Canad. Math. Bull. Vol. **16** (3), 1973

MATHEMATICAL NOTES

Manuscripts for this Department should be sent to R. D. Bercov and A. Meir, Editors-in-Chief, Canadian Mathematical Bulletin, Department of Mathematics, University of Alberta, Edmonton 7, Alberta.

A NOTE ON SEMIPRIME RINGS WITH TORSIONLESS INJECTIVE ENVELOPE

BY

EFRAIM P. ARMENDARIZ

Satyanarayana establishes in [6] that a semiprime right selfinjective ring with ACC on annihilator right ideals is semisimple Artinian, thereby extending a similar result of Koh [5] for prime rings. A theorem of Faith [3, Theorem 5.2], shows that the annihilator chain condition on either side implies that a right selfinjective semiprime ring is semisimple Artinian. Noting that any selfinjective ring has torsionless injective envelope we consider the possibility of replacing selfinjectivity by torsionless together with an annihilator condition. It turns out that we can get by with ACC on either principal left or principal right annihilators; specifically we have:

THEOREM 1. *Let R be a semiprime ring with torsionless injective envelope $E(R)$. If R has* ACC *on principal left annihilators or* ACC *on principal right annihilators then R is semisimple Artinian.*

In addition we provide a characterization of semiprime rings with zero (right) singular ideal and torsionless injective envelope in terms of the existence of injective right ideals and give an example of a prime non-selfinjective ring with zero singular ideal and torsionless injective envelope. We remark that our proofs are elementary in nature.

Concerning notation and terminology, R is always a ring with 1, modules are unital right R-modules, $E(R)$=injective envelope of R as a right R-module, and $Z(M)$=singular submodule of the R-module M. As in [2], an R-module M is *torsionless* if M is isomorphic to a submodule of a direct product of copies of R; thus M is torsionless if and only if for each $0\neq x \in M$, there exists $f \in \mathrm{Hom}_R(M, R)$ such that $f(x)\neq 0$. We will need the following two lemmas in the sequel.

LEMMA 1. [1, Lemma 1]. *If M is an injective R-module, N is an R-module with $Z(N)=0$ and $f \in \mathrm{Hom}_R(M, N)$ then $\mathrm{Im}\, f$ is an injective R-module.*

LEMMA 2. [7, Proposition 1.2]. *If R is a semiprime ring and M is a torsionless R-module then $\mathrm{Hom}_R(M, N)\neq 0$ for each nonzero submodule N of M.*

https://doi.org/10.4153/CMB-1973-067-7 Published online by Cambridge University Press

Proof of Theorem 1. We will first show that $Z(R)=0$ in the presence of either chain condition. If R has ACC on principal left annihilators and $Z(R)\neq 0$, choose $a\in Z(R)$ so that $l(a)$ is maximal in the set $\{l(x):0\neq x\in Z(R)\}$. Then $(aR)^2=0$, for if not, then $aras\neq 0$ for some $r, s\in R$ and $as\in Z(R)$. Hence $r(as)\cap asR\neq 0$ so $arasu=0$ for some $u\in R$ with $asu\neq 0$. Since $asu\in Z(R)$, $l(a)=l(asu)$ by maximality of $l(a)$ and so $ar\in l(asu)$ implies $ara=0$ hence $aras=0$, a contradiction. But then $(aR)^2=0$ and R semiprime implies $aR=0$ so $a=0$ and thus $Z(R)=0$. If R has ACC on principal right annihilators and $Z(R)\neq 0$, choose $a\in Z(R)$ so that $r(a)$ is maximal in $\{r(x):0\neq x\in Z(R)\}$. Then $aRa=0$, otherwise $axa\neq 0$ for some $x\in R$ implies $axR\cap r(ax)\neq 0$ so $axs\neq 0$ for some $s\in R$ with $axaxs=0$. But then $r(axa)=r(a)$ and so $axs=0$. It follows that $Z(R)=0$. Thus either chain condition implies $Z(R)=0$. Now by Lemmas 1 and 2 each nonzero right ideal contains a nonzero injective right ideal each of which is generated by an idempotent. Suppose $e_1R\supseteq e_2R\supseteq\cdots$ is a descending chain of injective right ideals with $e_i^2=e_i$ for each $i\geq 1$. If R has ACC on principal left annihilators then from $l(e_1)\subseteq l(e_2)\subseteq\cdots$ we have for some k, $l(e_k)=l(e_j)$ for all $j\geq k$, hence since $l(e_j)=R(1-e_j)$ for all $j\geq 1$ it follows that e_kR is a minimal right ideal. If R has ACC on principal right annihilators, write $R=J_1\oplus e_1R$, $e_1R=J_2\oplus e_2R$ hence $R=J_1\oplus J_2\oplus e_2R$. Continuing in this way we get $J_1\subseteq J_1\oplus J_2\subseteq\cdots$ and for each $k\geq 1$, $J_1\oplus\cdots\oplus J_k$, being a direct summand of R, is a principal right annihilator. As before it follows that e_kR is a minimal right ideal for some $k\geq 1$. Thus in the presence of either chain condition each nonzero right ideal contains an injective minimal right ideal. Then forming direct sums of minimal right ideals which are injective, a similar method as above shows such a sum is finite if either chain condition is assumed and so injective. But then R must be the sum of its minimal right ideals and so semisimple Artinian (e.g. [4, p. 56]). This completes the proof.

It should be noted that $Z(R)=0$ in the presence of the ACC on principal left annihilators is also a consequence of [4, p. 113, Theorem 3.1], however we have chosen to exhibit this directly.

In general, a semiprime ring R with $Z(R)=0$ and $E(R)$ torsionless is not self-injective even if R is a prime ring. Here is an example of such a ring. Let V be a countably infinite-dimensional vector space over a field D, let $K=\mathrm{Hom}_D(V, V)$ and let $S=\{f\in K:\text{rank }(f)\text{ is finite}\}$. If R consists of all matrices of the form

$$\begin{pmatrix} a & b \\ t & c \end{pmatrix} \quad \text{where } a, b, c\in K,\ t\in S$$

then R is a prime ring with $Z(R)=0$. Also the right ideal A of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \quad \text{where } a, b\in K$$

is an injective right ideal. By [1, Theorem 5], $E(R)=$all 2×2 matrices over K is a torsionless R-module.

https://doi.org/10.4153/CMB-1973-067-7 Published online by Cambridge University Press

THEOREM 2. *Let R be a semiprime ring. Then $Z(R)=0$ and $E(R)$ is torsionless if and only if each nonzero right ideal contains a nonzero injective right ideal.*

Proof. If $E(R)$ is torsionless and $Z(R)=0$ then each nonzero right ideal contains a nonzero injective right ideal by Lemmas 1 and 2. For the converse, $Z(R)=0$ since $Z(R)$ can contain no nonzero idempotents. Now let D be the sum of all injective right ideals of R. Then by assumption, D is an essential right ideal and so $l(D)=0$. If $U=r(D)$ then $(UD)^2=0$ so since R is semiprime $UD=0$ and hence $U=r(D)=0$. To show $E(R)$ is torsionless let $0\neq x \in E(R)$; then $0\neq xr \in R$ for some $r \in R$ and since $r(D)=0$, $dxr\neq 0$ for some $d \in D$. Then left multiplication by d gives a nonzero map $g:xrR\to dR \subseteq D$. Since $d \in D$, $d \in U_1+\cdots+U_k=V$ where each U_i is an injective right ideal. Since V is a factor of the direct sum of the U_i's, by Lemma 1, V is injective and $dR \subseteq V$. Thus g can be extended to $f:E(R)\to V \subseteq R$ and $f(x)\neq 0$, hence $E(R)$ is a torsionless R-module.

A similar proof establishes the following:

THEOREM 3. *Let R be a semiprime ring with $Z(R)=0$. If $E(R)$ is a torsionless R-module then $D=$sum of all injective right ideals of R is a two-sided ideal which is an essential left and right ideal of R. Moreover, every finitely-generated right ideal in D is contained in an injective right ideal.*

From this we have the

COROLLARY. *If R is a simple ring and $E(R)$ is torsionless, then $R=E(R)$.*

REFERENCES

1. E. P. ARMENDARIZ, *On finite-dimensional torsion-free modules and rings*, Proc. Amer. Math. Soc. **24** (1970), 566–571.

2. H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

3. C. Faith, *Rings with ascending condition on annihilators*, Nagoya Math. J. **27** (1966), 177–191.

4. ——, *Lectures on injective modules and quotient rings*, Lecture Notes in Math. no. 49, Springer-Verlag, Berlin-New York, 1967.

5. K. Koh, *A note on a self-injective ring*, Canad. Math. Bull. **8** (1965), 29–32.

6. M. Satyanarayana, *A note on a self-injective ring*, Canad. Math. Bull. **14** (1971), 271–272.

7. J. Zelmanowitz, *Endomorphism rings of torsionless modules*, J. Algebra **5** (1967), 325–341.

UNIVERSITY OF SOUTHWESTERN LOUISIANA,
LAFAYETTE, LOUISIANA