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THE HELGASON FOURIER TRANSFORM ON A CLASS OF NONSYMMETRIC HARMONIC SPACES

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Given a group N of Heisenberg type, we consider a one-dimensional solvable extension NA of N, equipped with the natural left-invariant Riemannian metric, which makes NA a harmonic (not necessarily symmetric) manifold. We define a Fourier transform for compactly supported smooth functions on NA, which, when NA is a symmetric space of rank one, reduces to the Helgason Fourier transform. The corresponding inversion formula and Plancherel Theorem are obtained. For radial functions, the Fourier transform reduces to the spherical transform considered by E. Damek and F. Ricci.

1. INTRODUCTION

Heisenberg type (or H–type) groups were introduced by Kaplan in [**9**]. Given a group N of Heisenberg type, one can construct a one-dimensional solvable extension NA of N, where $A = \mathbf{R}^+$ acts on N by anisotropic dilations. When NA is equipped with the appropriate left-invariant Riemannian structure, NA becomes a harmonic manifold [**4**]. This class of harmonic spaces includes all rank-one symmetric spaces of the noncompact type G/K as particular cases [**2**]. In this case N is the H–type group that appears in the Iwasawa decomposition $G = NAK$ of a connected noncompact semisimple Lie group G with finite centre and real rank one. On the other hand, there are many H–type groups N that do not appear in Iwasawa decompositions, see [**9**]; the corresponding NA manifolds are harmonic but nonsymmetric [**5**].

The analysis of radial functions on harmonic NA spaces (that is, functions that depend only on the geodesic distance from the identity), has been discussed in [**1, 4, 11**]. An important role is played by the spherical functions Φ, that is, the radial eigenfunctions of the Laplace–Beltrami operator $\mathcal{L}$, normalised by $\Phi(e) = 1$. The spherical transform for radial functions on NA and the corresponding inversion and Plancherel formulas were studied by F. Ricci in [**11**].

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In this paper we consider the analysis of arbitrary (smooth, compactly supported) functions on NA, not necessarily radial. If f is such a function, we define its Fourier transform to be the function $\widehat{f}$ on $\mathbf{C} \times N$ given by the rule

$$\widehat{f}(\lambda, n) = \int_{NA} f(x)\, \mathcal{P}_\lambda(x, n)\, dx, \qquad \lambda \in \mathbf{C}, \;\; n \in N,$$

where the kernel $\mathcal{P}_\lambda : NA \times N \to \mathbf{C}$ is an appropriate complex power of the Poisson kernel $\mathcal{P}(x, n)$ on NA, namely

$$\mathcal{P}_\lambda(x, n) = [\mathcal{P}(x, n)]^{1/2 - i\lambda/Q}.$$

If f is a radial function on NA, the Fourier transform reduces to the spherical transform. If NA is a rank-one symmetric space the Fourier transform coincides with the well-known Helgason Fourier transform [**8**].

In order to obtain the corresponding inversion formula, we generalise to the present case a formula for translated spherical functions, which in the symmetric case is referred to as "the symmetry of the spherical functions". In the symmetric case this formula is proved by changing variables in Harish-Chandra's representation of spherical functions as integrals over K [**8**, p.224].

In the general case this is not so easy, as there is no group K acting transitively on the distance spheres in NA. In order to generalise the symmetry of the spherical functions, we shall use a formula which expresses the spherical functions as matrix coefficients of suitable representations of NA on $L^2(N)$. This formula has been demonstrated in [**1, 7**].

The organisation of this paper is as follows. In section 2 we recall briefly the main definitions and the known results of spherical analysis on harmonic NA groups [**1, 4, 11**]. In section 3 we introduce the Fourier transform and establish its relation with the spherical transform. In section 4 we obtain the inversion formula. Specialising to radial functions, we verify that one re-obtains the spherical inversion formula. We also prove that the Fourier transform of a smooth function with compact support is a holomorphic function of uniform exponential type. Finally we show that, in the symmetric case, our results coincide with the known ones. In section 5 we prove the Plancherel Theorem.

2. Preliminaries on NA groups

Let $\mathbf{n}$ be a two-step real nilpotent Lie algebra endowed with an inner product $\langle\, ,\rangle$. Write $\mathbf{n}$ as an orthogonal sum $\mathbf{n} = \mathbf{v} \oplus \mathbf{z}$, where $\mathbf{z} = [\mathbf{n}, \mathbf{n}]$ is the centre of $\mathbf{n}$.

For each Z in $\mathbf{z}$, define the map $J_Z : \mathbf{v} \longrightarrow \mathbf{v}$ by

$$\langle J_Z X, Y\rangle = \langle [X, Y], Z\rangle, \qquad \forall X, Y \in \mathbf{v}.$$

DEFINITION: [9] The Lie algebra $\mathbf{n}$ is called an H–type algebra if, for every Z in $\mathbf{z}$,

$$J_Z^2 = -|Z|^2 I_{\mathbf{v}},$$

where $I_{\mathbf{v}}$ is the identity on $\mathbf{v}$. A connected and simply connected Lie group N is called an H–type group if its Lie algebra is an H–type algebra.

Note that for every unit Z in $\mathbf{z}$, J_Z is a complex structure on $\mathbf{v}$, so that $\mathbf{v}$ has even dimension $2m$.

Since $\mathbf{n}$ is a nilpotent Lie algebra, the exponential map is surjective. We can then parametrise the elements of $N = \exp \mathbf{n}$ by (X, Z), for X in $\mathbf{v}$ and Z in $\mathbf{z}$. By the Campbell–Hausdorff formula it follows that the product law in N is

$$(X, Z)(X', Z') = \left(X + X', Z + Z' + \frac{1}{2}[X, X']\right).$$

Let NA be the semidirect product of the Lie groups N and $A = \mathbf{R}^+$ with respect to the action of A on N given by the dilations $(X, Z) \longmapsto (a^{1/2}X, aZ)$. As customary we write (X, Z, a) for the element $na = \exp(X + Z)\,a$. It can easily be checked that the product in NA is

$$(X, Z, a)(X', Z', a') = \left(X + a^{1/2}X', Z + aZ' + \frac{1}{2}a^{1/2}[X, X'], aa'\right).$$

We denote by k the dimension of the centre $\mathbf{z}$, and by $Q = m + k$ the homogeneous dimension of N.

The left Haar measure on NA, unique up to a multiplicative constant, is given by

$$dx = a^{-Q-1}\, dX\, dZ\, da = a^{-Q-1}\, dn\, da,$$

where dX, dZ and da are the Lebesgue measures respectively on $\mathbf{v}$, $\mathbf{z}$ and $\mathbf{R}^+$. Note that the right Haar measure is $a^{-1}\, dX\, dZ\, da$, hence the group NA is not unimodular, with modular function δ given by the rule $\delta(X, Z, a) = a^{-Q}$.

We endow NA with the left-invariant Riemannian structure induced by the following inner product on the Lie algebra $\mathbf{n} \oplus \mathbf{R}$ of NA:

$$\langle (X, Z, \alpha), (X', Z', \alpha') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + \alpha\alpha',$$

where $\alpha = \log a$ $(a \in A)$. As a Riemannian manifold, NA is a harmonic space [4]. This class of harmonic spaces includes all rank-one symmetric spaces of the noncompact type $NA \simeq G/K = NAK/K$. In general NA is not symmetric, that is, the geodesic symmetry around the origin is not an isometry [2].

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DEFINITION: A function $f : NA \longrightarrow \mathbf{C}$ is said to be radial if, for all x in NA, $f(x)$ depends only on the geodesic distance $d(x,e)$ of x from the identity e of NA.

We denote by $\mathcal{D}(NA)$ the space of C^∞ functions on NA with compact support. If φ and ψ are functions in $\mathcal{D}(NA)$ we use the notation

$$\langle \varphi, \psi \rangle = \int_{NA} \varphi(x)\psi(x)\, dx,$$
$$(\varphi * \psi)(x) = \int_{NA} \varphi(y)\psi(y^{-1}x)\, dy.$$

Let $R : \mathcal{D}(NA) \longrightarrow \mathcal{D}(NA)$ be the linear operator defined by

$$(Rf)(x) = \int_{S_\rho} f(y)\, d\sigma_\rho(y), \qquad \rho = d(x,e), \tag{2.1}$$

where $d\sigma_\rho$ is the surface measure induced by the left-invariant Riemannian metric on the geodesic sphere $S_\rho = \{y \in NA : d(y,e) = \rho\}$, normalised by $\int_{S_\rho} d\sigma_\rho(y) = 1$. Let $\mathcal{D}^\natural(NA)$ denote the subspace of radial functions in $\mathcal{D}(NA)$. Then R is a projection from $\mathcal{D}(NA)$ onto $\mathcal{D}^\natural(NA)$. Damek and Ricci proved that the operator R is an averaging projector on NA according to the definition introduced in [**4**]. The following properties of R will be needed later:

$$\langle R\varphi, \psi \rangle = \langle \varphi, R\psi \rangle, \qquad \forall \varphi, \psi \in \mathcal{D}(NA), \tag{2.2}$$
$$R\varphi(e) = \varphi(e), \qquad \forall \varphi \in \mathcal{D}(NA). \tag{2.3}$$

It is proved in [**4**] that $\mathcal{D}^\natural(NA)$ is a commutative convolution algebra. Let $\mathbf{D}^\natural(NA)$ denote the algebra of all left-invariant differential operators D on NA which commute with R, that is, $R(Df) = D(Rf)$, for all f in $\mathcal{D}(NA)$. By [**4**, Lemma 2.1, Theorem 5.2] the algebra $\mathbf{D}^\natural(NA)$ is commutative and is generated by the Laplace–Beltrami operator $\mathcal{L}$ in the given Riemannian structure.

A *spherical function* Φ on NA is a radial eigenfunction of the Laplace–Beltrami operator normalised so that $\Phi(e) = 1$. For λ in $\mathbf{C}$, we denote by Φ_λ the spherical function with eigenvalue $-\left(\lambda^2 + Q^2/4\right)$.

The *spherical Fourier transform* of a function f in $\mathcal{D}^\natural(NA)$ is given by

$$\widetilde{f}(\lambda) = \int_{NA} f(x)\Phi_\lambda(x)\, dx, \qquad \lambda \in \mathbf{C}. \tag{2.4}$$

Ricci, in [**11**], determined the inversion formula for the spherical Fourier transform (the correct constant is given in [**1**]): if f is in $\mathcal{D}^\natural(NA)$, then

$$f(x) = \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \widetilde{f}(\lambda)\, \Phi_\lambda(x)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda, \tag{2.5}$$

where

$$c_{m,k} = \frac{2^{k-1}\,\Gamma\left(\dfrac{2m+k+1}{2}\right)}{\pi^{(2m+k+1)/2}}, \tag{2.6}$$

and

$$\mathbf{c}(\lambda) = \frac{2^{Q-2i\lambda}\,\Gamma(2i\lambda)\Gamma\left(\dfrac{2m+k+1}{2}\right)}{\Gamma\left(\dfrac{Q}{2}+i\lambda\right)\Gamma\left(\dfrac{m+1}{2}+i\lambda\right)}. \tag{2.7}$$

The function $\mathbf{c}(\lambda)$ generalises Harish-Chandra's $\mathbf{c}$-function. As in the symmetric case, $\mathbf{c}(\lambda)$ is determined by the asymptotic form of Φ_λ in A, according to

$$\mathbf{c}(\lambda) = \lim_{a\to 0^+} a^{i\lambda-Q/2}\,\Phi_\lambda(a) = \lim_{a\to+\infty} a^{-i\lambda+Q/2}\,\Phi_\lambda(a), \qquad \operatorname{Im}\lambda < 0.$$

3. The Fourier transform

In this section we define the Fourier transform and state its main properties. Since the definition involves the Poisson kernel, we recall briefly the basic facts regarding this kernel.

Damek [3] proved that if f is a bounded harmonic function on NA, then f can be represented as

$$f(x) = \int_N \mathcal{P}(x,n)F(n)\,dn, \qquad x \in NA,$$

where $F(n) = \lim_{a\to 0} f(na)$ and $\mathcal{P}$ is the Poisson kernel on NA. The expression of $\mathcal{P}$ is given by the formula

$$\mathcal{P}(na,n') = P_a(n^{-1}n'), \qquad na \in NA,\ n' \in N,$$

where, for any $a > 0$, $P_a(n)$ is the function on N defined by the rule

$$P_a(n) = P_a(X,Z) = c_{m,k}\,a^Q\left(\left(a+\frac{|X|^2}{4}\right)^2 + |Z|^2\right)^{-Q}.$$

Notice the following properties of $P_a(n)$:

$$P_a(n) = P_a(n^{-1}), \tag{3.1}$$

$$P_a(n) = a^{-Q}P_1(a^{-1}na), \tag{3.2}$$

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where $a^{-1}na \in N$ since A normalises N. Moreover we have $P_1(0,0) = c_{m,k}$, and

$$\int_N P_a(n)\,dn = 1, \qquad \forall a > 0.$$

We denote by $\mathcal{P}_\lambda(x,n)$ the function on $NA \times N$ given by the rule

$$\mathcal{P}_\lambda(x,n) = [\mathcal{P}(x,n)]^{1/2-i\lambda/Q},$$

that is, explicitly

$$\mathcal{P}_\lambda(na,n') = \left[P_a\left(n^{-1}n'\right)\right]^{1/2-i\lambda/Q}.$$

DEFINITION: The Fourier transform of the function f in $\mathcal{D}(NA)$ is the function $\widehat{f}$ on $\mathbf{C} \times N$ defined by the rule

$$\widehat{f}(\lambda,n) = \int_{NA} f(x)\,\mathcal{P}_\lambda(x,n)\,dx, \qquad \lambda \in \mathbf{C}, \ \ n \in N.$$

In the next proposition we compare the Fourier transform with the spherical Fourier transform for radial functions (2.4).

PROPOSITION 3.1. *Let f be a radial function in $\mathcal{D}(NA)$. Then*

$$\widehat{f}(\lambda,n) = \mathcal{P}_\lambda(e,n)\,\widetilde{f}(\lambda), \qquad \forall n \in N, \ \ \forall \lambda \in \mathbf{C},$$

PROOF: Since f is radial and by (2.2), we have

$$\begin{aligned}\widehat{f}(\lambda,n) &= \langle f, \mathcal{P}_\lambda(\cdot,n)\rangle \\ &= \langle Rf, \mathcal{P}_\lambda(\cdot,n)\rangle \\ &= \langle f, R\,\mathcal{P}_\lambda(\cdot,n)\rangle.\end{aligned}$$

Since $\widetilde{f}(\lambda) = \langle f, \Phi_\lambda\rangle$, we shall obtain the result by proving that

$$[R\,\mathcal{P}_\lambda(\cdot,n)]\,(x) = \mathcal{P}_\lambda(e,n)\,\Phi_\lambda(x).$$

For the sake of brevity, let us introduce the following notation. For every function g in $\mathcal{D}(NA)$ and x in NA we define the function $\tau_x g$ on NA by the rule

$$(\tau_x g)\,(y) = g\left(x^{-1}y\right), \qquad \forall y \in NA.$$

Moreover we denote by Ψ_λ the function on NA defined by

$$\Psi_\lambda(na) = [P_a(n)]^{1/2-i\lambda/Q} = \mathcal{P}_\lambda(a,n).$$

Using formula (3.1), one can easily check that, for x in NA and n in N, we have

$$\mathcal{P}_\lambda(x,n) = \Psi_\lambda(n^{-1}x) = (\tau_n \Psi_\lambda)(x). \tag{3.3}$$

One can prove that (see [**1**, **3**])

$$\mathcal{L}\Psi_\lambda = -\left(\lambda^2 + \frac{Q^2}{4}\right)\Psi_\lambda.$$

Since $\mathcal{L}$ is left-invariant, we also have

$$\mathcal{L}(\tau_n\Psi_\lambda) = \tau_n(\mathcal{L}\Psi_\lambda) = -\left(\lambda^2 + \frac{Q^2}{4}\right)\tau_n\Psi_\lambda.$$

Therefore, since $R\mathcal{L} = \mathcal{L}R$, the spherical function Φ_λ and $R(\tau_n\Psi_\lambda)$, for n in N, are two radial eigenfunctions of the Laplace–Beltrami operator with the same eigenvalue. By [**4**, Lemma 2.2] it follows that $R(\tau_n\Psi_\lambda)$ equals Φ_λ up to a multiplicative constant, depending on n:

$$R(\tau_n\Psi_\lambda) = c(n)\,\Phi_\lambda, \qquad \forall n \in N. \tag{3.4}$$

Evaluating both sides of the previous formula at the identity and using property (2.3) of the averaging projector R, we find

$$\begin{aligned} c(n) &= c(n)\,\Phi_\lambda(e) = [R(\tau_n\Psi_\lambda)](e) \\ &= (\tau_n\Psi_\lambda)(e) = \Psi_\lambda(n^{-1}) \\ &= \mathcal{P}_\lambda(e,n). \end{aligned} \tag{3.5}$$

Putting together formulas (3.3), (3.4) and (3.5), we obtain $R\mathcal{P}_\lambda(\cdot,n) = R(\tau_n\Psi_\lambda) = \mathcal{P}_\lambda(e,n)\Phi_\lambda$, as claimed. ∎

We remark that, if we define a normalised Fourier transform $\mathcal{H}$ on NA by the rule

$$\mathcal{H}f(\lambda,n) = \frac{\widehat{f}(\lambda,n)}{\mathcal{P}_\lambda(e,n)}, \qquad \lambda \in \mathbf{C},\ n \in N, \tag{3.6}$$

then, for radial f, $\mathcal{H}f(\lambda,n) = \widetilde{f}(\lambda)$ (the spherical transform) for every n in N, that is, $\mathcal{H}f$ does not depend on n.

Note that the convolution in $\mathcal{D}(NA)$ is not commutative (unlike in $\mathcal{D}^\natural(NA)$). Therefore the Fourier transform cannot convert it into multiplication. This holds, however, if the second factor is radial.

Proposition 3.2. *If f is in $\mathcal{D}(NA)$ and φ is in $\mathcal{D}^{\natural}(NA)$, then*

$$(f*\varphi)\widehat{\ }(\lambda,n)=\widehat{f}(\lambda,n)\,\widetilde{\varphi}(\lambda),\qquad \forall\lambda\in\mathbf{C},\ \forall n\in N.$$

Proof: As in Proposition 3.1, we denote by Ψ_λ the function on NA defined by

$$\Psi_\lambda(na)=\mathcal{P}_\lambda(a,n).$$

For every λ in $\mathbf{C}$ and n in N, we have

$$\begin{aligned}(f*\varphi)\widehat{\ }(\lambda,n)&=\int_{NA}(f*\varphi)(x)\,\mathcal{P}_\lambda(x,n)\,dx\\&=\int_{NA}\int_{NA}f(y)\,\varphi(y^{-1}x)\,dy\,\mathcal{P}_\lambda(x,n)\,dx\\&=\int_{NA}f(y)\int_{NA}\varphi(x)\,\mathcal{P}_\lambda(yx,n)\,dx\,dy\\&=\int_{NA}f(y)\int_{NA}\varphi(x)\,\left(\tau_{y^{-1}n}\Psi_\lambda\right)(x)\,dx\,dy\\&=\int_{NA}f(y)\,\langle\varphi,\tau_{y^{-1}n}\Psi_\lambda\rangle\,dy\\&=\int_{NA}f(y)\,\langle\varphi,R\left(\tau_{y^{-1}n}\Psi_\lambda\right)\rangle\,dy,\end{aligned}$$

where we have used (3.3).

By the same arguments as in Proposition 3.1, $\tau_{y^{-1}n}\Psi_\lambda$ is an eigenfunction of the Laplace–Beltrami operator with eigenvalue $-\left(\lambda^2+Q^2/4\right)$. It follows that

$$R\left(\tau_{y^{-1}n}\Psi_\lambda\right)(x)=c(y,n)\,\Phi_\lambda(x),$$

where

$$c(y,n)=R\left(\tau_{y^{-1}n}\Psi_\lambda\right)(e)=\tau_{y^{-1}n}\Psi_\lambda\,(e)=\mathcal{P}_\lambda(y,n)$$

Therefore

$$(f*\varphi)\widehat{\ }(\lambda,n)=\int_{NA}f(y)\,\mathcal{P}_\lambda(y,n)\,\langle\varphi,\Phi_\lambda\rangle\,dy=\widehat{f}(\lambda,n)\,\widetilde{\varphi}(\lambda),$$

as claimed. ∎

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4. The inversion formula

In order to obtain the inversion formula, we first generalise [8, Theorem 1.1, p.224] to the present case. We recall the following fact (see [1, 7]).

For λ in $\mathbf{C}$, let π_λ denote the representation of NA on $L^2(N)$ given by

$$[\pi_\lambda(na)\varphi]\,(n_1) = a^{-i\lambda-Q/2}\,\varphi(a^{-1}n^{-1}n_1a).$$

Note that the representation π_λ is unitary for real λ, with respect to the usual inner product on $L^2(N)$:

$$(\varphi,\psi) = \int_N \varphi(n)\,\overline{\psi(n)}\,dn.$$

Proposition 4.1. [1, 7] *The spherical function Φ_λ can be expressed in the form*

$$\Phi_\lambda(x) = \left(\pi_\lambda(x)P_1^{1/2+i\lambda/Q}\,,\,P_1^{1/2+i\bar{\lambda}/Q}\right), \qquad \forall\lambda\in\mathbf{C},$$

where $P_a^\sigma(n) = [P_a(n)]^\sigma$, for σ in $\mathbf{C}$.

The key ingredient to our inversion formula is the following property of the spherical functions.

Proposition 4.2. *Let λ be in $\mathbf{C}$. The spherical function Φ_λ satisfies the identity*

$$\Phi_\lambda(x^{-1}y) = \int_N \mathcal{P}_\lambda(x,n)\mathcal{P}_{-\lambda}(y,n)\,dn, \qquad \forall x,y\in NA.$$

Proof: Let $x = na$ and $y = n_1a_1$. By Proposition 4.1 and the equality $\pi_\lambda(x)^* = \pi_{\bar{\lambda}}(x^{-1})$ we have

$$\begin{aligned}
\Phi_\lambda(x^{-1}y) &= \Phi_\lambda\Big((na)^{-1}\,(n_1a_1)\Big)\\
&= \Big(\pi_\lambda\Big((na)^{-1}\,(n_1a_1)\Big)\,P_1^{1/2+i\lambda/Q}\,,\,P_1^{1/2+i\bar{\lambda}/Q}\Big)\\
&= \Big(\pi_\lambda(\,n_1a_1)\,P_1^{1/2+i\lambda/Q}\,,\pi_{\bar{\lambda}}(na)\,P_1^{1/2+i\bar{\lambda}/Q}\Big)\\
&= \int_N a_1^{-i\lambda-Q/2}P_1^{1/2+i\lambda/Q}\,(a_1^{-1}n_1^{-1}n_2a_1)\\
&\qquad\times\overline{\Big(a^{-i\bar{\lambda}-Q/2}P_1^{1/2+i\bar{\lambda}/Q}(a^{-1}n^{-1}n_2a)\Big)}\,dn_2\\
&= \int_N \Big[a_1^{-Q}P_1(a_1^{-1}n_1^{-1}n_2a_1)\Big]^{1/2+i\lambda/Q}\,\big[a^{-Q}P_1(a^{-1}n^{-1}n_2a)\big]^{1/2-i\lambda/Q}\,dn_2\\
&= \int_N \big[P_{a_1}(n_1^{-1}\,n_2)\big]^{1/2+i\lambda/Q}\,\big[P_a(n^{-1}\,n_2)\big]^{1/2-i\lambda/Q}\,dn_2
\end{aligned}$$

$$= \int_N \mathcal{P}_{-\lambda}(y, n_2)\, \mathcal{P}_\lambda(x, n_2)\, dn_2,$$

where we have used (3.2). This proves the proposition. ∎

Lemma 4.3. *If f is in $\mathcal{D}(NA)$ and λ is in $\mathbf{C}$ then*

$$(f * \Phi_\lambda)(x) = \int_N \mathcal{P}_{-\lambda}(x, n)\, \widehat{f}(\lambda, n)\, dn, \qquad \forall x \in NA.$$

Proof: By Proposition 4.2 we get

$$\begin{aligned}(f * \Phi_\lambda)(x) &= \int_{NA} f(y)\, \Phi_\lambda(y^{-1}x)\, dy \\ &= \int_{NA} f(y) \int_N \mathcal{P}_\lambda(y, n)\mathcal{P}_{-\lambda}(x, n)\, dn\, dy \\ &= \int_N \mathcal{P}_{-\lambda}(x, n) \left(\int_{NA} f(y)\, \mathcal{P}_\lambda(y, n)\, dy \right) dn \\ &= \int_N \mathcal{P}_{-\lambda}(x, n) \widehat{f}(\lambda, n)\, dn,\end{aligned}$$

thus proving the lemma. ∎

We are now ready to prove our main result, that is, the inversion formula for the Fourier transform.

Theorem 4.4. *Every function f in $\mathcal{D}(NA)$ can be written as*

$$f(x) = \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \int_N \mathcal{P}_{-\lambda}(x, n)\, \widehat{f}(\lambda, n)\; |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, dn, \qquad x \in NA,$$

where $\mathbf{c}(\lambda)$ is given by (2.7) and $c_{m,k}$ is the constant given by (2.6).

Proof: For every function f in $\mathcal{D}(NA)$ and every x in NA we define the function f_x on NA by the rule

$$f_x(y) = [R(\tau_{x^{-1}} f)]\,(y), \qquad \forall y \in NA,$$

where R is the averaging projector (2.1).

Since f_x is a radial function on NA, we can apply the inversion formula for the spherical Fourier transform, obtaining

$$f_x(y) = \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \widetilde{f}_x(\lambda)\, \Phi_\lambda(y)\; |\mathbf{c}(\lambda)|^{-2}\, d\lambda.$$

By equation (2.2) and by the equalities

$$R\Phi_\lambda = \Phi_\lambda,$$
$$\Phi_\lambda\big(x^{-1}\big) = \Phi_\lambda(x), \qquad \forall x \in NA,$$

we obtain

$$\begin{aligned}
\widetilde{f}_x(\lambda) &= \langle R(\tau_{x^{-1}} f)\,,\,\Phi_\lambda\rangle \\
&= \langle \tau_{x^{-1}} f\,,\,\Phi_\lambda\rangle \\
&= \int_{NA} f(xy)\,\Phi_\lambda(y)\,dy \\
&= \int_{NA} f(y)\,\Phi_\lambda\big(y^{-1}x\big)\,dy \\
&= (f * \Phi_\lambda)\,(x).
\end{aligned}$$

By the property (2.3) of the averaging projector R, we find

$$\begin{aligned}
f(x) &= (\tau_{x^{-1}}\, f)\,(e) \\
&= [R(\tau_{x^{-1}}\, f)]\,(e) \\
&= f_x(e) \\
&= \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \widetilde{f}_x(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&= \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} (f * \Phi_\lambda)\,(x)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&= \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \int_N \mathcal{P}_{-\lambda}(x,n)\, \widehat{f}(\lambda,n)\, dn\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda,
\end{aligned}$$

where we have used Lemma 4.3. □

When the function f in $\mathcal{D}(NA)$ is radial, we expect that our inversion formula for the Fourier transform reduces to the inversion formula for the spherical Fourier transform (2.4). Indeed, by Proposition 3.1 and by Proposition 4.2, our inversion formula in the case of a radial function f gives

$$\begin{aligned}
f(x) &= \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \left[\int_N \mathcal{P}_{-\lambda}(x,n)\,\mathcal{P}_\lambda(e,n)\,dn\right] \widetilde{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&= \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \Phi_\lambda(x)\widetilde{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda,
\end{aligned}$$

which is the known inversion formula (2.5).

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As in the symmetric case, the Fourier transform of a function in $\mathcal{D}(NA)$ is a holomorphic function of uniform exponential type, according to the following definition.

DEFINITION: Let $\rho > 0$ and denote by $\mathcal{H}_\rho(\mathbf{C} \times N)$ the space of C^∞ functions ψ on $\mathbf{C} \times N$ holomorphic in the first variable and such that for each j in $\mathbf{N}$

$$\sup_{(\lambda,n)\in\mathbf{C}\times N} [P_1(n)]^{-1/2-(\operatorname{Im}\lambda/Q)} e^{-\rho|\operatorname{Im}\lambda|} (1+|\lambda|)^j \, |\psi(\lambda,n)| < \infty.$$

Moreover let $\mathcal{H}(\mathbf{C} \times N)$ be the space of all functions ψ in $\bigcup_{\rho>0} \mathcal{H}_\rho(\mathbf{C} \times N)$ satisfying the condition

$$\int_N \mathcal{P}_{-\lambda}(x,n)\,\psi(\lambda,n)\,dn = \int_N \mathcal{P}_\lambda(x,n)\,\psi(-\lambda,n)\,dn, \qquad \forall x \in NA. \tag{4.1}$$

THEOREM 4.5. *If f is in $\mathcal{D}(NA)$, then $\widehat{f}$ belongs to $\mathcal{H}(\mathbf{C}\times N)$. Moreover if* $\operatorname{supp} f$ *is in $\{x \in NA \,:\, d(x,e) \leqslant \rho\}$, then $\widehat{f}$ is in $\mathcal{H}_\rho(\mathbf{C}\times N)$.*

PROOF: Let f be in $\mathcal{D}(NA)$; condition (4.1), with $\widehat{f} = \psi$, follows by Lemma 4.3 and by the equality $\Phi_\lambda = \Phi_{-\lambda}$. Applying the Morera Theorem one can see that $\widehat{f} : \mathbf{C} \times N \longrightarrow \mathbf{C}$ is holomorphic in the first variable. Suppose that the function f is supported in the geodesic ball $B_\rho = \{x \in NA \,:\, d(x,e) \leqslant \rho\}$. By the inequality (see [**1**, **6**])

$$|\log a| \leqslant d(na,e), \qquad \forall na \in NA,$$

we get, for na in B_ρ,

$$\left|a^{i\lambda}\right| = \left|e^{i\lambda\log a}\right| \leqslant e^{|\operatorname{Im}\lambda|\,|\log a|} \leqslant e^{|\operatorname{Im}\lambda|\,\rho}.$$

As already noted in Proposition 3.1, for every n in N, $\mathcal{P}_\lambda(\cdot,n)$ is an eigenfunction of the Laplace–Beltrami operator with eigenvalue $-(\lambda^2 + Q^2/4)$. Hence for every positive integer ℓ we have

$$(\mathcal{L}^\ell f)\widehat{\ }(\lambda,n) = \left[-(\lambda^2+Q^2/4)\right]^\ell \widehat{f}(\lambda,n).$$

Therefore

$$\begin{aligned}
\left|\lambda^2+\frac{Q^2}{4}\right|^\ell \left|\widehat{f}(\lambda,n_0)\right| &= \left|(\mathcal{L}^\ell f)\widehat{\ }(\lambda,n_0)\right| \\
&= \left|\int_{NA} \mathcal{L}^\ell f\,(na)\,[P_a(n^{-1}n_0)]^{1/2-i\lambda/Q}\,a^{-Q-1}\,dn\,da\right| \\
&= \left|\int_{NA} \mathcal{L}^\ell f\,(na)\,[P_1(a^{-1}n^{-1}n_0a)]^{1/2-i\lambda/Q}\,a^{-(3/2)Q-1}\,a^{i\lambda}\,dn\,da\right| \\
&\leqslant e^{|\operatorname{Im}\lambda|\rho}\int_{NA}\left|\mathcal{L}^\ell f\,(na)\right| a^{-(3/2)Q-1}[P_1(a^{-1}n^{-1}n_0a)]^{1/2+(\operatorname{Im}\lambda/Q)}dn\,da,
\end{aligned}$$

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so we shall get the result by proving the next lemma. ∎

Lemma 4.6. *For any $\rho > 0$ there exist two constants c and c', depending only on ρ, such that*

$$c\,[P_1(n_0)] \leqslant [P_1(a_1^{-1}n_1^{-1}n_0a_1)] \leqslant c'\,[P_1(n_0)], \qquad \forall n_0 \in N, \quad \forall n_1a_1 \in B_\rho.$$

Proof: If n_1a_1 belongs to B_ρ, then a_1 belongs to a closed interval I of $(0, +\infty)$. It is easy to see that

$$P_1(a_1^{-1}na_1) \asymp P_1(n), \qquad \forall n \in N, \quad \forall a_1 \in I.$$

We write $\alpha \asymp \beta$ if there exist constants c and c' such that $c\alpha \leqslant \beta \leqslant c'\alpha$.

Observe that there exists a positive number ρ_1 such that, for every n_1a_1 in B_ρ, the element n_1 belongs to B_{ρ_1}, so it is enough to prove that

$$P_1(n_1^{-1}n_0) \asymp P_1(n_0), \qquad \forall n_0 \in N, \quad \forall n_1 \in B_{\rho_1}. \tag{4.2}$$

The geodesic distance of $x = (X, Z, a)$ from the identity is (see [**2**, **4**])

$$\rho(x) = d(x, e) = \log \frac{1 + r(x)}{1 - r(x)},$$

where $r(x)$ is in $(0, 1)$ and is given by

$$1 - r(x)^2 = \frac{4a}{\left(1 + a + |X|^2/4\right)^2 + |Z|^2}.$$

Note that

$$r(x) = \tanh \frac{\rho(x)}{2} \qquad \text{and} \qquad 1 - r(x)^2 = \left(\cosh \frac{\rho(x)}{2}\right)^{-2}.$$

Since $P_1(X, Z) = c_{m,k}\left(\left(1 + |X|^2/4\right)^2 + |Z|^2\right)^{-Q}$, it is easy to check that

$$P_1(n) \asymp \left(1 - r(n)^2\right)^Q, \qquad \forall n \in N.$$

Thus, since $1 - r(n)^2 = (\cosh(\rho(n)/2))^{-2} \asymp e^{-\rho(n)}$, we get (as already observed in [1])

$$P_1(n) \asymp e^{-Q\rho(n)}, \qquad \forall n \in N. \tag{4.3}$$

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Let n_1 be in B_{ρ_1}. By the triangle inequality, we have, for every n_0 in N,

$$\rho(n_0) - \rho_1 \leqslant \rho(n_0) - \rho(n_1) \leqslant \rho\big(n_1^{-1}n_0\big) \leqslant \rho(n_1) + \rho(n_0) \leqslant \rho(n_0) + \rho_1.$$

Thus

$$e^{-\rho_1}\ e^{-\rho(n_0)} \leqslant e^{-\rho\left(n_1^{-1}n_0\right)} \leqslant e^{\rho_1}\ e^{-\rho(n_0)}.$$

Relation (4.2) follows by (4.3). ∎

Finally, we show that when NA is a rank-one symmetric space, our results reproduce the theory of Helgason [**8**].

Let G be a connected noncompact semisimple Lie group with finite centre and real rank one, K a maximal compact subgroup thereof, and $X = G/K$ the associated rank-one symmetric space. Fix an Iwasawa decomposition $G = NAK = KAN$, and for g in G write $g = k(g)\exp[H(g)]\,n(g)$, where $k(g) \in K$, $H(g) \in \mathbf{a}$ (the Lie algebra of A), and $n(g) \in N$. Let $\{\alpha, 2\alpha\}$ or $\{\alpha\}$ be the set of positive restricted roots, with multiplicities $m_\alpha \equiv 2m$ and $m_{2\alpha} \equiv k$. Let $\mathbf{g}_\alpha$, $\mathbf{g}_{2\alpha}$ be the corresponding root spaces; then $\mathbf{n} = \mathbf{g}_\alpha \oplus \mathbf{g}_{2\alpha}$ is a nilpotent H–type subalgebra of $\mathbf{g}$ (the Lie algebra of G), with centre $\mathbf{z} = \mathbf{g}_{2\alpha}$ and inner product given by

$$\langle X, Y\rangle = -\frac{1}{2m+4k} B(X, \theta Y), \qquad \forall X, Y \in \mathbf{n},$$

where B is the Killing form and θ is the Cartan involution (see Korányi [**10**]). The solvable subgroup NA of G is diffeomorphic to X under the identification $X = G/K \sim NAK/K \sim NA$. Therefore our results can be applied to the NA model of X.

In Helgason's theory one has the boundary $B = K/M$ of G/K, where M is the centraliser of A in K. The Poisson kernel on G/K is

$$\mathcal{Q}(x, b) = e^{2\rho(A(x,b))}, \qquad x \in G/K,\ b \in B,$$

where ρ is half the sum of the positive restricted roots and $A(x, b)$ is the function on $G/K \times B$ defined by $A(gK, kM) = -H\big(g^{-1}k\big)$ (see [**8**, p.118 and p.122]). Notice that $\mathcal{Q}$ is normalised so that $\mathcal{Q}(eK, b) = 1$ for every b in B.

In our model the boundary is seen as the group N and the Poisson kernel is normalised so that $\int_N \mathcal{P}(a, n)\,dn = 1$ for any a in A. One can easily check that

$$\mathcal{Q}(aK, k(\theta n)M) = \frac{\mathcal{P}\left(a^{-1}, n\right)}{\mathcal{P}(e, n)}, \qquad a \in A,\ n \in N$$

(see [**8**, p.180]). More generally one can show that the kernel on $G/K \times B$ given by $\mathcal{Q}_\lambda(x, b) = e^{(-i\lambda+\rho)(A(x,b))}$ is related to the kernel $\mathcal{P}_\lambda$ by

$$\mathcal{Q}_\lambda(\theta x\, K, k(\theta n)M) = \frac{\mathcal{P}_\lambda(x, n)}{\mathcal{P}_\lambda(e, n)}, \qquad \forall x \in NA.$$

It can easily be checked that the normalised transform $\mathcal{H}f(\lambda, n)$, defined in formula (3.6), equals the Helgason Fourier transform of the function $f \circ \theta$, evaluated at $(\lambda, k(\theta n)M)$.

Finally, the inversion formula of Theorem 4.4 on NA can be rewritten as

$$f(x) = \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \int_N \mathcal{Q}_{-\lambda}(\theta x\, K, k(\theta n)M)\, \mathcal{H}f(\lambda, n)\, P_1(n)\, dn\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda,$$

and is equivalent to Helgason's inversion formula in the rank-one case [**8**, Theorem 1.3, p.225] for the function $f \circ \theta$ evaluated at θx. Indeed, the integral over B in Helgason's inversion formula can be shifted to an integral over $\theta N \sim N$ and the invariant measure db on B satisfies $db = P_1(n)\, dn$.

5. The Plancherel Theorem

The purpose of this section is to prove the following theorem.

Theorem 5.1. *The Fourier transform extends to an isometry from $L^2(NA)$ onto the space $L^2\Big(\mathbf{R}^+ \times N, \frac{c_{m,k}}{2\pi}\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, dn\Big)$.*

The proof of Theorem 5.1 is divided in two steps. First we prove the Plancherel formula for the Fourier transform, which follows from Theorem 4.4 by a standard argument; then we prove that the Fourier transform is onto.

Let f_1 and f_2 be in $\mathcal{D}(NA)$; using the invariance property (4.1) for $\psi = \widehat{f}_1$, we have

$$\begin{aligned}
&\frac{c_{m,k}}{2\pi} \int_0^{+\infty} \int_N \widehat{f}_1(\lambda, n)\, \overline{\widehat{f}_2(\lambda, n)}\, |\mathbf{c}(\lambda)|^{-2}\, dn\, d\lambda \\
&= \frac{c_{m,k}}{2\pi} \int_0^{+\infty} \int_N \widehat{f}_1(\lambda, n) \int_{NA} \overline{f_2(x)}\, \mathcal{P}_{-\lambda}(x, n)\, dx\, |\mathbf{c}(\lambda)|^{-2}\, dn\, d\lambda \\
&= \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \int_{NA} \overline{f_2(x)} \int_N \widehat{f}_1(\lambda, n)\, \mathcal{P}_{-\lambda}(x, n)\, dn\, dx\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&= \int_{NA} \left[\frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \int_N \widehat{f}_1(\lambda, n) \mathcal{P}_{-\lambda}(x, n)\, |\mathbf{c}(\lambda)|^{-2}\, dn\, d\lambda \right] \overline{f_2(x)}\, dx \\
&= \int_{NA} f_1(x)\, \overline{f_2(x)}\, dx,
\end{aligned}$$

which is the desired formula.

Note that if f_1 and f_2 are both in $\mathcal{D}^\natural(NA)$, the Plancherel formula for the Fourier transform reduces to the spherical Plancherel formula [**11**]. Indeed, using Proposition

3.1 and observing that $\mathcal{P}_\lambda(e,n)\mathcal{P}_{-\lambda}(e,n) = P_1(n)$, we get

$$\int_{NA} f_1(x)\,\overline{f_2(x)}\,dx = \frac{c_{m,k}}{2\pi}\int_0^{+\infty}\left[\int_N \mathcal{P}_\lambda(e,n)\,\mathcal{P}_{-\lambda}(e,n)dn\right]\widetilde{f}_1(\lambda)\,\overline{\widetilde{f}_2(\lambda)}\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda$$
$$= \frac{c_{m,k}}{2\pi}\int_0^{+\infty}\widetilde{f}_1(\lambda)\,\overline{\widetilde{f}_2(\lambda)}\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda.$$

To prove that the Fourier transform is surjective, we need one more definition and a couple of lemmas. Let $\mathcal{E}(N)$ be the space of bounded smooth functions on the group N.

DEFINITION: We say that a complex number λ is simple if the map

$$\mathcal{E}(N) \longrightarrow C(NA)$$
$$F \mapsto \int_N \mathcal{P}_\lambda(\cdot,n)\,F(n)\,dn$$

is injective.

LEMMA 5.2. *Suppose λ is a real number. Then λ is simple.*

PROOF: By contradiction, suppose that λ is not simple, that is, there exists F in $\mathcal{E}(N)$, $F \neq 0$, such that

$$\int_N \mathcal{P}_\lambda(x,n)\,F(n)\,dn = 0, \qquad \forall x \in NA.$$

Notice that we may assume $F(0,0) \neq 0$. In particular, for any a in A, we have the equation

$$\int_N P_a^{1/2-i\lambda/Q}(n)\,F(n)\,dn = 0,$$

which, by property (3.2), we can also write as

$$\int_N P_1^{1/2-i\lambda/Q}\left(a^{-1}na\right)F(n)\,dn = 0.$$

Therefore for any a in $\mathbf{R}^+$ we have

$$\int_N P_1^{1/2-i\lambda/Q}\left(a^{1/2}X, aZ\right)F(X,Z)\,dX\,dZ = 0,$$

that is

$$\int_N \left[\left(a + |X|^2/4\right)^2 + |Z|^2\right]^{i\lambda - Q/2} F(X,Z)\,dX\,dZ = 0.$$

By differentiating under the integral sign with respect to a, we obtain

$$\int_N \left[\left(a+|X|^2/4\right)^2+|Z|^2\right]^{i\lambda-Q/2-1}\left(a+|X|^2/4\right) F(X,Z)\,dX\,dZ=0;$$

since, for any $a>0$, the integral in the previous equation is absolutely convergent, the interchange between the integral sign and the differentiation is justified.

Now we change variables, putting $X'=\left(a^{-1/2}/2\right)X$ and $Z'=a^{-1}Z$, so that

$$\int_N \left[\left(1+|X'|^2\right)^2+|Z'|^2\right]^{i\lambda-Q/2-1}\left(1+|X'|^2\right) F\left(2a^{1/2}X,aZ\right) dX'\,dZ'=0.$$

Since λ is real, by dominated convergence, letting a go to 0, we obtain the equation

$$F(0,0)\int_N \left[\left(1+|X'|^2\right)^2+|Z'|^2\right]^{i\lambda-Q/2-1}\left(1+|X'|^2\right) dX'\,dZ'=0.$$

Since we have supposed $F(0,0)\neq 0$, we shall prove the lemma by showing that the last integral is nonzero for real λ. Indeed, passing to polar coordinates (r,θ) in $\mathbf{v}$ and (s,φ) in $\mathbf{z}$ and by the change of variables $\rho=r^2$, $\sigma=s^2$, the last integral is a constant multiple of

$$\int_0^{+\infty}\int_0^{+\infty}\left((1+\rho)^2+\sigma\right)^{i\lambda-Q/2-1}(1+\rho)\,\rho^{m-1}\,\sigma^{k/2-1}\,d\sigma\,d\rho.$$

Now again change coordinates, letting $\sigma=(1+\rho)^2\tau$; then the last integral equals

$$\int_0^{+\infty}\tau^{k/2-1}(1+\tau)^{i\lambda-Q/2-1}\,d\tau\int_0^{+\infty}\rho^{m-1}(1+\rho)^{2i\lambda-1-m}\,d\rho$$
$$=\frac{\Gamma(k/2)\,\Gamma(m/2+1-i\lambda)}{\Gamma(Q/2+1-i\lambda)}\;\frac{\Gamma(m)\,\Gamma(1-2i\lambda)}{\Gamma(m+1-2i\lambda)},$$

which is different from 0 for all real λ. ∎

Lemma 5.3. *Suppose λ is simple. Then the space of functions on N of the form $\widehat{f}(\lambda,\cdot)$, as f runs in $\mathcal{D}(NA)$, is dense in $L^2(N)$.*

Proof: Let F in $L^2(N)$ be such that

$$\int_N \widehat{f}(\lambda,n)\overline{F(n)}\,dn=0,\qquad \forall f\in\mathcal{D}(NA).$$

This means that

$$\int_{NA} f(x)\left(\int_N \mathcal{P}_\lambda(x,n)\overline{F(n)}\,dn\right)dx=0,\qquad \forall f\in\mathcal{D}(NA),$$

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hence for every x we have

$$\int_N \mathcal{P}_\lambda(x,n)\overline{F(n)}\,dn = 0.$$

Therefore, for any smooth, compactly supported function G on N, we have

$$\int_N G(n_1)\int_N \mathcal{P}_\lambda(n_1 a, n)\overline{F(n)}\,dn\,dn_1 = 0,$$

that is

$$\begin{aligned}\int_N G(n_1)\int_N \mathcal{P}_\lambda\big(a, n_1^{-1}n\big)\overline{F(n)}\,dn\,dn_1 &= \int_N \mathcal{P}_\lambda(a,n_2)\int_N G\big(nn_2^{-1}\big)\overline{F(n)}\,dn\,dn_2\\ &= \int_N \mathcal{P}_\lambda(a,n_2)\,\big(\check{G} * \overline{F}\big)(n_2)\,dn_2 = 0,\end{aligned}$$

where $\check{G}(n) = G\big(n^{-1}\big)$. Since the function $\check{G} * \overline{F}$ is in $\mathcal{E}(N)$, by the simplicity of λ, we get $\check{G} * \overline{F} = 0$.

Since G is arbitrary, we conclude that $F = 0$ almost everywhere. □

We are now ready to prove our result.

Proof of Theorem 5.1: All we need to prove is that the Fourier transform is surjective. Suppose that the function F in $L^2\Big(\mathbf{R}^+ \times N, (c_{m,k}/2\pi)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,dn\Big)$ is orthogonal to the range, that is,

$$\int_0^{+\infty}\int_N \widehat{f}(\lambda,n)\,F(\lambda,n)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,dn = 0, \qquad \forall f \in \mathcal{D}(NA).$$

By Proposition 3.2, for every φ in $\mathcal{D}^\natural(NA)$, we can rewrite the previous equation as

$$\int_0^{+\infty}\widetilde{\varphi}(\lambda)\left(\int_N \widehat{f}(\lambda,n)F(\lambda,n)\,dn\right)|\mathbf{c}(\lambda)|^{-2}\,d\lambda = 0.$$

By the Stone–Weierstrass Theorem, the algebra of functions of the form $\widetilde{\varphi}$, as φ runs through $\mathcal{D}^\natural(NA)$, is dense in the space of even continuous functions on $\mathbf{R}$ vanishing at infinity. Hence, for every f in $\mathcal{D}(NA)$, there exists a set E_f of measure zero in $\mathbf{R}^+$ such that

$$\int_N \widehat{f}(\lambda,n)F(\lambda,n)\,dn = 0, \qquad \forall\lambda \in \mathbf{R}^+ \setminus E_f,$$

and we may suppose also that the function $n \mapsto F(\lambda,n)$ is in $L^2(N)$ for any such λ.

For every j in $\mathbf{N}$, let φ_j be a smooth function with the following properties:

(1) $\varphi_j(x) = 1$ for every x in the ball of radius $j - (1/j)$ (centred at the identity);
(2) the support of φ_j is contained in the ball of radius $j + (1/j)$;
(3) for every x in NA, we have $0 \leqslant \varphi_j(x) \leqslant 1$.

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Now fix a global coordinate system on NA and let M be the set of all functions g on NA of the form $g = \varphi_j\, p$, for some j, where p is a polynomial in the coordinates with rational coefficients. Since the set M is countable, the set

$$E = \bigcup_{g \in M} E_g$$

has measure zero; moreover we have

$$\int_N \widehat{g}(\lambda, n)\, F(\lambda, n)\, dn = 0, \qquad \forall \lambda \in \mathbf{R}^+ \setminus E,\, \forall g \in M.$$

Since any function f in $\mathcal{D}(NA)$ has compact support, there exists j such that $f = f\,\varphi_j$ and we can uniformly approximate f with a sequence (g_ℓ) of functions in M.

Therefore we have

$$\int_N \widehat{f}(\lambda, n)\, F(\lambda, n)\, dn = 0, \qquad \forall \lambda \in \mathbf{R}^+ \setminus E,\, \forall f \in \mathcal{D}(NA),$$

and, by Lemma 5.3, we deduce $F = 0$ almost everywhere. ∎

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