k_\lambda \ast b)(\sigma)) V_\sigma^a \right\| = 0.$$
That this limit exists and is equal to 0 is a consequence of the inequality
\[
\left\| \sum_{\sigma \in F} \sum_{a \in \mathcal{A}} V_{a}^{a} \left[ \hat{b} (\sigma) - (k_{x} \ast b) \right] V_{a}^{a} \mathcal{P}_{a} \right\|_{\mathcal{K}}^{2} = \sum_{\sigma \in F} \sum_{a \in \mathcal{A}} \left\| V_{a}^{a} \left[ \hat{b} (\sigma) - (k_{x} \ast b) \right] V_{a}^{a} \mathcal{P}_{a} \right\|_{\mathcal{K}}^{2}.
\]
\[
\leq \sup_{a \in \mathcal{A}} \left\| V_{a}^{a} \left[ \hat{b} (\sigma) - (k_{x} \ast b) \right] V_{a}^{a} \mathcal{P}_{a} \right\|_{\mathcal{K}} \sum_{\sigma \in F} \sum_{a \in \mathcal{A}} \left\| \mathcal{P}_{a} \right\|_{\mathcal{K}}^{2}
\]
\[
\leq \left\| b - k_{x} \ast b \right\|_{B}^{2} \left\| \mathcal{K} \right\|_{\mathcal{K}}^{2}
\]
for all \( b \in B \).

Theorem 1 together with Equation (1) establishes that for each \( b \in B \),
\[
R(b) = \sum_{\sigma \in F} \sum_{a \in \mathcal{A}} V_{a}^{a} \hat{b} (\sigma) V_{a}^{a} \mathcal{P}_{a},
\]
the series being convergent in the strong operator topology on \( \mathcal{B}(\mathcal{H}) \).

If we recall our intention to generalize Segal's result, this suggests that the sums
\[
\sum_{\sigma \in F} \sum_{a \in \mathcal{A}} V_{a}^{a} U_{a}(x) V_{a}^{a} \mathcal{P}_{a}
\]
may be useful. Each is convergent in the strong operator topology on \( \mathcal{B}(\mathcal{H}) \) since, for each \( x \in G \), we have
\[
\sum_{\sigma \in F} \sum_{a \in \mathcal{A}} V_{a}^{a} U_{a}(x) V_{a}^{a} \mathcal{P}_{a} \mathcal{P}_{a}^{a}
\]
\[
\leq \sum_{\sigma \in F} \sum_{a \in \mathcal{A}} \left\| \mathcal{P}_{a} \mathcal{P}_{a}^{a} \right\|_{\mathcal{K}}^{2} \mathcal{K} \leq \left\| \mathcal{K} \right\|_{\mathcal{K}}^{2}
\]
for each \( \mathcal{K} \in \mathcal{H} \).

For each \( x \in G \) we define \( W(x) \) by
\[
W(x) = \bigoplus_{\sigma \in F} \bigoplus_{a \in \mathcal{A}} V_{a}^{a} U_{a}(x) V_{a}^{a},
\]
which is in \( \mathcal{B}(\mathcal{H}) \) because \( \sup_{a \in \mathcal{A}} \left\| V_{a}^{a} U_{a}(x) V_{a}^{a} \right\| \leq 1 \). Then, for each \( \mathcal{K} \in \mathcal{H} \), we have
\[
W(x) \mathcal{K} = \sum_{\sigma \in F} \sum_{a \in \mathcal{A}} V_{a}^{a} U_{a}(x) V_{a}^{a} \mathcal{P}_{a},
\]
where, by (4), the series must be uniformly convergent in \( \mathcal{H} \) over \( G \).

**Lemma 2.** The map \( W : G \to \mathcal{B}(\mathcal{H}) \), defined by \( x \mapsto W(x) \), is a continuous unitary representation of \( G \) on \( \mathcal{H} \).

**Proof.** First observe that since \( R \) is non-degenerate we have
\[
\bigoplus_{\sigma \in F} \bigoplus_{a \in \mathcal{A}} \mathcal{H}_{a}^{a} = \mathcal{H},
\]
and so $\sum_{a \in F} \sum_{a \in \mathcal{A}} P_a^a = I$, the identity map on $\mathcal{H}$. Consequently each operator $W(x)$ is unitary. Furthermore, by Lemma 1, $W$ must be multiplicative, and so it remains only to prove that $W$ is continuous. In fact, as $W$ is unitary it is sufficient to prove that it is weakly continuous.

Now, for each $\sigma \in F$, $a \in \mathcal{A}$, $\mathcal{A} \in \mathcal{H}$ and $x$, $y \in G$, we have

$$\|V_\sigma^a [U_\sigma(x) - U_\sigma(y)] V_\sigma^a P_\sigma^a \|_{\mathcal{H}} \leq \|U_\sigma(x) - U_\sigma(y)\| \|\mathcal{A}\|_{\mathcal{H}}$$

so that the functions $x \mapsto V_\sigma^a U_\sigma(x) V_\sigma^a P_\sigma^a$ are continuous from $G$ to $\mathcal{H}$. Since the series $\sum_{a \in F} \sum_{a \in \mathcal{A}} V_\sigma^a U_\sigma(x) V_\sigma^a P_\sigma^a$ must be uniformly convergent in $\mathcal{H}$ over $G$, this guarantees weak continuity.

The theory of vector-valued integrals, as presented in [1, Sections 8.14 and 8.15], can be used to demonstrate that $R$ can be recovered from $W$. To this end, we again let $(k_\lambda)_{\lambda \in \Lambda}$ denote an approximate identity, consisting of trigonometric polynomials of $L^1$. We then prove:

**Theorem 2.** For each $b \in B$, $R(b) = \lim_{\lambda \in \Lambda} \int_G k_\lambda \ast b(x)W(x) \, dx$ in $\mathcal{B}(\mathcal{H})$.

**Proof.** We first take $b \in T_F$ and use some familiar properties of vector-valued integrals to prove that $\int_G b(x)W(x) \, dx$ is an element of $\mathcal{B}(\mathcal{H})$. By [1, 8.15.2 and 8.14.4], it is sufficient to prove that the function $g: G \to \mathcal{B}(\mathcal{H})$, $x \mapsto b(x)W(x)$, is measurable and that the real integral $\int_G \|g(x)\| \, dx$ is finite. However, observe that the first of these conditions follows from Lemma 2, and the second from Lemma 2 and the compactness of $G$.

We now prove that for such $b$,

$$R(b) = \int_G b(x)W(x) \, dx. \tag{5}$$

We know that, for each $\mathcal{A} \in \mathcal{H}$,

$$R(b) \mathcal{A} = \sum_{\sigma \in F} Q_\sigma (b(\sigma)) \mathcal{A} \tag{6}$$

$$= \sum_{\sigma \in F} \sum_{a \in \mathcal{A}} V_\sigma^a \left[ \int_G b(x) U_\sigma(x) \, dx \right] V_\sigma^a P_\sigma^a \mathcal{A},$$

and that each of the functions $x \mapsto V_\sigma^a b(x) U_\sigma(x) V_\sigma^a P_\sigma^a \mathcal{A}$ is continuous from $G$ to $\mathcal{H}$. Since the series is uniformly convergent in $\mathcal{H}$ over $G$, we can interchange the order of summation and integration in (6), giving (5).

To extend the result to the whole of $B$, let $b$ denote an arbitrary element of $B$ and $(k_\lambda)_{\lambda \in \Lambda}$ an approximate identity, consisting of trigonometric polynomials, of $L^1$. Then for each $\lambda \in \Lambda$, $k_\lambda \ast b$ is a trigonometric polynomial, and so it
follows from (3) and (5) that
\[ \lim_{\lambda \to \Lambda} \left\| R(b) - \int_G k_\lambda \ast b(x) W(x) \, dx \right\| = 0. \]

In particular we obtain the obvious extension of Segal's result.

**Corollary.** For each \( b \in B \) and \( \mathcal{A} \in \mathcal{H} \),
\[ \langle R(b) \mathcal{A}, \mathcal{A} \rangle = \int \langle W(x) \mathcal{A}, \mathcal{A} \rangle b(x) \, dx, \]
where \( b \) acts as a continuous linear functional on \( A \). Moreover, if \( b \in L^1 \) then
\[ \langle R(b) \mathcal{A}, \mathcal{A} \rangle = \int \langle W(x) \mathcal{A}, \mathcal{A} \rangle b(x) \, dx. \]

We also obtain the result of Hewitt and Ross as a corollary.

**Corollary.** If \( B = L^p \) for some \( p \), \( 1 < p < \infty \), then for all \( b \in B \),
\[ R(b) = \int_G b(x) W(x) \, dx \]
in \( \mathcal{B}(\mathcal{H}) \).

**Proof.** It follows from [1, 8.14.6] that for each \( b \in B \) and \( \lambda \in \Lambda \),
\[ \left\| \int_G (k_\lambda \ast b - b)(x) W(x) \, dx \right\| \leq k_\lambda \ast b - b \|_{L^p} \sup_{x \in G} \| W(x) \|. \]
Therefore, since
\[ \left\| R(b) - \int_G b(x) W(x) \, dx \right\| \leq \left\| R(b) - \int_G k_\lambda \ast b(x) W(x) \, dx \right\| \]
\[ + \left\| \int_G (k_\lambda \ast b - b)(x) W(x) \, dx \right\|, \]
Theorem 2 ensures (7).

In the case of either group or of algebra representations, of fundamental importance are the so-called irreducible representations. In either case, a representation is called irreducible if the only subspaces of \( \mathcal{H} \) which are invariant under the action of the representation are the trivial ones \( \{0\} \) and \( \mathcal{H} \). In other words, the only projections commuting with the action of the representation are
the 0 and the identity operators. The continuous irreducible unitary representations of a group $G$ are identified, up to equivalence, with $\Sigma$. We now prove that the continuous irreducible unitary representations of $B$ are identified, up to equivalence, with $F$.

**Theorem 3.** $R$ is irreducible if and only if $W$ is irreducible.

**Proof.** Assume that $R$ is reducible and let $\mathcal{N}$ denote a proper closed subspace of $\mathcal{H}$ which is invariant under the action of $R$. For each finite subset $E$ of $F$ define the complex-valued map $\omega_E$ on $G$ by $\omega_E(x) = \sum_{a \in E} d_a \text{tr}(U_a(x)^*)$. Then $\omega_E \in T_F$, and by Theorem 1 we have

$$\lim_{E} \| W(x) \mathcal{N} - R(x\omega_E) \mathcal{N} \|_{\mathcal{N}} = 0$$

for each $\mathcal{N} \in \mathcal{H}$. But $R(x\omega_E)\mathcal{N} \subseteq \mathcal{H}$ for each $x \in G$ and finite subset $E$ of $F$, and so $W(x)\mathcal{N} \subseteq \mathcal{H}$ for all $x \in G$. Hence $W$ is reducible. On the other hand, if $W$ is reducible, and $P$ denotes a non-trivial projection commuting with $W$, then $PW(x) = W(x)P$ for all $x \in G$. Hence, by Theorem 2 $PR(b) = R(b)P$ for each $b \in B$, and so $R$ is also reducible.

So if $R$ is an irreducible unitary representation of $B$, then Theorem 3 ensures that $W = U_a$ for some $a \in F$.

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**References**


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