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ARTICLE

# New upper bounds for the Erdős-Gyárfás problem on generalized Ramsey numbers

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**Abstract**

A $(p, q)$-colouring of a graph $G$ is an edge-colouring of $G$ which assigns at least $q$ colours to each $p$-clique. The problem of determining the minimum number of colours, $f(n, p, q)$, needed to give a $(p, q)$-colouring of the complete graph $K_n$ is a natural generalization of the well-known problem of identifying the diagonal Ramsey numbers $r_k(p)$. The best-known general upper bound on $f(n, p, q)$ was given by Erdős and Gyárfás in 1997 using a probabilistic argument. Since then, improved bounds in the cases where $p = q$ have been obtained only for $p \in \{4, 5\}$, each of which was proved by giving a deterministic construction which combined a $(p, p-1)$-colouring using few colours with an algebraic colouring.

In this paper, we provide a framework for proving new upper bounds on $f(n, p, p)$ in the style of these earlier constructions. We characterize all colourings of $p$-cliques with $p-1$ colours which can appear in our modified version of the $(p, p-1)$-colouring of Conlon, Fox, Lee, and Sudakov. This allows us to greatly reduce the amount of case-checking required in identifying $(p, p)$-colourings, which would otherwise make this problem intractable for large values of $p$. In addition, we generalize our algebraic colouring from the $p = 5$ setting and use this to give improved upper bounds on $f(n, 6, 6)$ and $f(n, 8, 8)$.

**Keywords:** Ramsey theory; graph colouring; generalized Ramsey number

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## 1. Introduction

Let $p$ and $q$ be positive integers such that $1 \le q \le \binom{p}{2}$. We say that an edge-colouring of a graph $G$ is a $(p, q)$-*colouring* if any $p$-clique of $G$ contains edges of at least $q$ distinct colours. In 1975, Erdős and Shelah [5] posed the question of determining $f(n, p, q)$, the minimum number of colours needed to give a $(p, q)$-colouring of the complete graph on $n$ vertices, $K_n$.

This function $f(n, p, q)$ is known as the Erdős-Gyárfás function after the authors of the first paper [6] to systematically study $(p, q)$-colourings. The majority of their work focused on understanding the asymptotic behaviour of this function as $n \to \infty$ for fixed values of $p$ and $q$. One of their primary results was a general upper bound of

$$f(n, p, q) = O\left(n^{\frac{p-2}{\binom{p}{2}-q+1}}\right)$$

obtained using the Lovász Local Lemma, while one of the main problems they left open was the determination of $q$, given a fixed value of $p$, for which $f(n, p, q) = \Omega(n^{\varepsilon})$ for some constant $\varepsilon$, but

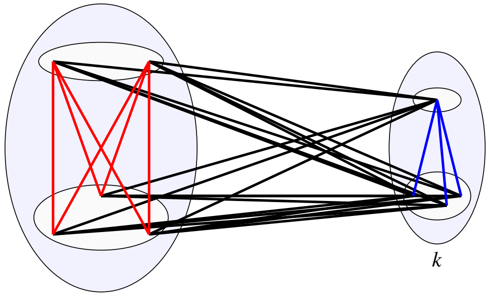


**Figure 1.** An example of a $p$-clique with a leftover structure.

$f(n,p,q-1)=n^{o(1)}$. Towards this end, they found that

$$n^{\frac{1}{p-2}}-1\leq f(n,p,p)\leq cn^{\frac{2}{p-1}},$$

where the lower bound is given by a simple induction argument and the upper bound is a special case of their general upper bound. However, they did not determine whether $f(n,p,p-1)=n^{o(1)}$.

In 2015, Conlon, Fox, Lee, and Sudakov [3], building on work done on small cases by Mubayi and Eichhorn [4, 7], showed that $f(n,p,p-1)=n^{o(1)}$ by constructing an explicit $(p,p-1)$-colouring using very few colours. In [2], we slightly modified their colouring, which we call the *CFLS colouring*, and paired it with an 'algebraic' construction to show $f(n,5,5)\leq n^{1/3+o(1)}$. This improves on the general upper bound found by Erdős and Gyárfás and comes close to matching their lower bound in terms of order of growth. Our construction built on the ideas of Mubayi in [8], where he gave an explicit construction showing $f(n,4,4)\leq n^{1/2+o(1)}$.

In this paper, we push these ideas further. In Section 2, we prove the following result.

**Theorem 1.1.** *For any $p\geq 3$, there is a $(p,p-1)$-colouring of $K_n$ using $n^{o(1)}$ colours such that the only $p$-cliques that contain exactly $p-1$ distinct edge colours are isomorphic (as edge-coloured graphs) to one of the edge-coloured $p$-cliques given in the definition below.*

**Definition 1.1.** Given an edge-colouring $f:E(K_n)\to C$, we say that a subset $S\subseteq V(K_n)$ has a *leftover structure under $f$* if either $|S|=1$ or there exists a bipartition (which we will call the *initial bipartition*) of $S$ into nonempty sets $A$ and $B$ for which

- $A$ and $B$ each have a leftover structure under $f$;
- $f(A)\cap f(B)=\emptyset$; and
- there is a fixed colour $\alpha\in C$ such that $f(a,b)=\alpha$ for all $a\in A$ and all $b\in B$ and $\alpha\notin f(A)$ and $\alpha\notin f(B)$.

See Figure 1 for an example of a leftover structure. Alternatively, a more constructive definition is to say that a $p$-clique $S$ is leftover if either $p=1$ or if it can be formed from a leftover $(p-1)$-clique by taking one of its vertices $x$, making a copy $x'$, colouring $xx'$ with a new colour, and colouring $x'y$ with the same colour as $xy$ for each $y\in S$ for which $y\neq x$. Note that it is easy to see by induction that these $p$-cliques always contain exactly $p-1$ colours.

One of the general difficulties in producing explicit $(p,q)$-colourings is dealing with the large number of possible non-isomorphic ways to colour the edges of a $p$-clique with fewer than $q$ colours in order to demonstrate that a construction avoids them. By identifying the 'bad' structures that are leftover after using only $n^{o(1)}$ colours, we are able to greatly reduce the amount of case-checking required in identifying $(p,p)$-colourings, which would otherwise make this problem intractable for large $p$.

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More precisely, one of the nice properties of these leftover structures is that any subset of vertices of a leftover clique induces a clique that is itself leftover. Therefore, any edge-colouring of $K_n$ that eliminates leftover $p$-cliques also eliminates all leftover $P$-cliques for any $P \geq p$. Moreover, by Theorem 1.1, if this colouring uses $n^{\varepsilon+o(1)}$ colours, then $f(n, P, P) \leq n^{\varepsilon+o(1)}$, as the product of this colouring with the one guaranteed in Theorem 1.1 will avoid any $P$-clique with fewer than $P$ colours for each $P \geq p$.

As a specific example, in [2] we gave a $(5, 5)$-colouring of $K_n$ that used $n^{1/3+o(1)}$ colours. Since this colouring avoids leftover 5-cliques, then it also avoids leftover $P$-cliques for all $P \geq 5$. Therefore, if we take the product of this colouring with the appropriate one developed in Section 2 that eliminates all 6-cliques with 5 or fewer colours other than leftover 6-cliques, then we have a $(6, 6)$-colouring that uses only $n^{1/3+o(1)}$ colours, improving the best-known upper bound given above, $O(n^{2/5})$.

In Section 3, we generalize the algebraic portion of our colouring in [2], the 'Modified Dot Product' colouring, to a version that eliminates leftover 6-cliques with $O(n^{1/3})$ colours (making the above example redundant) and eliminates leftover 8-cliques with $O(n^{1/4})$ colours. By taking the product of these colourings with the appropriate ones developed in Section 2, this gives us the following theorem.

**Theorem 1.2.** *We have the following upper bounds:*

$$f(n, 6, 6) = n^{1/3+o(1)}; \quad f(n, 8, 8) = n^{1/4+o(1)}.$$

This improves the best-known upper bound $f(n, 8, 8) = O(n^{2/7})$ as well.

## 2. Modified CFLS colouring

In this section, we define an edge-colouring $\psi_p$ of the complete graph with vertex set $\{0, 1\}^\alpha$ for some positive integer $\alpha$. This construction is the product of two colourings, $\psi_p = c_p \times \Delta_p$, where $c_p$ is the $(p+3, p+2)$-colouring defined in [3]. In many places, this section tracks parts of the proof given in [3], and we have attempted to keep the notation consistent with that paper to make cross-referencing easier.

We will prove the following lemma about the colouring $c_p$.

**Lemma 2.1.** *Let $p$ be a fixed positive integer. Any subset $S \subseteq \{0, 1\}^\alpha$ with $|S| \leq p+3$ vertices that contains exactly $|S|-1$ distinct colours under the edge-colouring $c_p$ either has a leftover structure under $c_p$ or contains a striped $K_4$ under $c_p$.*

A striped $K_4$, as described by the following definition, was first defined in [8].

**Definition 2.1.** Let $f : E(G) \to C$ be an edge-colouring of a graph $G$. We call any 4-clique of $G$, $\{a, b, c, d\} \subseteq V(G)$, for which $f(ab) = f(cd)$, $f(ac) = f(bd)$, $f(ad) = f(bc)$, $f(ab) \neq f(ac)$, $f(ab) \neq f(ad)$, and $f(ac) \neq f(ad)$ a *striped* $K_4$.

We will also prove the following result about the colouring $\psi_p$.

**Lemma 2.2.** *There is no striped $K_4$ under the edge-colouring $\psi_p$.*

These two lemmas are enough to conclude that $\psi_p$ is a $(p+3, p+2)$-colouring for which any clique $S$ with $|S| \leq p+3$ that contains exactly $|S|-1$ colours must have a leftover structure.

### *2.1 The construction*

For some positive integer $p$, let

$$1 \leq r_1 \leq r_2 \leq \cdots \leq r_p$$

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be fixed positive integers such that $r_d | r_{d+1}$ for each $d = 1, \dots, p-1$. These $r_i$ will be called the parameters of our edge-colouring.

For any $\alpha \geq r_p$, let $n = 2^\alpha$, and associate each vertex of the complete graph $K_n$ with its own unique binary string of length $\alpha$. For each $d = 1, \dots, p$, let $\alpha = a_d r_d + b_d$ for positive integers $a_d, b_d$ such that $1 \leq b_d \leq r_d$. For each string $x \in \{0, 1\}^\alpha$, we let

$$x = \left(x_1^{(d)}, x_2^{(d)}, \dots, x_{a_d+1}^{(d)}\right)$$

where $x_i^{(d)}$ denotes a binary string in $\{0, 1\}^{r_d}$ for each $i = 1, \dots, a_d$ and $x_{a_d+1}^{(d)}$ denotes a binary string from $\{0, 1\}^{b_d}$. We will call these substrings $r_d$-blocks of $x$, including the final one which may or may not actually have length equal to $r_d$.

In the following definitions, we let $r_0 = 1$ and $r_{p+1} = \alpha$. First, we define a function $\eta_d$ for any $d = 0, \dots, p$ on domain $\{0, 1\}^\beta \times \{0, 1\}^\beta$ where $\beta$ is any positive integer as

$$\eta_d(x, y) = \begin{cases} \left(i, \{x_i^{(d)}, y_i^{(d)}\}\right) & x \neq y \\ 0 & x = y \end{cases}$$

where $i$ denotes the minimum index for which $x_i^{(d)} \neq y_i^{(d)}$.

For $x, y \in \{0, 1\}^\alpha$ and $0 \leq d \leq p$, let

$$\xi_d(x, y) = \left(\eta_d\left(x_1^{(d+1)}, y_1^{(d+1)}\right), \dots, \eta_d\left(x_{a_{d+1}+1}^{(d+1)}, y_{a_{d+1}+1}^{(d+1)}\right)\right).$$

And let

$$c_p(x, y) = \left(\xi_p(x, y), \dots, \xi_0(x, y)\right).$$

Next, we assume that the binary strings of $\{0, 1\}^\beta$ are lexicographically ordered for every positive integer $\beta$. For $1 \leq i \leq a_p + 1$ and binary strings $x < y$, define

$$\delta_{p,i}(x, y) = \begin{cases} +1 & \text{if } x_i^{(p)} \leq y_i^{(p)} \\ -1 & \text{if } x_i^{(p)} > y_i^{(p)}. \end{cases}$$

Let

$$\Delta_p(x, y) = \left(\delta_{p,1}(x, y), \dots, \delta_{p,a_p+1}(x, y)\right).$$

Finally, let

$$\psi_p(x, y) = \left(c_p(x, y), \Delta_p(x, y)\right).$$

### *2.2 Number of colours*

For any positive integer $n$, let $\beta$ be the positive integer for which

$$2^{(\beta-1)^{p+1}} < n \leq 2^{\beta^{p+1}}.$$

For each $d = 1, \dots, p+1$, let $r_d = \beta^d$ in the construction of $\psi_p$. Specifically, this means we are constructing the colouring on the complete graph with vertex set $\{0, 1\}^\alpha$ where $\alpha = \beta^{p+1}$. We can apply this colouring to $K_n$ by arbitrarily associating each vertex of $K_n$ with a unique binary string from $\{0, 1\}^\alpha$ and taking the induced colouring.

As shown in [3], for these choices of parameters $r_d$, the colouring $c_p$ uses at most $2^{4(p+1)\beta^p \log_2 \beta}$ colours. On the other hand, $\Delta_p$ uses

$$2^{a_p+1} \leq 2^\beta$$

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colours. So all together, $\psi_p$ uses at most $2^{4(p+1)\beta^p \log_2 \beta + \beta}$ colours, where

$$(\log_2 n)^{1/(p+1)} \leq \beta < (\log_2 n)^{1/(p+1)} + 1.$$

Thus, for any fixed $p$, $\psi_p$ uses a total of $n^{o(1)}$ colours.

### 2.3 Refinement of functions

Before we prove Lemma 2.1, it will be helpful to give the following definition and results about refinement of functions. The definition and Lemma 2.3 are paraphrased from [3].

**Definition 2.2.** Let $f : A \to B$ and $g : A \to C$. We say that $f$ *refines* $g$ if $f(a_1) = f(a_2)$ implies that $g(a_1) = g(a_2)$ for all $a_1, a_2 \in A$.

**Lemma 2.3** (Lemma 4.1(vi) from [3]). *Let $f, g$ be functions on domain $A$. If $f$ refines $g$, then for all $A' \subseteq A$, we have $|f(A')| \geq |g(A')|$.*

**Lemma 2.4.** *Let $f, g$ be functions on domain $A$. If $f$ refines $g$ and $S \subseteq A$ is a finite subset for which $|f(S)| = |g(S)|$, then*

$$f(s_1) = f(s_2) \iff g(s_1) = g(s_2)$$

*for all $s_1, s_2 \in S$.*

**Proof.** The forward direction follows from the definition of $f$ refining $g$. Conversely, if we have $g(s_1) = g(s_2)$ but $f(s_1) \neq f(s_2)$ for some $s_1, s_2 \in S$, then $|f(S)| \geq |g(S)| + 1$, a contradiction. □

In particular, Lemma 2.4 implies that if some edge-colouring of a clique $S$ is refined by another edge-colouring, but $S$ contains the same number of colours under each, then the edge-colourings must be isomorphic.

### 2.4 Proof of Lemma 2.1

Let $S \subseteq \{0, 1\}^\alpha$ be a set of $|S| \leq p + 3$ vertices which contains exactly $|S| - 1$ distinct edge colours under $c_p$. We will prove that $S$ either has a leftover structure or contains a striped $K_4$ by induction on $\alpha$, similar to the proof of Theorem 2.2 from [3].

For the base case, consider $\alpha \leq r_p$. Then for any $x, y \in S$, the first component of $c_p(x, y)$ is

$$\xi_p(x, y) = \big(\eta_p(x, y)\big) = \big((1, \{x, y\})\big) .$$

Therefore, all of the edges of $S$ receive distinct colours. So it must be that $|S| - 1 = \binom{|S|}{2}$, which happens only when $|S| = 1, 2$. In either case, $S$ trivially has a leftover structure.

Now assume that $\alpha > r_p$ and that the statement is true for shorter binary strings. For each $d = 1, \ldots, p$, let $\alpha_d$ be the largest integer strictly less than $\alpha$ that is divisible by $r_d$. For any $x \in S$, let $x = (x'_d, x''_d)$ for $x'_d \in \{0, 1\}^{\alpha_d}$ and $x''_d \in \{0, 1\}^{\alpha - \alpha_d}$.

Let $S_d$ denote the set of $\alpha_d$-prefixes of $S$,

$$S_d = \left\{x'_d \in \{0, 1\}^{\alpha_d} \,|\, \exists x \in S, x = (x'_d, x''_d)\right\} .$$

For each $x'_d \in S_d$, let

$$T_{x'_d} = \{x \in S | x = (x'_d, x''_d)\}.$$

Let $\Lambda_I^{(d)}$ be the set of colours contained in $S$ found on edges that go between vertices from two distinct $T$-sets,

$$\Lambda_I^{(d)} = \{c_p(x, y) | x, y \in S; x'_d \neq y'_d\}.$$

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Similarly, let $\Lambda_E^{(d)}$ denote the set of colours contained in $S$ found on edges between vertices from the same $T$-set,

$$\Lambda_E^{(d)} = \{c_p(x, y) | x, y \in S; x \neq y; x'_d = y'_d\}.$$

Note that these sets of colours, $\Lambda_I^{(d)}$ and $\Lambda_E^{(d)}$, partition all of the colours contained in $S$. Therefore,

$$|S| - 1 = |\Lambda_I^{(d)}| + |\Lambda_E^{(d)}|.$$

Next, define

$$C_I^{(d)} = \left\{(c_p(x'_d, y'_d), \eta_{d-1}(x''_d, y''_d)) | x, y \in S; x'_d \neq y'_d\right\}$$
$$C_E^{(d)} = \left\{\{x''_d, y''_d\} | x, y \in S; x \neq y; x'_d = y'_d\right\}.$$

It is shown in [3] that $|\Lambda_I^{(d)}| \geq |C_I^{(d)}|$ and that $|\Lambda_E^{(d)}| \geq |C_E^{(d)}|$. The second inequality is easier to see since any distinct $x, y \in S$ for which $x'_d = y'_d$ give $\xi_d = \left(0, \ldots, 0, (i, \{x''_d, y''_d\})\right)$ as the appropriate component of $c_p(x, y)$. Although the first inequality seems intuitively true, its proof is a bit more subtle. The following fact (proved in [3]) together with Lemma 2.3 gives us the desired inequality.

**Fact 2.1** (*Lemma 4.3 from [3]*). For $x, y \in \{0, 1\}^\alpha$, let

$$\gamma_d(x, y) = (c_p(x'_d, y'_d), \eta_{d-1}(x''_d, y''_d)).$$

Then $c_p$ refines $\gamma_d$ as functions on domain $\{0, 1\}^\alpha \times \{0, 1\}^\alpha$.

We will also use the following fact which is proven in [3], although not stated as a claim or lemma that can be easily cited. (See the final sentence of the second-to-final paragraph on page 11.)

**Fact 2.2** (*proved in [3]*). There exists an integer $1 \leq d \leq p$ for which

$$|C_I^{(d)}| + |C_E^{(d)}| \geq |S| - 1.$$

Therefore,

$$|S| - 1 = |\Lambda_I^{(d)}| + |\Lambda_E^{(d)}| \geq |C_I^{(d)}| + |C_E^{(d)}| \geq |S| - 1,$$

which implies that

$$|S| - 1 = |\Lambda_I^{(d)}| + |\Lambda_E^{(d)}| = |C_I^{(d)}| + |C_E^{(d)}|.$$

Let

$$\tilde{c}_p(x, y) = \begin{cases} (c_p(x'_d, y'_d), \eta_{d-1}(x''_d, y''_d)) & \text{if } x'_d \neq y'_d \\ \{x''_d, y''_d\} & \text{otherwise.} \end{cases}$$

Then by Fact 2.1, we know that $\tilde{c}_p$ refines $c_p$. And since $|\Lambda_I^{(d)}| + |\Lambda_E^{(d)}| = |C_I^{(d)}| + |C_E^{(d)}|$, then by Lemma 2.4, we know that the structure of $S$ under $\tilde{c}_p$ must be the same as the structure of $S$ under $c_p$. Therefore, we need only show that $S$ either has a leftover structure or contains a striped $K_4$ under $\tilde{c}_p$ to complete the proof. We consider two cases: either there exists some $\omega \in C_E^{(d)}$ that appears more than once in $S$ under $\tilde{c}_p$ or each $\omega \in C_E^{(d)}$ appears exactly once in $S$ under $\tilde{c}_p$.

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*Case 1*: Let $\omega \in C_E^{(d)}$ appear on at least two edges in $S$. This implies that $\omega = \{x''_d, y''_d\}$ and so there must exist $a, b, c, e \in S$ such that $a = (x'_d, x''_d)$, $b = (x'_d, y''_d)$, $c = (y'_d, x''_d)$, and $e = (y'_d, y''_d)$ for some $x'_d \neq y'_d$. Therefore,

$$\begin{aligned}\tilde{c}_p(a, b) &= \tilde{c}_p(c, e) = \{x''_d, y''_d\}\\ \tilde{c}_p(a, c) &= \tilde{c}_p(b, e) = (c_p(x'_d, y'_d), 0)\\ \tilde{c}_p(a, e) &= \tilde{c}_p(b, c) = (c_p(x'_d, y'_d), \eta_{d-1}(x''_d, y''_d)),\end{aligned}$$

and all three colours are distinct. Hence, $S$ contains a striped $K_4$ under $\tilde{c}_p$.

*Case 2*: If each $\omega \in C_E^{(d)}$ appears exactly once in $S$ under $\tilde{c}_p$, then we know that

$$|C_E^{(d)}| = \sum_{x'_d \in S_d} \binom{|T_{x'_d}|}{2}$$

since each edge within a given $T$-set receives a unique colour. Moreover, if we let

$$C_B^{(d)} = \{c_p(x'_d, y'_d) | x'_d, y'_d \in S_d\},$$

then we know that

$$|C_I^{(d)}| \geq |C_B^{(d)}| \geq |S_d| - 1.$$

Therefore,

$$\begin{aligned}|S_d| - 1 + \sum_{x'_d \in S_d} \binom{|T_{x'_d}|}{2} &\leq |S| - 1\\ \sum_{x'_d \in S_d} \binom{|T_{x'_d}|}{2} &\leq |S| - |S_d|\\ \sum_{x'_d \in S_d} \binom{|T_{x'_d}|}{2} &\leq \sum_{x'_d \in S_d} (|T_{x'_d}| - 1)\\ \sum_{x'_d \in S_d} (|T_{x'_d}| - 1)(|T_{x'_d}| - 2) &\leq 0.\end{aligned}$$

Hence, we have $|T_{x'_d}| = 1, 2$ for each $x'_d \in S_d$. This implies that $|C_E^{(d)}| = \sum_{x'_d \in S_d} (|T_{x'_d}| - 1)$ and $|C_I^{(d)}| = |C_B^{(d)}| = |S_d| - 1$. So by induction, $S_d$ either has a leftover structure or contains a striped $K_4$ under $c_p$. Furthermore, the colouring defined by

$$c'_p(x, y) = \begin{cases} c_p(x'_d, y'_d) & \text{if } x'_d \neq y'_d \\ \{x''_d, y''_d\} & \text{otherwise} \end{cases}$$

is refined by $\tilde{c}_p$, and $S$ contains exactly $|S| - 1$ colours under both $c'_p$ and $\tilde{c}_p$. So by Lemma 2.4, the edge-colouring of $S$ under $\tilde{c}_p$ is isomorphic to the one under $c'_p$, and hence it is sufficient to show that $S$ has either a leftover structure or contains a striped $K_4$ under $c'_p$.

If $S_d$ has a leftover structure under $c_p$, then we see that $S$ also has a leftover structure under $c'_p$ since we can form $S$ under $c'_p$ from $S_d$ under $c_p$ by a sequence of splits as described in the definition of a leftover structure. That is, for each $x'_d \in S_d$ for which $|T_{x'_d}| = 2$, we replace $x'_d$ with two vertices with a new edge colour between them, and the same edge colours that $x'_d$ already had to the rest of the vertices.

On the other hand, if $S_d$ contains a striped $K_4$ under $c_p$, then $S$ must contain a striped $K_4$ under $c'_p$ with colours entirely from $C_B^{(d)}$. This concludes the proof.

### 2.5 Proof of Lemma 2.2

Let $a, b, c, d \in \{0, 1\}^\alpha$ be four distinct vertices, and assume towards a contradiction that they form a striped $K_4$ under $\psi_p$. Specifically, assume that $\psi_p(a, b) = \psi_p(c, d)$, $\psi_p(a, c) = \psi_p(b, d)$, and $\psi_p(a, d) = \psi_p(b, c)$.

Without loss of generality, we may assume the following: that $a$ is the minimum element of the four under the lexicographic ordering of $\{0, 1\}^\alpha$; that for some $i \leq j, k$,

$$\begin{aligned}\eta_p(a, b) &= \eta_p(c, d) = (i, \{x, y\})\\ \eta_p(a, c) &= \eta_p(b, d) = (j, \{z, w\})\\ \eta_p(a, d) &= \eta_p(b, c) = (k, \{s, t\});\end{aligned}$$

and that $a_i^{(p)} = c_i^{(p)} = x$ while $b_i^{(p)} = d_i^{(p)} = y$. It follows from the ordering that $x < y$ and that $a < c < b, d$. Furthermore, we have $i < j$ since $a$ and $c$ do not differ in the $i^{th}$ block. Similarly, we see that $(k, \{s, t\}) = (i, \{x, y\})$. Without loss of generality, we may assume $a_j^{(p)} = b_j^{(p)} = z$ and $c_j^{(p)} = d_j^{(p)} = w$. Therefore, $z < w$ and $a < c < b < d$.

Now, it follows that $\delta_j(a, d) = +1$ and that $\delta_j(c, b) = -1$, contradicting our assumption that $\psi_p(a, d) = \psi_p(c, b)$.

## 3. Modified Dot Product colouring

Fix an odd prime power $q$ and a positive integer $d$. In this section, we prove Theorem 1.2 by giving an edge-colouring $\varphi_d$ for the complete graph on $n = (q-1)^d$ vertices that uses $(3d+1)q - 1$ colours and contains no leftover 6-cliques when $d = 3$ and no leftover 8-cliques when $d = 4$.

In what follows, we make use of several standard concepts and results from linear algebra without providing explicit definitions or proofs. We highly recommend *Linear Algebra Methods in Combinatorics* by László Babai and Péter Frankl [1] for a detailed treatment of these ideas. In particular, Chapter 2 covers all of the necessary background for our argument.

### 3.1 The construction

Let $\mathbb{F}_q^*$ denote the nonzero elements of the finite field with $q$ elements, and let $(\mathbb{F}_q^*)^d$ denote the set of ordered $d$-tuples of elements from $\mathbb{F}_q^*$. In other words, $(\mathbb{F}_q^*)^d$ is the set of $d$-dimensional vectors over the field $\mathbb{F}_q$ without zero components. In what follows, we will assume that the set $\mathbb{F}_q^*$ is endowed with a linear order which can be arbitrarily chosen. We then order the set $(\mathbb{F}_q^*)^d$ with lexicographic ordering based on the order applied to $\mathbb{F}_q^*$.

Define a set of colours $C_d$ as the disjoint union

$$C_d = \mathrm{DOT} \sqcup \mathrm{ZERO} \sqcup \mathrm{UP} \sqcup \mathrm{DOWN},$$

where $\mathrm{DOT} = \mathbb{F}_q^*$, and ZERO, UP, and DOWN are each copies of the set $\{1, \ldots, d\} \times \mathbb{F}_q$. Let

$$\varphi_d : \binom{(\mathbb{F}_q^*)^d}{2} \to C_d$$

be a colouring function of pairs of distinct vectors, $x < y$, defined by

$$\varphi_d(x, y) = \begin{cases} (i, x_i + y_i)_{\mathrm{ZERO}} & \text{if } x \cdot y = 0 \\ (i, x_i + y_i)_{\mathrm{UP}} & \text{if } x \cdot y \neq 0 \text{ and } x \cdot y = x \cdot x \\ (i, x_i + y_i)_{\mathrm{DOWN}} & \text{if } x \cdot y \notin \{0, x \cdot x\} \text{ and } x \cdot y = y \cdot y \\ x \cdot y & \text{otherwise} \end{cases}$$

where $i$ is the first coordinate for which $x = (x_1, \ldots, x_d)$ differs from $y = (y_1, \ldots, y_d)$ and $x \cdot y$ denotes the standard inner product (dot product).

### 3.2 Number of colours

Let $n$ be a positive integer. Let $q$ be the smallest odd prime power for which $n \leq (q-1)^d$. Then we can colour the edges of $K_n$ by arbitrarily associating each vertex with a unique vector from $(\mathbb{F}_q^*)^d$ and taking the colouring induced by $\varphi_d$. By Bertrand's Postulate, $q \leq 2(n^{1/d} + 1)$. Therefore, the number of colours used by $\varphi_d$ on $K_n$ is at most

$$(3d+1)q - 1 \leq (6d+2)n^{1/d} + (6d+1).$$

### 3.3 Proof of Theorem 1.2

**Definition 3.1.** Given a subset of vectors $S \subseteq \mathbb{F}^d$, let rk$(S)$ denote the *rank* of the subset, the dimension of the linear subspace spanned by the vectors of $S$. Let af$(S)$ denote the *affine dimension* of $S$, the dimension of the affine subspace (also known as the affine hull) spanned by $S$.

**Definition 3.2.** A colour $\alpha \in C_d$ has the *dot property* if $\alpha \in \mathrm{DOT} \cup \mathrm{ZERO}$. Note that if $\alpha$ has the dot property, then $\varphi_d(a, b) = \varphi_d(e, f) = \alpha$ implies that $a \cdot b = e \cdot f$ for any $a, b, e, f \in (\mathbb{F}_q^*)^d$.

**Lemma 3.1.** *Let $\{s_1, \ldots, s_t\} \subseteq (\mathbb{F}_q^*)^d$ be a set of linearly independent vectors and let $a, b \in (\mathbb{F}_q^*)^d$ such that*

$$\varphi_d(a, b) = \varphi_d(a, s_i) = \alpha$$
$$\varphi_d(b, s_i) = \beta$$

*for some $\alpha, \beta \in C_d$ and for each $1 \leq i \leq t$. Then $s_1, \ldots, s_t, b$ are linearly independent.*

**Proof.** Assume towards a contradiction that $b = \sum_{j=1}^t \lambda_j s_j$ for some scalars $\lambda_1, \ldots, \lambda_t \in \mathbb{F}_q$. We will first show that $\sum_{j=1}^t \lambda_j = 1$.

If $\alpha \in \mathrm{DOT}$, then $b = \sum_{j=1}^t \lambda_j s_j$ implies that

$$\alpha = a \cdot b = \sum_{j=1}^t \lambda_j (a \cdot s_j) = \sum_{j=1}^t \lambda_j \alpha.$$

Therefore, $\sum_{j=1}^t \lambda_j = 1$ since $\alpha \notin \mathrm{ZERO}$.

If $\alpha \notin \mathrm{DOT}$, then

$$a_i + b_i = a_i + s_{1,i} = \cdots = a_i + s_{t,i}$$

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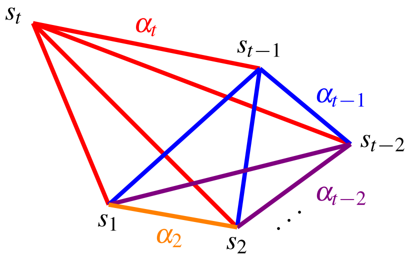


**Figure 2.** A $t$-falling star.

where $i$ is the first index of difference between $a$ and $b$. Thus, $s_{j,i} = b_i$ for all $1 \leq j \leq t$. But then $b = \sum_{j=1}^{t} \lambda_j s_j$ implies that

$$b_i = \sum_{j=1}^{t} \lambda_j s_{j,i} = \sum_{j=1}^{t} \lambda_j b_i.$$

Hence, $\sum_{j=1}^{t} \lambda_j = 1$ since $b_i \neq 0$. Therefore, for any $\alpha \in C_d$ we have $\sum_{j=1}^{t} \lambda_j = 1$.

Now, if $\beta$ has the dot property, then let $\beta'$ denote $b \cdot s_j$ for all $j = 1, \ldots, t$. We have

$$b \cdot b = \sum_{j=1}^{t} \lambda_j (b \cdot s_j) = \sum_{j=1}^{t} \lambda_j \beta' = \beta'.$$

But this implies that $\beta \in \mathrm{UP} \cup \mathrm{DOWN}$, contradicting that $\beta$ has the dot property.

So we must assume that $\beta$ does not have the dot property. It follows that

$$b_k + s_{1,k} = \cdots = b_k + s_{t,k}$$

where $k$ is the first index of difference between $b$ and $s_1$. Therefore, $s_{1,k} = \cdots = s_{t,k}$, and so

$$b_k = \sum_{j=1}^{t} \lambda_j s_{j,k} = \sum_{j=1}^{t} \lambda_j s_{1,k} = s_{1,k},$$

contradicting our choice of $k$.

Since we reach a contradiction for all colours $\beta$, it must be the case that $s_1, \ldots, s_t, b$ are linearly independent vectors, as desired. $\square$

We now define a particular instance of leftover structure that will be useful in our arguments.

**Definition 3.3.** We call the set of vectors $S = \{s_1, \ldots, s_t\} \subseteq (\mathbb{F}_q^*)^d$ a *t-falling star* under the colouring $\varphi_d$ if $\varphi_d\left(s_i, s_j\right) = \alpha_i$ for all $1 \leq j < i \leq t$, as shown in Figure 2. For any set of vectors $T \subseteq (\mathbb{F}_q^*)^d$ under $\varphi_d$, let $FS(T)$ denote the maximum $t$ such that $T$ contains a $t$-falling star.

The following result about these falling stars is an easy consequence of Lemma 3.1 which can be shown by induction on the number of vectors.

**Corollary 3.2.** *Let $S = \{s_1, \ldots, s_t\} \subseteq (\mathbb{F}_q^*)^d$ be a t-falling star under $\varphi_d$. Then the vectors $s_1, \ldots, s_{t-1}$ are linearly independent. Consequently, for any subset $T \subseteq (\mathbb{F}_q^*)^d$,*

$$\mathrm{rk}(T) \geq FS(T) - 1.$$

*Moreover, if T is contained within a monochromatic neighbourhood of some other vector, then*

$$\mathrm{rk}(T) \geq FS(T).$$

**Definition 3.4.** Let $A, B \subseteq \mathbb{F}_q^d$ be disjoint sets of vectors. We say that $A$ *confines* $B$ if for each $a \in A$, $a \cdot x = a \cdot y$ for every $x, y \in B$.

**Lemma 3.3.** *Let $A, B \subseteq \mathbb{F}_q^d$ be disjoint sets of vectors such that $A$ confines $B$. Then*

$$\mathrm{af}(B) \leq d - \mathrm{rk}(A).$$

**Proof.** Let $t = \mathrm{rk}(A)$, and let $a_1, \ldots, a_t$ be linearly independent vectors from $A$. Since $A$ confines $B$, then for each $a_i$, there exists an $\alpha_i \in \mathbb{F}_q$ such that $a_i \cdot b = \alpha_i$ for all $b \in B$. Therefore, $B$ is a subset of the solution space for the matrix equation,

$$\begin{pmatrix} -a_1- \\ \vdots \\ -a_t- \end{pmatrix} \begin{pmatrix} | \\ x \\ | \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{pmatrix}.$$

Since $a_1, \ldots, a_t$ are linearly independent, the matrix of these $t$ vectors has full rank, and hence, the solution set is an affine space of dimension $d - t$, as desired. □

**Lemma 3.4.** *Let $A, B \subseteq (\mathbb{F}_q^*)^d$ be disjoint sets of vectors and $\alpha \in C_d$ such that $\varphi_d(a, b) = \alpha$ for all $a \in A$ and $b \in B$. Then either $A$ confines $B$ or $B$ confines $A$ (or both).*

**Proof.** If $\alpha$ has the dot property, then it is trivial that $A$ and $B$ confine one another. So assume that $\alpha \in \mathrm{UP} \cup \mathrm{DOWN}$. It follows that the first position of difference $i$ is the same between any $a \in A$ and any $b \in B$. Moreover, every vector of $A$ has the same $i^{th}$ component, every vector of $B$ has the same $i^{th}$ component, and every vector of $A \cup B$ has the same $j^{th}$ component for each $1 \leq j < i$ if $i > 1$. Since the vectors are ordered lexicographically based on an underlying linear order of $\mathbb{F}_q^*$, it follows that either $a < b$ for all $a \in A$ and $b \in B$, or $b < a$ for all $a \in A$ and $b \in B$.

Without loss of generality, assume that $a < b$ for all $a \in A$ and $b \in B$. If $\alpha \in \mathrm{UP}$, then for any particular $a \in A$, $a \cdot b = a \cdot a$ for every $b \in B$. Therefore, $A$ confines $B$. Similarly, if $\alpha \in \mathrm{DOWN}$, then for any particular $b \in B$, $b \cdot a = b \cdot b$ for every $a \in A$, so $B$ confines $A$. □

**Lemma 3.5.** *Let $t \geq 2$ be an integer. An affine subspace of $\mathbb{F}_q^d$ of dimension $t - 2$ will contain no $t$-falling stars of $(\mathbb{F}_q^*)^d$ under $\varphi_d$. Therefore,*

$$\mathrm{af}(S) \geq FS(S) - 1$$

*for any subset of vectors $S \subseteq (\mathbb{F}_q^*)^d$.*

**Proof.** We will proceed by induction on $t$. The base case $t = 2$ is trivial since an affine subspace of dimension 0 is just one vector while a 2-falling star contains two distinct vectors.

So assume that $t \geq 3$ and that the statement is true for $t - 1$. Let $s_1, \ldots, s_t$ be $t$ distinct vectors that form a $t$-falling star. That is, let $\alpha_1, \ldots, \alpha_{t-1} \in C_d$ and let $\varphi_d\left(s_i, s_j\right) = \alpha_i$ when $1 \leq i < j \leq t$. Assume towards a contradiction that these vectors are contained inside an affine subspace of dimension $t - 2$. Then there exist scalars $\lambda_1, \ldots, \lambda_{t-1} \in \mathbb{F}_q$ such that $s_t = \sum_{j=1}^{t-1} \lambda_j s_j$ and $\sum_{j=1}^{t-1} \lambda_j = 1$.

First, note that if $\lambda_1 = 0$, then the vectors $s_2, \ldots, s_t$ form a $(t-1)$-falling star and are contained in an affine subspace of dimension $t - 3$, a contradiction of the inductive hypothesis. So we must assume in what follows that $\lambda_1 \neq 0$.

Now, we consider two cases: either $\alpha_1 \in \mathrm{DOT}$ or $\alpha_1 \notin \mathrm{DOT}$. If $\alpha_1 \in \mathrm{DOT}$, then

$$\alpha_1 = s_1 \cdot s_t = s_1 \cdot \sum_{j=1}^{t-1} \lambda_j s_j = \lambda_1 (s_1 \cdot s_1) + \alpha_1 \sum_{j=2}^{t-1} \lambda_j = \lambda_1 (s_1 \cdot s_1) + \alpha_1 (1 - \lambda_1).$$

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Therefore, $\lambda_1(s_1 \cdot s_1 - \alpha_1) = 0$. Since $\lambda_1 \neq 0$, it follows that

$$s_1 \cdot s_1 = \alpha_1 = s_1 \cdot s_2,$$

which that implies $\alpha_1 \notin \mathrm{DOT}$, a contradiction.

So assume that $\alpha_1 \notin \mathrm{DOT}$, and let $i$ denote the index of the first component where $s_1$ differs from the other vectors. In this case,

$$s_{1,i} + s_{2,i} = \cdots = s_{1,i} + s_{t,i},$$

and hence $s_{2,i} = \cdots = s_{t,i}$. Therefore,

$$s_{t,i} = \sum_{j=1}^{t-1} \lambda_j s_{j,i} = \lambda_1 s_{1,i} + s_{t,i} \sum_{j=2}^{t-1} \lambda_j = \lambda_1 s_{1,i} + s_{t,i}(1 - \lambda_1).$$

So $\lambda_1(s_{1,i} - s_{t,i}) = 0$. Since $\lambda_1 \neq 0$, we have $s_{1,i} = s_{t,i}$, a contradiction of our choice of $i$. □

**Lemma 3.6.** *Let $S \subseteq (\mathbb{F}_q^*)^d$ be a set of $p \geq 1$ vectors with a leftover structure under the colouring $\varphi_d$. Then*

$$FS(S) \geq \lceil \log_2 p \rceil + 1.$$

**Proof.** We will prove this by induction on $p$. The base case when $p = 1$ is trivial, so assume that $S$ has $p \geq 2$ vectors. Then $S$ has an initial bipartition, $S = A \cup B$, and we note that

$$FS(S) \geq 1 + \max\left(FS(A), FS(B)\right).$$

Since $|A|, |B| < p$, then by induction $FS(T) \geq \lceil \log_2(|T|) \rceil + 1$ for $T = A, B$. Thus, we have

$$FS(S) \geq \lceil \log_2(\max(|A|, |B|)) \rceil + 2,$$

and since $\max(|A|, |B|) \geq \lceil \frac{p}{2} \rceil$, then

$$FS(S) \geq \left\lceil \log_2\left(\left\lceil \frac{p}{2} \right\rceil\right) \right\rceil + 2 = \lceil \log_2 p \rceil + 1.$$

□

**Lemma 3.7.** *Let $p \geq 2$ and $T \geq 0$ be integers. Let $S \subseteq (\mathbb{F}_q^*)^d$ be a subset of $p$ vectors with a leftover structure under $\varphi_d$. If $T \geq 1$, let $a_1, \ldots, a_T \in (\mathbb{F}_q^*)^d$ and $\alpha_1, \ldots, \alpha_T \in C_d$ such that $\varphi_d\left(a_i, a_j\right) = \alpha_i$ for all $1 \leq i < j \leq T$ and $\varphi_d(a_i, s) = \alpha_i$ for all $1 \leq i \leq T$ and all $s \in S$.*

*Then there exists a sequence of positive integers, $x_1, \ldots, x_t$ such that $\sum_{i=1}^{t} x_i = p - 1$ and for each $i = 1, \ldots, t$, the following three conditions hold:*

*(1)* $1 \leq x_i \leq \left\lfloor \frac{p - s_i}{2} \right\rfloor$;

*(2)* $\lceil \log_2(x_i) \rceil + \lceil \log_2(p - s_i - x_i) \rceil \leq d - 1$;

*(3)* $\lceil \log_2(p - s_i - x_i) \rceil \leq d - i - T$,

*where $s_i = 0$ if $i = 1$ and $s_i = \sum_{j=1}^{i-1} x_j$ otherwise.*

**Proof.** We will prove this by induction on $p$. For the base case, let $p = 2$. Let $x_1 = 1$ be the entire sequence. Then the first two conditions hold trivially since the sum of the sequence is 1, and since

$$\lceil \log_2(1) \rceil + \lceil \log_2(1) \rceil = 0 \leq d - 1$$

for any $d \geq 1$. For the third condition, since $\lceil \log_2(1) \rceil = 0$, it suffices to show that $T + 1 \leq d$. This follows from Corollary 3.2, since $S \cup \{a_1, \ldots, a_T\}$ forms a $(T + 2)$-falling star, and hence $d \geq \mathrm{rk}\,(S \cup \{a_1, \ldots, a_T\}) \geq T + 1$.

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So assume that $S$ is a set of $p$ vertices for $p \geq 3$ and that the statement is true for smaller sets. Let the initial bipartition of $S$ be $S = A \cup B$. By Lemma 3.4, we may assume without loss of generality that $A$ confines $B$. Therefore, $\mathrm{af}(B) \leq d - \mathrm{rk}(A)$ by Lemma 3.3. By Corollary 3.2, we know that $\mathrm{rk}(A) \geq FS(A)$ since $A$ is in a monochromatic neighbourhood of any vector from $B$. And by Lemma 3.5, we know that $\mathrm{af}(B) \geq FS(B) - 1$. Thus, $FS(A) + FS(B) - 1 \leq d$. So by Lemma 3.6, we can conclude that

$$\lceil \log_2 (|A|) \rceil + \lceil \log_2 (|B|) \rceil \leq d - 1.$$

Therefore, setting $x_1 = \min\{|A|, |B|\}$ guarantees that $1 \leq x_1 \leq \left\lfloor \frac{p}{2} \right\rfloor$ and that

$$\lceil \log_2 (x_1) \rceil + \lceil \log_2 (p - x_1) \rceil \leq d - 1.$$

This gives us a positive integer $x_1$ which satisfies the first two conditions. Moreover, by Corollary 3.2 and Lemma 3.6,

$$\begin{aligned} d \geq \mathrm{rk}\,(S \cup \{a_1, \ldots, a_T\}) &\geq FS(S \cup \{a_1, \ldots, a_T\}) - 1 \\ &\geq (T + 1 + \max(FS(A), FS(B))\,) - 1 \\ &\geq T + \lceil \log_2 (p - x_1) \rceil + 1. \end{aligned}$$

Thus, $x_1$ also satisfies the third condition.

Let $S'$ denote the larger of the two parts $A$ and $B$, and let $a_{T+1}$ denote an arbitrary vector from $S \setminus S'$. Then $S'$ contains $p - x_1 < p$ vectors and has a leftover structure under $\varphi_d$. Moreover, $S'$ and $a_1, \ldots, a_T, a_{T+1}$ satisfy the monochromatic neighbourhood conditions of the hypothesis. Hence, by induction there exists a sequence of positive integers $x_1{}', \ldots, x'_{t'}$ such that $\sum_{i=1}^{t'} x'_i = p - x_1 - 1$ and for each $i = 1, \ldots, t'$, the following three conditions hold:

(1) $1 \leq x'_i \leq \left\lfloor \frac{p - x_1 - s'_i}{2} \right\rfloor$;

(2) $\lceil \log_2 (x'_i) \rceil + \lceil \log_2 (p - x_1 - s'_i - x'_i) \rceil \leq d - 1$;

(3) $\lceil \log_2 (p - x_1 - s'_i - x'_i) \rceil \leq d - i - (T + 1)$,

where $s'_i = 0$ if $i = 1$ and $s'_i = \sum_{j=1}^{i-1} x'_j$ otherwise.

Let $x_i = x'{}_{i-1}$ for $2 \leq i \leq t' + 1$ and let $t = t' + 1$ to get a sequence $x_1, \ldots, x_t$ for which

$$\sum_{i=1}^{t} x_i = x_1 + \sum_{i=1}^{t'} x_i{}' = x_1 + p - x_1 - 1 = p - 1.$$

For each $i = 2, \ldots, t$, the first two conditions are satisfied since $x_1 + s'_i = s_{i+1}$, and the third condition is satisfied since $d - i - (T + 1) = d - (i + 1) - T$. □

**Corollary 3.8.** *Let $S \subseteq (\mathbb{F}_q^*)^3$ be a set of 6 vectors. Then $S$ cannot have a leftover structure under the colouring $\varphi_3$.*

**Proof.** If such a set exists, then by Lemma 3.7 with $T = 0$, a positive integer $x_1$ exists such that $1 \leq x_1 \leq 3$ and

$$\lceil \log_2 (x_1) \rceil + \lceil \log_2 (6 - x_1) \rceil \leq 2.$$

It is simple to check that no such integer exists. □

**Corollary 3.9.** *Let $S \subseteq (\mathbb{F}_q^*)^4$ be a set of 8 vectors. Then $S$ cannot have a leftover structure under the colouring $\varphi_4$.*

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**Proof.** If such a set exists, then by Lemma 3.7 with $T=0$, we must be able to find a sequence of positive integers $x_1, x_2, \ldots, x_t$ that satisfy the conditions given in the Lemma. In particular, $1 \leq x_1 \leq 4$ and

$$\lceil \log_2(x_1) \rceil + \lceil \log_2(8-x_1) \rceil \leq 3.$$

We can check and find that $x_1 = 1$ is the only possibility. Therefore, $1 \leq x_2 \leq 3$ such that

$$\lceil \log_2(7-x_2) \rceil \leq 2$$
$$\lceil \log_2(x_2) \rceil + \lceil \log_2(7-x_2) \rceil \leq 3.$$

A quick check reveals that no such integer exists. □

Theorem 1.2 follows from Theorem 1.1 and Corollaries 3.8 and 3.9.

## 4. Conclusion

The proof of Lemma 3.7 actually shows which leftover $p$-cliques can appear under $\varphi_d$ for a particular $d$. For example, this proof implies that the only leftover 5-clique that can appear under $\varphi_3$ is a monochromatic $C_4$ contained inside a monochromatic neighbourhood of one vertex (that is, an initial $(1, 4)$-bipartition with a $(2, 2)$-bipartition inside the part with four vertices). In [2], we handled this specific leftover structure by splitting each colour class of $\varphi_3$ into four new colours determined by certain relations between vectors. While the current paper can be viewed as our attempt to fully generalize the colouring techniques used in [2] and [8], it does not generalize the splitting that was crucial for handling the final leftover 5-clique. Perhaps such a generalized splitting would be enough to give $f(n, p, p) \leq n^{1/(p-2)+o(1)}$ for $p \geq 6$ or at least improve the best-known upper bounds for values of $p$ other than the two addressed in this paper.

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