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REVERSE INEQUALITIES FOR THE NUMERICAL RADIUS OF LINEAR OPERATORS IN HILBERT SPACES

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Some elementary inequalities providing upper bounds for the difference of the norm and the numerical radius of a bounded linear operator on Hilbert spaces under appropriate conditions are given.

1. INTRODUCTION

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [**1**, p. 1]:

$$W(T) = \left\{\langle Tx, x\rangle,\ x \in H,\ \|x\| = 1\right\}.$$

The following properties of $W(T)$ are immediate:

(i) $W(\alpha I + \beta T) = \alpha + \beta W(T)$ for $\alpha, \beta \in \mathbb{C}$;

(ii) $W(T^*) = \left\{\overline{\lambda}, \lambda \in W(T)\right\}$, where T^* is the *adjoint operator* of T;

(iii) $W(U^*TU) = W(T)$ for any *unitary* operator U.

The following classical fact about the geometry of the numerical range [**1**, p. 4] may be stated:

THEOREM 1. (Toeplitz-Hausdorff.) *The numerical range of an operator is convex.*

An important use of $W(T)$ is to bound the *spectrum* $\sigma(T)$ of the operator T [**1**, p. 6]:

THEOREM 2. (Spectral inclusion.) *The spectrum of an operator is contained in the closure of its numerical range.*

The self-adjoint operators have their spectra bounded sharply by the numerical range [**1**, p. 7]:

THEOREM 3. *The following statements hold true:*

(i) *T is self-adjoint if and only if $W(T)$ is real;*

(ii) *If T is self-adjoint and $W(T) = [m, M]$ (the closed interval of real numbers m, M), then $\|T\| = \max\{|m|, |M|\}$.*

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(iii) *If* $W(T) = [m, M]$, *then* $m, M \in \sigma(T)$.

The *numerical radius* $w(T)$ of an operator T on H is given by [**1**, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\left\{|\langle Tx, x\rangle|, \|x\| = 1\right\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leqslant w(T)\|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, that is,

(i) $w(T) \geqslant 0$ for any $T \in B(H)$ and $w(T) = 0$ if and only if $T = 0$;

(ii) $w(\lambda T) = |\lambda| w(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;

(iii) $w(T + V) \leqslant w(T) + w(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds [**1**, p. 9]:

Theorem 4. (Equivalent norm.) *For any* $T \in B(H)$ *one has*

$$w(T) \leqslant \|T\| \leqslant 2w(T). \tag{1.3}$$

Let us now look at two extreme cases of the inequality (1.3). In the following $r(t) := \sup\{|\lambda|, \lambda \in \sigma(T)\}$ will denote the *spectral radius* of T and $\sigma_p(T) = \{\lambda \in \sigma(T),\ Tf = \lambda f \text{ for some } f \in H\}$ the *point spectrum* of T.

The following results hold [**1**, p. 10]:

Theorem 5. *We have*

(i) *If* $w(T) = \|T\|$, *then* $r(T) = \|T\|$.

(ii) *If* $\lambda \in W(T)$ *and* $|\lambda| = \|T\|$, *then* $\lambda \in \sigma_p(T)$.

To address the other extreme case $w(T) = \|T\|/2$, we can state the following sufficient condition in terms of (see [**1**, p. 11])

$$R(T) := \{Tf,\ f \in H\} \quad \text{and} \quad R(T^*) := \{T^*f,\ f \in H\}.$$

Theorem 6. *If* $R(T) \perp R(T^*)$, *then* $w(T) = \|T\|/2$.

It is well-known that the two-dimensional shift

$$S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

has the property that $w(T) = \|T\|/2$.

The following theorem shows that some operators T with $w(T) = \|T\|/2$ have S_2 as a component [**1**, p. 11]:

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Theorem 7. *If $w(T) = \|T\|/2$ and T attains its norm, then T has a two-dimensional reducing subspace on which it is the shift S_2.*

For other results on numerical radius, see [**2**, Chapter 11].

The main aim of the present paper is to point out some upper bounds for the nonnegative difference

$$\|T\| - w(T) \qquad \left(\|T\|^2 - (w(T))^2\right)$$

under appropriate assumptions for the bounded linear operator $T : H \to H$.

2. The Results

The following results may be stated:

Theorem 8. *Let $T : H \to H$ be a bounded linear operator on the complex Hilbert space H. If $\lambda \in \mathbb{C}\backslash\{0\}$ and $r > 0$ are such that*

$$\|T - \lambda I\| \leqslant r, \tag{2.1}$$

where $I : H \to H$ is the identity operator on H, then

$$(0 \leqslant)\|T\| - w(T) \leqslant \frac{1}{2} \cdot \frac{r^2}{|\lambda|}. \tag{2.2}$$

Proof: For $x \in H$ with $\|x\| = 1$, we have from (2.1) that

$$\|Tx - \lambda x\| \leqslant \|T - \lambda I\| \leqslant r,$$

giving

$$\|Tx\|^2 + |\lambda|^2 \leqslant 2\,\mathrm{Re}\left[\overline{\lambda}\langle Tx, x\rangle\right] + r^2 \leqslant 2|\lambda|\left|\langle Tx, x\rangle\right| + r^2. \tag{2.3}$$

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.3) we get the following inequality that is of interest in itself:

$$\|T\|^2 + |\lambda|^2 \leqslant 2w(T)|\lambda| + r^2. \tag{2.4}$$

Since, obviously,

$$\|T\|^2 + |\lambda|^2 \geqslant 2\|T\||\lambda|, \tag{2.5}$$

hence by (2.4) and (2.5) we deduce the desired inequality (2.2). $\square$

Remark 1. If the operator $T : H \to H$ is such that $R(T) \perp R(T^*)$, $\|T\| = 1$ and $\|T - I\| \leqslant 1$, then the equality holds in (2.2). Indeed, by Theorem 6, we have in this case $w(T) = \|T\|/2 = 1/2$ and since we can choose $\lambda = 1$, $r = 1$ in Theorem 8, then we get in both sides of (2.2) the same quantity $1/2$.

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PROBLEM 1. Find bounded linear operators $T : H \to H$ with $\|T\| = 1$, $R(T) \perp R(T^*)$ and $\|T - \lambda I\| \leqslant |\lambda|^{1/2}$.

The following corollary may be stated:

COROLLARY 1. *Let $A : H \to H$ be a bounded linear operator and $\varphi, \psi \in \mathbb{C}$ with $\psi \notin \{-\varphi, \varphi\}$. If*

$$\operatorname{Re}\langle \psi x - Ax, Ax - \varphi x\rangle \geqslant 0 \quad \text{for any} \quad x \in H,\ \|x\| = 1 \tag{2.6}$$

then

$$(0 \leqslant)\|A\| - w(A) \leqslant \frac{1}{4} \cdot \frac{|\psi - \varphi|^2}{|\psi + \varphi|}. \tag{2.7}$$

PROOF: Utilising the fact that in any Hilbert space the following two statements are equivalent:

(i) $\operatorname{Re}\langle u - x, x - z\rangle \geqslant 0$, $x, z, u \in H$;
(ii) $\|x - (z+u)/2\| \leqslant \|u - z\|/2$,

we deduce that (2.6) is equivalent to

$$\left\|Ax - \frac{\psi + \varphi}{2} \cdot Ix\right\| \leqslant \frac{1}{2}|\psi - \varphi| \tag{2.8}$$

for any $x \in H$, $\|x\| = 1$, which in its turn is equivalent with the operator norm inequality:

$$\left\|A - \frac{\psi + \varphi}{2} \cdot I\right\| \leqslant \frac{1}{2}|\psi - \varphi|. \tag{2.9}$$

Now, applying Theorem 8 for $T = A$, $\lambda = (\varphi + \psi)/2$ and $r = |\varphi - \psi|/2$, we deduce the desired result (2.7). ∎

REMARK 2. Following [1, p. 25], we say that an operator $B : H \to H$ is accretive, if $\operatorname{Re}\langle Bx, x\rangle \geqslant 0$ for any $x \in H$. One may observe that the assumption (2.6) above is then equivalent with the fact that the operator $(A^* - \overline{\varphi} I)(\psi I - A)$ is accretive.

Perhaps a more convenient sufficient condition in terms of positive operators is the following one:

COROLLARY 2. *Let $\varphi, \psi \in \mathbb{C}$ with $\psi \notin \{-\varphi, \varphi\}$ and $A : H \to H$ a bounded linear operator in H. If $(A^* - \overline{\varphi} I)(\psi I - A)$ is self-adjoint and*

$$(A^* - \overline{\varphi} I)(\psi I - A) \geqslant 0 \tag{2.10}$$

in the operator partial order, then

$$(0 \leqslant)\|A\| - w(A) \leqslant \frac{1}{4} \cdot \frac{|\psi - \varphi|^2}{|\psi + \varphi|}. \tag{2.11}$$

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Corollary 3. *Assume that T, λ, r are as in Theorem 8. If, in addition,*

$$\big||\lambda| - w(T)\big| \geqslant \rho, \tag{2.12}$$

for some $\rho \geqslant 0$, then

$$(0 \leqslant)\|T\|^2 - w^2(T) \leqslant r^2 - \rho^2. \tag{2.13}$$

Proof: From (2.4) of Theorem 8, we have

$$\begin{aligned} \|T\|^2 - w^2(T) &\leqslant r^2 - w^2(T) + 2w(T)|\lambda| - |\lambda|^2 \\ &= r^2 - \big(|\lambda| - w(T)\big)^2. \end{aligned} \tag{2.14}$$

The desired inequality follows from (2.12). □

Remark 3. In particular, if $\|T - \lambda I\| \leqslant r$ and $|\lambda| = w(T)$, $\lambda \in \mathbb{C}$, then

$$(0 \leqslant)\|T\|^2 - w^2(T) \leqslant r^2. \tag{2.15}$$

The following result may be stated as well.

Theorem 9. *Let $T : H \to H$ be a nonzero bounded linear operator on H and $\lambda \in \mathbb{C} \setminus \{0\}$, $r > 0$ with $|\lambda| > r$. If*

$$\|T - \lambda I\| \leqslant r, \tag{2.16}$$

then

$$\sqrt{1 - \frac{r^2}{|\lambda|^2}} \leqslant \frac{w(T)}{\|T\|} \quad (\leqslant 1). \tag{2.17}$$

Proof: From (2.4) of Theorem 8, we have

$$\|T\|^2 + |\lambda|^2 - r^2 \leqslant 2|\lambda| w(T),$$

which implies, on dividing with $\sqrt{|\lambda|^2 - r^2} > 0$ that

$$\frac{\|T\|^2}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \leqslant \frac{2|\lambda| w(T)}{\sqrt{|\lambda|^2 - r^2}}. \tag{2.18}$$

By the elementary inequality

$$2\|T\| \leqslant \frac{\|T\|^2}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \tag{2.19}$$

and by (2.18) we deduce

$$\|T\| \leqslant \frac{w(T)|\lambda|}{\sqrt{|\lambda|^2 - r^2}},$$

which is equivalent to (2.17). □

REMARK 4. Squaring (2.17), we get the inequality

$$(0 \leqslant)\|T\|^2 - w^2(T) \leqslant \frac{r^2}{|\lambda|^2}\|T\|^2. \tag{2.20}$$

REMARK 5. For any bounded linear operator $T : H \to H$ we have the relation $w(T) \geqslant \|T\|/2$. Inequality (2.17) would produce a refinement of this classic fact only in the case when

$$\frac{1}{2} \leqslant \left(1 - \frac{r^2}{|\lambda|^2}\right)^{1/2},$$

which is equivalent to $r/|\lambda| \leqslant \sqrt{3}/2$.

The following corollary holds.

COROLLARY 4. *Let $\varphi, \psi \in \mathbb{C}$ with $\mathrm{Re}(\psi\overline{\varphi}) > 0$. If $T : H \to H$ is a bounded linear operator such that either (2.6) or (2.10) holds true, then:*

$$\frac{2\sqrt{\mathrm{Re}(\psi\overline{\varphi})}}{|\psi + \varphi|} \leqslant \frac{w(T)}{\|T\|}(\leqslant 1) \tag{2.21}$$

and

$$(0 \leqslant)\|T\|^2 - w^2(T) \leqslant \left|\frac{\psi - \varphi}{\psi + \varphi}\right|^2 \|T\|^2. \tag{2.22}$$

PROOF: If we consider $\lambda = (\psi + \varphi)/2$ and $r = |\psi - \varphi|/2$, then

$$|\lambda|^2 - r^2 = \left|(\psi + \varphi)/2\right|^2 - \left|(\psi - \varphi)/2\right|^2 = \mathrm{Re}(\psi\overline{\varphi}) > 0.$$

Now, on applying Theorem 9, we deduce the desired result. ∎

REMARK 6. If $|\psi - \varphi| \leqslant (\sqrt{3}/2)|\psi + \varphi|$, $\mathrm{Re}(\psi\overline{\varphi}) > 0$, then (2.21) is a refinement of the inequality $w(T) \geqslant \|T\|/2$.

The following result may be of interest as well.

THEOREM 10. *Let $T : H \to H$ be a nonzero bounded linear operator on H and $\lambda \in \mathbb{C}\backslash\{0\}$, $r > 0$ with $|\lambda| > r$. If*

$$\|T - \lambda I\| \leqslant r, \tag{2.23}$$

then

$$(0 \leqslant)\|T\|^2 - w^2(T) \leqslant \frac{2r^2}{|\lambda| + \sqrt{|\lambda|^2 - r^2}} w(T). \tag{2.24}$$

PROOF: From the proof of Theorem 8, we have

$$\|Tx\|^2 + |\lambda|^2 \leqslant 2\,\mathrm{Re}\left[\overline{\lambda}\langle Tx, x\rangle\right] + r^2 \tag{2.25}$$

for any $x \in H$, $\|x\| = 1$.

If we divide (2.25) by $|\lambda|\left|\langle Tx, x\rangle\right|$, (which, by (2.25), is positive) then we obtain

$$\frac{\|Tx\|^2}{|\lambda|\left|\langle Tx,x\rangle\right|} \leqslant \frac{2\operatorname{Re}[\overline{\lambda}\langle Tx,x\rangle]}{|\lambda|\left|\langle Tx,x\rangle\right|} + \frac{r^2}{|\lambda|\left|\langle Tx,x\rangle\right|} - \frac{|\lambda|}{\left|\langle Tx,x\rangle\right|} \tag{2.26}$$

for any $x \in H$, $\|x\| = 1$.

If we subtract in (2.26) the same quantity $\left|\langle Tx, x\rangle\right|/|\lambda|$ from both sides, then we get

$$\begin{aligned}
&\frac{\|Tx\|^2}{|\lambda|\left|\langle Tx,x\rangle\right|} - \frac{\left|\langle Tx,x\rangle\right|}{|\lambda|} \\
&\quad\leqslant \frac{2\operatorname{Re}[\overline{\lambda}\langle Tx,xt\rangle]}{|\lambda|\left|\langle Tx,x\rangle\right|} + \frac{r^2}{|\lambda|\left|\langle Tx,x\rangle\right|} - \frac{\left|\langle Tx,x\rangle\right|}{|\lambda|} - \frac{|\lambda|}{\left|\langle Tx,x\rangle\right|} \\
&\quad= \frac{2\operatorname{Re}[\overline{\lambda}\langle Tx,x\rangle]}{|\lambda|\left|\langle Tx,x\rangle\right|} - \frac{|\lambda|^2 - r^2}{|\lambda|\left|\langle Tx,x\rangle\right|} - \frac{\left|\langle Tx,x\rangle\right|}{|\lambda|} \\
&\quad= \frac{2\operatorname{Re}[\overline{\lambda}\langle Tx,x\rangle]}{|\lambda|\left|\langle Tx,x\rangle\right|} - \left(\frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{|\lambda|\left|\langle Tx,x\rangle\right|}} - \frac{\sqrt{\left|\langle Tx,x\rangle\right|}}{\sqrt{|\lambda|}}\right)^2 - 2\frac{\sqrt{|\lambda|^2 - r^2}}{|\lambda|}.
\end{aligned} \tag{2.27}$$

Since

$$\operatorname{Re}\left[\overline{\lambda}\langle Tx,x\rangle\right] \leqslant |\lambda|\left|\langle Tx,x\rangle\right|$$

and

$$\left(\frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{|\lambda|\,\left|\langle Tx,x\rangle\right|}} - \frac{\sqrt{\left|\langle Tx,x\rangle\right|}}{\sqrt{|\lambda|}}\right)^2 \geqslant 0,$$

by (2.27) we get

$$\frac{\|Tx\|^2}{|\lambda|\left|\langle Tx,x\rangle\right|} - \frac{\left|\langle Tx,x\rangle\right|}{|\lambda|} \leqslant \frac{2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)}{|\lambda|}$$

which gives the inequality

$$\|Tx\|^2 \leqslant \left|\langle Tx,x\rangle\right|^2 + 2\left|\langle Tx,x\rangle\right|\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right) \tag{2.28}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get

$$\begin{aligned}
\|T\|^2 &\leqslant \sup\left\{\left|\langle Tx,x\rangle\right|^2 + 2\left|\langle Tx,x\rangle\right|\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)\right\} \\
&\leqslant \sup\left\{\left|\langle Tx,x\rangle\right|^2\right\} + 2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)\sup\left\{\left|\langle Tx,x\rangle\right|\right\} \\
&= w^2(T) + 2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)w(T),
\end{aligned}$$

which is clearly equivalent to (2.24). $\square$

Corollary 5. *Let $\varphi, \psi \in \mathbb{C}$ with $\operatorname{Re}(\psi\overline{\varphi}) > 0$. If $A : H \to H$ is a bounded linear operator such that either (2.6) or (2.10) hold true, then:*

$$(0 \leqslant)\|A\|^2 - w^2(A) \leqslant \left[|\psi + \varphi| - 2\sqrt{\operatorname{Re}(\psi\overline{\varphi})}\right]w(A). \tag{2.29}$$

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REMARK 7. If $M \geqslant m > 0$ are such that either $(A^* - mI)(MI - A)$ is accretive, or, sufficiently, $(A^* - mI)(MI - A)$ is self-adjoint and

$$(A^* - mI)(MI - A) \geqslant 0 \quad \text{in the operator partial order,} \tag{2.30}$$

then, by (2.21) we have:

$$(1 \leqslant) \frac{\|A\|}{w(A)} \leqslant \frac{M+m}{2\sqrt{mM}}, \tag{2.31}$$

which is equivalent to

$$(0 \leqslant) \|A\| - w(A) \leqslant \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} w(A), \tag{2.32}$$

while from (2.24) we have

$$(0 \leqslant) \|A\|^2 - w^2(A) \leqslant (\sqrt{M} - \sqrt{m})^2 w(A). \tag{2.33}$$

Also, the inequality (2.7) becomes

$$(0 \leqslant) \|A\| - w(A) \leqslant \frac{1}{4} \cdot \frac{(M-m)^2}{M+m}. \tag{2.34}$$

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