A NOTE AS TO A PERTURBATION OF HILL'S CURVES

V.R. Matas
Astronomical Institute, Czechoslovak Academy of Sciences,
Praha, Czechoslovakia

Let us consider motion of an infinitesimal body in the gravitational field of two homogeneous spheroids with coincident equatorial planes. Let mass of the infinitesimal body M - a material point - be negligible if compared with masses of the spheroids S_1, S_2 respectively. We will study the motion of M in the common equatorial plane provided that the spheroids move along circular orbits (in an inertial frame of reference) - see Kondurar', 1952. System of the equations of motion of M posseses Jacobi's integral and Hill's generalized curves of zero relative velocity of M are consequently described by the equation

$$H(r_1) = (1 + \frac{3}{10}(a_1^2e_1^2 + a_2^2e_2^2))\frac{1-m}{2}r_1^2 + \frac{1-m}{r_1} + \frac{1-m}{10}a_1^2e_1^2r_1^{-3} = \qquad (1)$$

$$C - (1 + \frac{3}{10}(a_1^2e_1^2 + a_2^2e_2^2))\frac{m}{2}r_2^2 - \frac{m}{r_2} - \frac{m}{10}a_2^2e_2^2r_2^{-3} = C - B(r_2) = A$$

where C is a "Jacobian" constant of integration, r_1, r_2 are the dimensionless distances between M and the centers of mass of the spheroids S_1, S_2; $1 - m$, m are the dimensionless masses of the spheroids; a_i, e_i are their respective semimajor axes and eccentricities ($i = 1,2$).

Let r_2 be given. Then solving (1) is equivalent to seeking positive roots of the function $G(x) = H(x) - A$, where A is a constant parameter. If $A \leqq 0$ then there does not exist any solution of the equation $G(x) = 0$. In the case $A > 0$ let us study the functions $G(x)$, $H(x)$ in the interval $(0,\infty)$. It follows that the functions G, H are convex in $(0,\infty)$ and have there an absolute minimum at a point $x_o \leqq 1$ (1 is the dimensionless distance of the centers of mass of the spheroids) which is the only root of the derivative G' (or H') - an increasing and concave function in $(0,\infty)$. If the minimum $G(x_o)$ is positive, equation $G(x) = 0$ evidently has no real solution. Provided that $G(x_o) = 0$ we put $r_1 = x_o$ and consequently pair r_1, r_2 (r_2 was given) satisfies Eq. (1) of the zero relative

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velocity curve. Eventually in the case $G(x_o) < 0$ there are two different positive roots r_{11}, r_{12}, $(r_{11} < x_o < r_{12})$ of the equation $G(x) = 0$. If moreover

$$r_{1k} + r_2 \geqq 1, \; r_{1k} + 1 \geqq r_2, \; r_2 + 1 \geqq r_{1k}, \tag{2}$$

where k equals either 1 or 2 (r_2 was given), then the pair of distances r_{1k} and r_2 (from S_1 and S_2 respectively) defines two points (or one point) on the zero relative velocity curve (1).

An analogical examination of the function $B(x)$ in the interval $(0,\infty)$ immediately gives that B is concave in $(0,\infty)$ and has there an absolute maximum at a point $x_1 \leqq 1$ which is the only root of the derivative B' (that is decreasing and convex in $(0,\infty)$). The above considerations as to the functions H, B imply that the curves of the zero relative velocity of M are real if, and only if, $H(x_o) \leqq C + B(x_1)$. In this case the Jacobian constants C have a lower bound, viz. $C \geqq H(x_o) - B(x_1) = C_o$. Consequently if $C = C_o$ equation (1) posseses only one solution $r_1 = x_o$, $r_2 = x_1$ which defines, if moreover $x_o + x_1 > 1$, two equilibrium solutions L_T, L_T' of the equations of motion of M.

Let $C \geqq C_o$ and $x_o + x_1 > 1$ be valid. It follows from the preceding study of the function B that the equation $H(x_o) = C + B(r_2)$ has two solutions $r_{2min} \geqq r_{2max}$ in the most. The pair x_o, r_{2min} defines actual points on the zero relative velocity curve in case $r_{2min} \geqq 1 - x_o$. The pair x_o, r_{2max} represents actual points on the curve if $r_{2max} \leqq 1 + x_o$. Analogically the equation $H(r_1) = C + B(x_1)$ posseses two solutions $r_{1min} \leqq r_{1max}$ in the most. The pair r_{1min}, x_1 characterizes actual points on the curve examined when $r_{1min} \geqq 1 - x_1$. The pair r_{1max}, x_1 determines actual points on the curve if $r_{1max} \leqq 1 + x_1$. r_{1min}, r_{2min} are decreasing functions of C and r_{1max}, r_{2max} are increasing functions of C, all of them in $\langle C_o,\infty)$. Let C_1, C_2, D_1, D_2 be values of C for which

$$\begin{aligned} r_{1min}(C_1) &= 1 - x_1, \; r_{1max}(D_1) = 1 + x_1, \\ r_{2min}(C_2) &= 1 - x_o, \; r_{2max}(D_2) = 1 + x_o. \end{aligned} \tag{3}$$

C_1, C_2, D_1, D_2 are to be determined uniquely (if $x_1 < 1$, $x_o < 1$). We easily find

$$\begin{aligned} C_1 &= H(1 - x_1) - B(x_1), \; C_2 = H(x_o) - B(1 - x_o), \\ D_1 &= H(1 + x_1) - B(x_1), \; D_2 = H(x_o) - B(1 + x_o). \end{aligned} \tag{4}$$

Let us first consider the zero relative velocity curves determined by $C \in \langle C_o, C_1\rangle$ (or $C \in \langle C_o, C_2\rangle$). Then the properties found as to the functions H, B justify us to state that the locus of all the points least distant from the spheroid S_1 (or S_2) on the

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concerned zero relative velocity curves is the minor arc $L_T L_T'$ (intersecting the line segment S_1S_2 at a point P_1 (or P_2)) of a circle center of which is the center of mass of the other spheroid. As the Jacobian constant C approaches C_o, the points of the determined locus approach L_T, L_T' ($C = C_o$ implies that L_T, L_T' are on the locus). As C approaches C_1 (or C_2), the points of the locus approach the intersection of the locus and the line segment S_1S_2. If $C \geqq C_1$ (or $C \geqq C_2$) then, with respect to (3), the points – on the corresponding generalized Hill's curves – which are nearest to the spheroid S_1 (or S_2) fill the line segment S_1P_1, excluding S_1 (or S_2P_2, excluding S_2); the finite dimensions of the spheroids are not considered now. As C becomes equal to C_1 (or C_2) the concerned points approach the intersection P_1 (or P_2) – these points approach the center of mass of S_1 (or S_2) if the constant C becomes infinite.

If, for instance, S_2 approaches a sphere with a spherical density distribution then $x_o \to 1$ and C_2 becomes infinite. If $x_o = 1$ there does not exist any real C_2 so that $r_{2min}(C_2) = 1 - x_o = 0$. Instead of this we have $r_{2min}(C) \to 0$ for $C \to \infty$. In this case the concerned minor arc $L_T L_T'$ of the circle centered at S_1 goes through S_2 (provided that S_2 is considered to be a material point) but S_2 is not a point of the locus. The points of this minor arc merely approach S_2 if C becomes infinite. The same considerations may be made for the body S_1. It is to be seen that the locuses found represent a generalization of a similar geometric property of the Hill's curves in the restricted three-body problem (see Matas, 1978).

Notice now the zero relative velocity curves defined by $C \in \langle C_o, D_1 \rangle$ (or $C \in \langle C_o, D_2 \rangle$). The derived characteristics of the functions H, B imply that the locus of all the points most distant from the spheroid S_1 (or S_2) on the considered zero relative velocity curves is the major arc $L_T L_T'$ of the circle centered at the remaining spheroid. As the Jacobian constant C approaches C_o, the points of this locus approach L_T, L_T' ($C = C_o$ corresponds to the equilibrium points L_T, L_T'). As C approaches D_1 (or D_2), the points of the locus approach an intersection R_1 (or R_2) of the locus and the ray S_1S_2 (or S_2S_1) extended from S_1 (or S_2). In the case when $C \geqq D_1$ (or $C \geqq D_2$) obviously (see (3)) the locus of all the most distant points – on the given generalized Hill's curves – from the spheroid S_1 (or S_2) is a ray which: (i) is situated on the straight line S_1S_2, (ii) has endpoint R_1 (or R_2) and (iii) does not go through the spheroids. As the Jacobian constant C approaches D_1 (or D_2) points of the ray approach R_1 (or R_2); the points of the ray have unlimited extent in the opposite direction if the constant C becomes infinite.

An analytical approach gives (see the precision adopted in (1))

$$x_o = 1 - \frac{a_2^2 e_2^2}{10}, \quad x_1 = 1 - \frac{a_1^2 e_1^2}{10}, \tag{5}$$

$$
\begin{aligned}
&C_0 = \frac{3}{2} + \frac{1}{4} a_1^2 e_1^2 + \frac{3}{20} a_2^2 e_2^2 + \frac{m}{10}(a_2^2 e_2^2 - a_1^2 e_1^2) \geqq \frac{3}{2} , \\
&C_0 \rightarrow \frac{3}{2} \text{ for } e_1, e_2 \rightarrow 0 ; \\
&C_1 \approx \frac{100(1 - m)}{a_1^2 e_1^2} \rightarrow \infty \text{ for } e_1 \rightarrow 0 ; \\
&C_2 \approx \frac{100m}{a_2^2 e_2^2} \rightarrow \infty \text{ for } e_2 \rightarrow 0 ; \\
&D_1 = \frac{5}{2} - m + \frac{7}{16} a_1^2 e_1^2 + \frac{3}{5} a_2^2 e_2^2 - \frac{m}{20}(\frac{23}{4} a_1^2 e_1^2 + 7a_2^2 e_2^2) , \\
&D_1 \rightarrow \frac{5}{2} - m \text{ for } e_1, e_2 \rightarrow 0 ; \\
&D_2 = \frac{3}{2} + m + \frac{1}{4} a_1^2 e_1^2 + \frac{3}{20} a_2^2 e_2^2 + \frac{m}{20}(7a_1^2 e_1^2 + \frac{23}{4} a_2^2 e_2^2) , \\
&D_2 \rightarrow \frac{3}{2} + m \text{ for } e_1, e_2 \rightarrow 0 .
\end{aligned} \tag{6}
$$

REFERENCES

Kondurar', V.T.: 1952, "Trudy Gos. Astron. Inst. Shternberga" 21, pp.135-158.

Matas, V.: 1978, Celes. Mech. (in press).

DISCUSSION

Garfinkel: Are the two spheroids of zero obliquity with respect to the orbital plane?

Hori: Yes.

Garfinkel: Can the restriction that the spheroids are of constant density be removed by introducing the moments of interia in the place of a_1, a_2, e_1, e_2?

Hori: Very likely.

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