INITIAL AND RELATIVE LIMITING BEHAVIOUR
OF TEMPERATURES ON A STRIP

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Abstract

Let \( u \) be a solution of the heat equation which can be written as the difference of two non-negative solutions, and let \( v \) be a non-negative solution. A study is made of the behaviour of \( u(x, t)/v(x, t) \) as \( t \to 0^+ \). The methods are based on the Gauss-Weierstrass integral representation of solutions on \( \mathbb{R}^n \times ]0, a[ \) and results on the relative differentiation of measures, which are employed in a novel way to obtain several domination, non-negativity, uniqueness and representation theorems.


Let \( W \) denote the Gauss-Weierstrass kernel, defined, for all \( (x, t) \in \mathbb{R}^n \times ]0, \infty[ \), by \( W(x, t) = (4\pi t)^{-n/2} \exp(-\|x\|^2/4t) \), and let \( \mu \) be a locally finite, signed Borel measure on \( \mathbb{R}^n \). Then \( u \), given by the convolution

\[
u(x, t) = \int_{\mathbb{R}^n} W(x - y, t) d\mu(y),
\]

is called the Gauss-Weierstrass integral of \( \mu \), provided that the integral exists. If the integral exists and is finite at a point \( (x_0, t_0) \), then \( u \) is a temperature, that is, a solution of the heat equation, on \( \mathbb{R}^n \times ]0, t_0[ \). Conversely, if \( v \) is a temperature on a strip \( \mathbb{R}^n \times ]0, c[ \), or on a half-space \( \mathbb{R}^n \times ]0, \infty[ \), and \( v \) can be written as the difference of two non-negative temperatures, then \( v \) has a representation as the Gauss-Weierstrass integral of some signed measure \( \nu \). For details and references, see [14]. We write \( u = W\mu \) if \( u \) and \( \mu \) are related by (1), and always assume that such integrals are finite on some strip or half-space \( \mathbb{R}^n \times ]0, c[ \), where \( 0 < c \leq \infty \).

In [5, Theorem 5.2], Doob proved that, if \( u = W\mu \) and \( v = W\nu \), then

\[
\lim_{t \to 0} \frac{u(x, t)}{v(x, t)}
\]

\( \square \)

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exists a.e. \(|v|\) on \(\mathbb{R}^n\), and is then equal to the Radon-Nikodym derivative of \(\mu\) with respect to \(\nu\). Similar results have been proved for harmonic functions, and in more general situations with different limits (see [3] for references), but further study of the behaviour of \(u/v\), and application of the results about \(u/v\), have apparently been neglected. In [3], Brelot mentioned one simple application of an analogous result for harmonic functions. In [19], new results and applications were given for Gauss-Weierstrass integrals, and the present paper contains further theorems, but generally of a different nature. We use the following basic result [19, Theorem 1].

Let \(u = W\mu\) and \(v = W\nu\), where \(\nu\) is non-negative, and let \(x \in \mathbb{R}^n\). If \(\nu(B(x, r)) > 0\) for all closed balls \(B(x, r)\) in \(\mathbb{R}^n\) with positive radius \(r\), then

\[
\liminf_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} \leq \liminf_{t \to 0} \frac{u(x, t)}{v(x, t)} \leq \limsup_{t \to 0} \frac{u(x, t)}{v(x, t)} \leq \limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}.
\]

The first theorem of the present paper is concerned with the upper and lower limits of the quotient \(\mu(B(x, r))/\nu(B(x, r))\) as \(r \to 0\) and, in view of the above result, it has immediate application to the relative behaviour of temperatures. We are thus able to prove some new domination, non-negativity, uniqueness and representation theorems for temperatures. These results include a multi-variable version of a theorem of Gehring [6, Theorem 10], analogues of results for harmonic functions on a disc in the plane due to Bruckner, Lohwater and Ryan [4, Theorems 2 and 3], Hall [8, Theorem 4], and Lohwater [12], and a much more general version of a recent improvement for temperatures [15, Theorem 5] of a result of Krzyżański [11, Theorem 5].

In addition, we are able to compare the strengths of singularities of Gauss-Weierstrass integrals of singular and absolutely continuous measures. For example, it is well-known that, if \(\mu(\{x\}) = \lambda \neq 0\) and \(u = W\mu\), then \(u(x, t) \sim (4\pi t)^{-n/2}\lambda\) as \(t \to 0\), whereas if \(\mu(\{x\}) = 0\) then \(u(x, t) = o(t^{-n/2})\) as \(t \to 0\). We shall show that, if \(\nu\) is non-negative and absolutely continuous, \(v = W\nu\), \(\mu\) is non-negative and concentrated on the set where \(\nu(x, t)\) is unbounded as \(t \to 0\), and \(u = W\mu\), then \(\nu(x, t) = o(u(x, t))\) as \(t \to 0\) for \(\mu\)-almost every point \(x\) in \(\mathbb{R}^n\).

We also give two theorems which show that we can sometimes deduce from the behaviour of \(u/v\) that \(\mu\) or \(\nu\) must be concentrated on some particular set.

Given \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(r > 0\), we put \(\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}\) and \(B(x, r) = \{y \in \mathbb{R}^n: \|x - y\| \leq r\}\). Every measure in this paper is a locally finite, signed Borel measure on \(\mathbb{R}^n\). The letter \(m\) is used to denote Lebesgue measure on \(\mathbb{R}^n\).
We shall call a measure \( \nu \) **strictly positive** if \( \nu(B(x, r)) > 0 \) for all \( x \in \mathbb{R}^n \) and \( r > 0 \). The positive, negative and total variations of a measure \( \mu \) are denoted by \( \mu^+ \), \( \mu^- \) and \( |\mu| \).

The following temperature occurs in several of our theorems. Given a number \( \kappa \geq 0 \), we let \( V_\kappa \) denote the Gauss-Weierstrass integral of the function \( x \mapsto \exp(\kappa \|x\|^2) \), that is,

\[
V_\kappa(x, t) = (1 - 4\kappa t)^{-n/2} \exp\{\kappa \|x\|^2 / (1 - 4\kappa t)\}
\]

for all \((x, t)\) in \( \mathbb{R}^n \times ]0,(4\kappa)^{-1}[ \) if \( \kappa > 0 \), in \( \mathbb{R}^n \times ]0,\infty[ \) if \( \kappa = 0 \). Of course, \( V_0(x, t) = 1 \).

Finally, if \( u \) is a temperature and \( v \) is a non-negative temperature such that \( u < v \) on \( \mathbb{R}^n \times ]0,\infty[ \), then \( v \) is called a **positive thermic majorant** of \( u \) on \( \mathbb{R}^n \times ]0,\infty[ \). For details and references, see [18].

### 2. Relative differentiation of measures

In this section we present several results on the behaviour of \( \frac{\mu(B(x, r))}{v(B(x, r))} \) as \( r \to 0 \), which we require later. The lemmas are all due to Besicovitch [1,2], but one new theorem is also given.

**Lemma 1.** If \( \mu \) and \( v \) are non-negative measures on \( \mathbb{R}^n \), then

\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{v(B(x, r))}
\]

exists and is finite for \( v \)-almost all \( x \) in \( \mathbb{R}^n \).

This result is proved, in the case \( n = 2 \), in [1, Theorem 2]. As with all the results in [1,2], the proof carries over to the general case.

**Lemma 2.** Let \( \mu \) and \( v \) be non-negative measures on \( \mathbb{R}^n \), and let \( Y \) be a Borel set such that \( \mu(Y) = 0 \). Then

\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{v(B(x, r))} = 0
\]

for \( v \)-almost all \( x \) in \( Y \).

See [1, Theorem 3].
Lemma 3. If \( \mu \) is a non-negative measure, and if a family \( F \) of balls covers a Borel set \( E \) in such a way that, for each \( x \in E \), there is a ball \( B(x, r) \) in \( F \) with arbitrarily small \( r \), then \( F \) contains a subfamily of disjoint balls whose union \( H \) has the property that \( \mu(E \setminus H) = 0 \).

This is a special case of [2, Theorem 3].

We now come to a new theorem, which generalizes and strengthens a result which was stated, without proof and for the case \( \nu = m \) only, by Rosenbloom in [13].

Theorem 1. Let \( \mu \) and \( \nu \) be measures on \( \mathbb{R}^n \), \( \nu \) being strictly positive. If
\[
\limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} > -\infty
\]
for all \( x \in \mathbb{R}^n \), and
\[
\limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} \geq 0
\]
for \( \nu \)-almost all \( x \in \mathbb{R}^n \), then \( \mu \) is non-negative.

Proof. For each non-negative integer \( k \), let \( P_k \) denote the set of all \( x \) for which
\[
\limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} \geq -k.
\]
Then (3) implies that
\[
\bigcup_{k=0}^{\infty} P_k = \mathbb{R}^n
\]
and (4) shows that
\[
\nu(\mathbb{R}^n \setminus P_0) = 0.
\]

Let \( \epsilon > 0 \). To each \( x \) in \( P_0 \) there corresponds a positive null sequence \( \{r_i\} \) such that
\[
\mu(B(x, r_i)) \geq -\epsilon \nu(B(x, r_i))
\]
for all \( i \). For each \( k > 0 \) we have \( \nu(P_k \setminus P_{k-1}) = 0 \), by (6), so that there is an open set \( V_k \supseteq P_k \setminus P_{k-1} \) such that
\[
\nu(V_k) < 2^{-k} \epsilon.
\]
To each \( x \in P_k \setminus P_{k-1} \) there corresponds a positive null sequence \( \{r_i\} \) such that
\[
B(x, r_i) \subseteq V_k \quad \text{and} \quad \mu(B(x, r_i)) \geq -(k + \epsilon) \nu(B(x, r_i))
\]
for all \( i \).
Let $E$ be any bounded open set in $\mathbb{R}^n$. Consider the family $F$ of all balls $B(x, r) \subseteq E$ such that either $x \in E \cap P_0$ and (7) holds, or $x \in E \cap (P_k \setminus P_{k-1})$ and (8) holds. In view of (5) the family $F$ covers $E$, and for each $x \in E$ there is a ball $B(x, r)$ in $F$ with arbitrarily small $r$. Therefore, by Lemma 3, there is a sequence $\{C_j\}$ of disjoint members of $F$ such that

$$|\mu|(E \setminus \bigcup_j C_j) = 0.$$ 

For each $k \geq 0$, let $\{\Gamma_{kj}\}$ denote the (possibly finite or empty) subsequence consisting of those $C_j$ whose centres lie in $P_k \setminus P_{k-1}$ if $k > 0$, in $P_0$ if $k = 0$. Then

$$\mu(E) = \mu\left(\bigcup_j C_j\right) = \sum_{k=0}^{\infty} \left(\sum_j \mu(\Gamma_{kj})\right) \geq -\sum_{k=0}^{\infty} (k + \epsilon)\left(\sum_j \nu(\Gamma_{kj})\right) \geq -\epsilon \nu(E) - \sum_{k=1}^{\infty} (k + \epsilon)\nu(V_k) \geq -\epsilon \left(\nu(E) + \sum_{k=1}^{\infty} 2^{-k}(k + \epsilon)\right).$$

Since $\epsilon$ is arbitrary, it follows that $\mu(E) \geq 0$.

Therefore $\mu^+(E) \geq \mu^-(E)$ for all bounded open sets $E$. Using the regularity properties of $\mu^+$ and $\mu^-$, we deduce that $\mu^+(S) \geq \mu^-(S)$ for every $\mu$-measurable set $S$. This proves the theorem.

**Corollary.** Let $\mu$ and $\nu$ be measures on $\mathbb{R}^n$ such that $\nu$ is strictly positive. If

$$\lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

is finite whenever it exists, and is zero $\nu$-almost everywhere, then $\mu = 0$.

**Proof.** By Lemma 1, the limit in (9) exists and is finite $\nu$-almost everywhere. Therefore the hypotheses of Theorem 1 are satisfied with $\mu$ itself, and also with $\mu$ replaced by $-\mu$ throughout. Hence both $\mu$ and $-\mu$ are non-negative, and the corollary is proved.
3. Some applications of Besicovitch's results

The results presented here are all consequences of the above lemmas and the fundamental inequalities in (2).

**Theorem 2.** Let \( \mu \) and \( \nu \) be non-negative measures on \( \mathbb{R}^n \), and let \( Y \) be a Borel set such that \( \mu(Y) = 0 \). If \( u = W\mu \) and \( v = W\nu \) on \( \mathbb{R}^n \times ]0, \infty[ \), then

\[
\lim_{t \to 0} u(x,t) = o(\nu(x,t)) \quad \text{as } t \to 0
\]

for \( \nu \)-almost all \( x \in Y \). In particular, if \( \mu \) and \( \nu \) are mutually singular, then (10) holds for \( \nu \)-almost every \( x \in \mathbb{R}^n \).

**Proof.** By (2) and Lemma 2, we have

\[
\lim_{t \to 0} \frac{u(x,t)}{v(x,t)} = \lim_{r \to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} = 0
\]

for \( \nu \)-almost all \( x \in Y \). This proves the first part, and the second now follows by taking \( Y \) to be any Borel set such that \( \mu(Y) = 0 \) and \( \nu(\mathbb{R}^n \setminus Y) = 0 \).

We now use Theorem 2 to show that the initial singularities of \( W\mu \), where \( \mu \) is absolutely continuous with respect to \( m \), are milder than those of a corresponding \( W\nu \) with \( \nu \) singular with respect to \( m \), at least \( \nu \)-a.e.

**Theorem 3.** Let \( u = W\mu \), where \( \mu \) is non-negative and absolutely continuous with respect to \( m \), and put

\[
Z = \left\{ x : \lim_{t \to 0} u(x,t) = \infty \right\}.
\]

If \( \nu \) is a non-negative measure concentrated on \( Z \), and \( v = W\nu \), then

\[
u(x,t) = o(\nu(x,t)) \quad \text{as } t \to 0
\]

for \( \nu \)-almost every \( x \in \mathbb{R}^n \).

**Proof.** Since \( u(x,t) \) tends to a finite limit as \( t \to 0 \) for \( m \)-almost every \( x \) in \( \mathbb{R}^n \), we see that \( m(Z) = 0 \) and hence that \( \mu(Z) = 0 \). Since \( \nu \) is concentrated on \( Z \), we deduce that \( \mu \) and \( \nu \) are mutually singular, and the result now follows from Theorem 2.

The next theorem is analogous to certain results of Brelot [3] on various limits of quotients of positive harmonic or superharmonic functions.
THEOREM 4. Let $u = W\mu$ and $v = W\nu$, where $\nu$ is non-negative on $\mathbb{R}^n$. The limit

$$\lim_{t \to 0} \frac{v(x, t)}{u(x, t)}$$

exists and is non-zero $\nu$-a.e. In particular

$$\lim_{t \to 0} v(x, t)$$

exists and is strictly positive $\nu$-a.e.

PROOF. Let $N = \{x: \nu(B(x, r)) = 0 \text{ for some } r > 0\}$. Then $N$ is an open set and $\nu(N) = 0$. Since the inequalities in (2) are applicable to any $x$ in $\mathbb{R}^n \setminus N$, it follows from (2) and Lemma 1 that

$$\lim_{t \to 0} \frac{u(x, t)}{v(x, t)} = \lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

exists and is finite for $\nu$-almost all $x$ in $\mathbb{R}^n$. Hence the limit in (11) exists and is non-zero $\nu$-a.e. in $\mathbb{R}^n$. The second part of the theorem follows from the first by taking $u = 1$.

COROLLARY. Let $u = W\mu$ and $v = W\nu$, where $\nu$ is non-negative on $\mathbb{R}^n$. The set of $x$ for which

$$\liminf_{t \to 0} \frac{v(x, t)}{u(x, t)} = 0$$

has $\nu$-measure zero. In particular,

$$\nu\left(\left\{x: \liminf_{t \to 0} v(x, t) = 0\right\}\right) = 0.$$

Our final result in this section is a generalization of [15, Corollary, page 278], which corresponds to the case where $\nu = m$ and $S = \emptyset$. In view of Theorem 2, it is essentially a sharpened form of the above Corollary for the case where $\mu$ and $\nu$ are mutually singular.

THEOREM 5. Let $u = W\mu$ and $v = W\nu$, where $\mu$ is non-negative and $\nu$ is strictly positive, and put

$$E = \{x: u(x, t)/v(x, t) \text{ tends to a finite limit as } t \to 0\}$$

and

$$S = \{x: u(x, t)/v(x, t) \text{ tends to } \infty \text{ as } t \to 0\}.$$

If $u(x, t) = o(\nu(x, t))$ as $t \to 0$, for $\nu$-almost all $x$ in $E$, then $\mu$ is concentrated on $S$.
PROOF. If \( x \notin S \), then either
(i) \( u(x, t)/v(x, t) \) tends to zero as \( t \to 0 \), or
(ii) \( u(x, t)/v(x, t) \) tends to a finite, non-zero limit as \( t \to 0 \), or
(iii) \( u(x, t)/v(x, t) \) tends neither to a limit nor to infinity.

Let \( A, B \) and \( C \) denote the sets where (i), (ii) and (iii) hold respectively. By the Corollary to Theorem 4, \( \mu(A) = 0 \). By hypothesis, \( \nu(B) = 0 \). By [5, Theorem 5.2], \( \nu(\mathbb{R}^n \setminus E) = 0 \) and hence \( \nu(C) = 0 \). Therefore \( \nu(B \cup C) = 0 \), and hence Lemma 2 implies that
\[
\lim_{r \to 0} \frac{\nu(B(x, r))}{\mu(B(x, r))} = 0
\]
for \( \mu \)-almost all \( x \in B \cup C \). The inequalities in (2) now show that
\[
\lim_{t \to 0} \frac{u(x, t)}{v(x, t)} = \lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} = \infty
\]
\( \mu \)-a.e. on \( B \cup C \). The definitions of \( B \) and \( C \) now imply that \( \mu(B \cup C) = 0 \), and hence \( \mu(\mathbb{R}^n \setminus S) = 0 \), as required.

4. Domination, non-negativity and uniqueness theorems
for temperatures

We now present some immediate consequences of Theorem 1.

THEOREM 6. Let \( u = W\mu \) and \( v = W\nu \) on \( \mathbb{R}^n \times ]0, c[ \), where \( \nu \) is strictly positive and \( 0 < c \leq \infty \). If
\[
(12) \quad \limsup_{t \to 0} \frac{u(x, t)}{v(x, t)} > -\infty
\]
for all \( x \in \mathbb{R}^n \), and
\[
(13) \quad \limsup_{t \to 0} \frac{u(x, t)}{v(x, t)} \geq A
\]
for \( \nu \)-almost every \( x \in \mathbb{R}^n \), then \( u \geq Av \) on \( \mathbb{R}^n \times ]0, c[ \).

PROOF. We may suppose that \( A = 0 \), since we could replace \( u \) by \( u - Av \) throughout. By (2),
\[
\limsup_{t \to 0} \frac{u(x, t)}{v(x, t)} \leq \limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}
\]
for all \( x \in \mathbb{R}^n \), so that (12) and (13) imply that the hypotheses of Theorem 1 are satisfied. Hence \( \mu \geq 0 \), and therefore \( u \geq 0 \).
As a consequence of Theorem 6, we can extend a result of Gehring [6, Theorem 10] to the case of an arbitrary $n$, and thus sharpen [18, Theorem 3] and extend [17, Theorem 5] to $\mathbb{R}^n$ for all $n$.

**Theorem 7.** Let $u = W_\mu$ on $\mathbb{R}^n \times ]0, c[$. If

$$\lim_{t \to 0} u(x, t) > -\infty$$

for all $x$ at which the limit exists, and

$$\lim_{t \to 0} u(x, t) \geq A$$

for $\mu$-almost every $x \in \mathbb{R}^n$, then $u \geq A$ on $\mathbb{R}^n \times ]0, c[$.

**Proof.** Take $v = \mu$ in Theorem 6.

Theorem 6 also gives rise to the following uniqueness result.

**Theorem 8.** Let $u = W_\mu$ and $v = W_\nu$ on $\mathbb{R}^n \times ]0, c[$, where $\nu$ is strictly positive. If

$$\liminf_{t \to 0} \frac{|u(x, t)|}{\nu(x, t)} < \infty$$

for all $x \in \mathbb{R}^n$, and

$$\left(\text{14}\right) \liminf_{t \to 0} \frac{u(x, t)}{\nu(x, t)} = 0$$

for $\nu$-almost every $x \in \mathbb{R}^n$, then $u = 0$ throughout $\mathbb{R}^n \times ]0, c[$.

**Proof.** By applying Theorem 6 to $-u$ and $\nu$, we deduce that $u \leq 0$. Hence (14) implies that

$$\lim_{t \to 0} \frac{u(x, t)}{\nu(x, t)} = 0$$

for $\nu$-almost every $x \in \mathbb{R}^n$. Another application of Theorem 6 now shows that $u \geq 0$, and the result is proved.

If we put $\nu = \mu$ in Theorem 8, we obtain a strengthened form of a result which was announced, without proof, in [13], and incorrectly demonstrated in [7]. See [15, page 278] for further details. The result is also analogous to one due to Lohwater [12, Corollary] for harmonic functions on a disc in the plane.
THEOREM 9. Let \( u = W_\mu \) on \( \mathbb{R}^n \times ]0, c[ \). If

\[ \liminf_{t \to 0} |u(x, t)| < \infty \]

for all \( x \in \mathbb{R}^n \), and

\[ \liminf_{t \to 0} u(x, t) = 0 \]

for \( m \)-almost every \( x \in \mathbb{R}^n \), then \( u = 0 \) throughout \( \mathbb{R}^n \times ]0, c[ \).

Another interesting consequence of Theorem 6 is motivated by analogy with recent work of Kuran [10]. It implies that condition (15) in Theorem 9 can be weakened in a particular way, without affecting the conclusion of the theorem (cf. the proof of Theorem 8).

We first recall [16, Theorem 11]. If \( Z \subseteq \mathbb{R}^n \) and \( m(Z) = 0 \), then there exists a positive temperature \( v \) on \( \mathbb{R}^n \times ]0, \infty[ \) such that \( v(x, t) \to \infty \) as \( (x, t) \to (y, 0) \) for all \( y \in Z \). We can obviously suppose that \( v \geq 1 \), since \( v + 1 \) has similar properties.

THEOREM 10. Let \( u = W_\mu \) on \( \mathbb{R}^n \times ]0, c[ \), and suppose that

\[ \liminf_{t \to 0} u(x, t) \leq A \]

for all \( x \in \mathbb{R}^n \setminus Z \), where \( m(Z) = 0 \). Let \( v \) be a temperature such that \( v \geq 1 \) on \( \mathbb{R}^n \times ]0, c[ \) and \( v(x, t) \to \infty \) as \( t \to 0 \) for all \( x \in Z \). If

\[ \liminf_{t \to 0} \frac{u(x, t)}{v(x, t)} \leq 0 \]

for all \( x \in Z \), then \( u \leq A \) on \( \mathbb{R}^n \times ]0, c[ \).

PROOF. There is a non-negative measure \( \nu \) on \( \mathbb{R}^n \) such that \( \nu = W_\nu \) on \( \mathbb{R}^n \times ]0, c[ \). Since \( v \geq 1 \), we have \( \nu \geq m \) and hence \( \nu \) is strictly positive. We can suppose that \( A = 0 \), since we could replace \( u \) by \( u - A \) throughout. It follows from (16) and (17) that

\[ \liminf_{t \to 0} \frac{u(x, t)}{v(x, t)} \leq 0 \]

for all \( x \in \mathbb{R}^n \), so that Theorem 6 gives the desired result.

In the next section we shall use Theorem 10 to prove some new representation theorems for temperatures.
5. Representation theorems

The theorems of this section feature a countable set \( C \). We allow this set to be finite or empty, but retain the notation for a countably infinite set.

The first result is analogous to one due to Bruckner, Lohwater and Ryan [4, Theorem 3] for harmonic functions on the unit disc in \( \mathbb{R}^2 \), at least when \( A = 0 \). Another special case, in which \( C = \emptyset \), parallels [4, Theorem 2].

**Theorem 11.** Let \( u = W\mu \) on \( \mathbb{R}^n \times ]0, c[ \), and let \( C = \{ x_j \}_{j \geq 1} \) be a sequence of points in \( \mathbb{R}^n \). If there is a real constant \( A \), and a non-negative constant \( \kappa \), such that

\[
\liminf_{t \to 0} u(x, t) \leq A \exp(\kappa \| x \|^2)
\]

for \( m \)-almost all \( x \in \mathbb{R}^n \), and

\[
\liminf_{t \to 0} u(x, t) < \infty
\]

for all \( x \in \mathbb{R}^n \setminus C \), then \( u \) can be written in the form

\[
u(x, t) = AV_x(x, t) - h(x, t) + \sum_{j=1}^{\infty} \mu^+ \{ x_j \} W(x - x_j, t)
\]

on \( \mathbb{R}^n \times ]0, \min\{ c, (4\kappa)^{-1} \} \), if \( \kappa > 0 \), on \( \mathbb{R}^n \times ]0, c[ \) if \( \kappa = 0 \), where \( h \) is a non-negative temperature and \( V_x \) is as defined in Section 1.

**Proof.** If we put \( u^* = u - AV_x \), then (18) becomes

\[
\liminf_{t \to 0} u^*(x, t) \leq 0
\]

for \( m \)-almost all \( x \in \mathbb{R}^n \), and (19) holds with \( u^* \) in place of \( u \). If we prove the result for \( u^* \), then the result for \( u \) will follow immediately. We may therefore suppose that \( A = 0 \) and \( \kappa = 0 \).

Let \( \varepsilon > 0 \). For each \( j \), put \( \lambda_j = \mu^+ \{ x_j \} + \varepsilon 2^{-j} \), and let

\[
w(x, t) = u(x, t) - \sum_{j=1}^{\infty} \lambda_j W(x - x_j, t)
\]

for all \( (x, t) \in \mathbb{R}^n \times ]0, c[ \). Since

\[
\sum_{j=1}^{\infty} \lambda_j W(x - x_j, t) \leq \sum_{j=1}^{\infty} \mu^+ \{ x_j \} W(x - x_j, t) + \varepsilon (4\pi t)^{-n/2} \sum_{j=1}^{\infty} 2^{-j}
\]

\[
\leq \int_{\mathbb{R}^n} W(x - y, t) d\mu^+ (y) + \varepsilon (4\pi t)^{-n/2} < \infty
\]

for all \( (x, t) \in \mathbb{R}^n \times ]0, \infty[ \), it follows from [14, Lemma 1] that \( w \) is a temperature.
Let $Z$ denote the set of points where (18) fails to hold, so that $m(Z) = 0$. Let $v$ be a temperature such that $v \geq 1$ on $\mathbb{R}^n \times ]0, c[$ and $v(x, t) \to \infty$ as $t \to 0$ for all $x \in Z$. Since $w \leq u$, for all $x \in \mathbb{R}^n \setminus Z$ we have

\begin{equation}
\liminf_{t \to 0} w(x, t) \leq 0.
\end{equation}

Next, for each $j$ let $\delta_j$ denote the Dirac $\delta$-measure concentrated at $x_j$. Then $w = W\eta$, where $\eta = \mu - \sum_{j=1}^{\infty} \lambda_j \delta_j$, and for each $j$ we have $\eta(x_j) = \mu(x_j) - \lambda_j < 0$. Therefore $w(x_j, t) \sim \eta(x_j)(4\pi t)^{-n/2}$ as $t \to 0$, in view of [19, Examples 1 and 2]. Thus we see that

\begin{equation}
\lim_{t \to 0} w(x, t) = -\infty
\end{equation}

for all $x \in C$. Finally, if $x \in Z \setminus C$ we have

\begin{equation}
\liminf_{t \to 0} w(x, t) < \liminf_{t \to 0} u(x, t) < \infty
\end{equation}

by (19), so that

\begin{equation}
\liminf_{t \to 0} \frac{w(x, t)}{v(x, t)} \leq 0.
\end{equation}

It follows from (21), (22), (23) and Theorem 10 that $w \leq 0$ on $\mathbb{R}^n \times ]0, c[$.

Therefore, in view of (20),

\[ u(x, t) \leq \sum_{j=1}^{\infty} \mu^+\{x_j\} W(x - x_j, t) + e(4\pi t)^{-n/2} \]

for all $(x, t) \in \mathbb{R}^n \times ]0, c[$ and all $e > 0$. Making $e \to 0$, we obtain

\[ u(x, t) \leq \sum_{j=1}^{\infty} \mu^+\{x_j\} W(x - x_j, t). \]

The sum on the right is therefore a positive thermic majorant of $u$ on $\mathbb{R}^n \times ]0, c[$, and hence majorizes the least such majorant. Hence, by [18, Theorem 2],

\[ \int_{\mathbb{R}^n} W(x - y, t) d\mu^+ (y) \leq \sum_{j=1}^{\infty} \mu^+\{x_j\} W(x - x_j, t), \]

so that

\[ u(x, t) - \sum_{j=1}^{\infty} \mu^+\{x_j\} W(x - x_j, t) \leq -\int_{\mathbb{R}^n} W(x - y, t) d\mu^-(y) \leq 0, \]

and the result is proved.

Theorem 11 gives rise to another representation theorem, as follows.
Theorem 12. Let \( u = W_\mu \) on \( \mathbb{R}^n \times ]0, c[ \), and let \( C = \{x_j\}_{j \geq 1} \) be a sequence in \( \mathbb{R}^n \). If there exist non-negative constants \( A \) and \( \kappa \) such that
\[
\liminf_{t \to 0} |u(x, t)| \leq A \exp(\kappa \|x\|^2)
\]
m-a.e. on \( \mathbb{R}^n \), and
\[
\liminf_{t \to 0} |u(x, t)| < \infty
\]
for all \( x \in \mathbb{R}^n \setminus C \), then \( u \) can be written in the form
\[
u(x, t) = h(x, t) + \sum_{j=1}^{\infty} \mu(\{x_j\}) W(x - x_j, t)
\]
on \( \mathbb{R}^n \times ]0, \min\{c, (4\kappa)^{-1}\} \) if \( \kappa > 0 \), on \( \mathbb{R}^n \times ]0, c[ \) if \( \kappa = 0 \), where \( h \) is a temperature which satisfies
\[
|h| \leq AV_\kappa.
\]

Proof. Applying Theorem 11 to \( u \) we obtain
\[
u(x, t) \leq AV_\kappa(x, t) + \sum_{j=1}^{\infty} \mu^+(\{x_j\}) W(x - x_j, t),
\]
so that \( u \) has a positive thermic majorant given by the expression on the right. This expression therefore majorizes the least positive thermic majorant of \( u \), so that by [18, Theorem 2],
\[
\int_{\mathbb{R}^n} W(x - y, t)d\mu^+(y) \leq AV_\kappa(x, t) + \sum_{j=1}^{\infty} \mu^+(\{x_j\}) W(x - x_j, t).
\]
Therefore
\[
0 \leq \int_{\mathbb{R}^n} W(x - y, t)d\mu^+(y) - \sum_{j=1}^{\infty} \mu^+(\{x_j\}) W(x - x_j, t) \leq AV_\kappa(x, t),
\]
and a similar argument applied to \(-u\) gives
\[
0 \leq \int_{\mathbb{R}^n} W(x - y, t)d\mu^-(y) - \sum_{j=1}^{\infty} \mu^-(\{x_j\}) W(x - x_j, t) \leq AV_\kappa(x, t).
\]
It follows that
\[
-AV_\kappa(x, t) \leq u(x, t) - \sum_{j=1}^{\infty} \mu(\{x_j\}) W(x - x_j, t) \leq AV_\kappa(x, t),
\]
which shows that \( |h| \leq AV_\kappa \), as required.

There is a known representation theorem for a temperature \( h \) which satisfies (25). For \( n = 1 \), it is proved in [9, page 206]. Combining this with Theorem 12, we obtain a more explicit representation of \( u \).
COROLLARY 1. If \( u \) satisfies the hypotheses of Theorem 12, then there exists a function \( f \) on \( \mathbb{R}^n \) such that

\[
|f(x)| \leq A \exp(\kappa \|x\|^2)
\]

for all \( x \), and

\[
u(x, t) = \int_{\mathbb{R}^n} W(x - y, t) f(y) dy + \sum_{j=1}^{\infty} \mu(\{x_j\}) W(x - x_j, t)
\]
on \( \mathbb{R}^n \times [0, \min\{c, (4\kappa)^{-1}\}] \) if \( \kappa > 0 \), on \( \mathbb{R}^n \times [0, c] \) if \( \kappa = 0 \).

PROOF. By Theorem 12, \( u \) has the representation (24). Define \( f \) on \( \mathbb{R}^n \) by

\[
f(x) = \limsup_{t \to 0} \nu(x, t).
\]

Since \( |\nu| \leq AV_\kappa \), it is obvious that \( \nu \) has a positive thermic majorant \( \psi \) such that

\[
\limsup_{t \to 0} \psi(x, t) < \infty
\]

for all \( x \), and that \( f(x) > -\infty \) for all \( x \). The result now follows from [18, Theorem 1].

The special case of Theorem 12 in which \( \kappa = 0 \) gives us the following analogue of a theorem on harmonic functions on a disc in \( \mathbb{R}^2 \) due to Lohwater [12]. This corollary also contains, as the special case where \( \mu \) is non-negative and \( C \) is a singleton, a recent improvement [15, Theorem 5] of a theorem of Krzyżaniski [11, Theorem 5].

COROLLARY 2. Let \( u = W\mu \) on \( \mathbb{R}^n \times [0, c] \), let

\[
E = \left\{ x \in \mathbb{R}^n : \lim_{t \to 0} u(x, t) \text{ exists} \right\},
\]

and let \( C = \{x_j\}_{j \geq 1} \) be a sequence of points in \( E \). If \( \lim_{t \to 0} u(x, t) = 0 \) m-a.e. on \( E \), and \( \lim_{t \to 0} u(x, t) \) is finite on \( E \setminus C \), then

\[
u(x, t) = \sum_{j=1}^{\infty} \mu(\{x_j\}) W(x - x_j, t)
\]

for all \( (x, t) \in \mathbb{R}^n \times [0, c] \).

PROOF. By [5, Theorem 5.2], \( m(\mathbb{R}^n \setminus E) = 0 \). It now follows that the hypotheses of Theorem 12 are satisfied, with \( \kappa = 0 \), so that \( u \) can be written in the form (24). Since \( |\nu| \leq AV_\kappa = 0 \), the corollary is proved.
Another consequence of Theorem 12 is roughly analogous to a result of Hall [8, Theorem 4] on holomorphic functions on a disc. His hypotheses allow approach to the boundary along arbitrary Jordan arcs, not just along radii, but require a uniform rate of growth where the values of the modulus are unbounded.

**Theorem 13.** Let \( u = W\mu \) on \( \mathbb{R}^n \times \mathbb{R}^+ \), and suppose that there are non-negative constants \( A \) and \( \kappa \) such that

(i) \( \lim_{t \to 0} |u(x, t)| \leq A \exp(\kappa \|x\|^2) \) a.e. on \( \mathbb{R}^n \),
(ii) \( \lim_{t \to 0} |u(x, t)| = \infty \) on a countable set \( C \), and
(iii) \( \lim_{t \to 0} t^{n/2} u(x, t) = 0 \) for all \( x \in C \).

Then \( |u| \leq A V_\kappa \) on \( \mathbb{R}^n \times \mathbb{R}^+ \), \( \min\{c, (4\kappa)^{-1}\} \) if \( \kappa > 0 \), on \( \mathbb{R}^n \times \mathbb{R}^+ \), \( c \) if \( \kappa = 0 \), so that \( u \) has a representation in the form

\[
(26) \quad u(x, t) = \int_{\mathbb{R}^n} W(x - y, t) f(y) dy
\]

for some function \( f \) such that \( |f(y)| \leq A \exp(\kappa \|y\|^2) \) for all \( y \).

**Proof.** Hypotheses (i) and (ii), together with Theorem 12, imply that \( u \) has the representation (24), where \( \{x_j\}_{j \geq 1} = C \) and (25) holds. Using (iii) and (25), we obtain

\[
(27) \quad \lim_{t \to 0} t^{n/2} \sum_{j=1}^{\infty} \mu(\{x_j\}) W(x - x_j, t) = \lim_{t \to 0} t^{n/2} (u(x, t) - h(x, t)) = 0
\]

for all \( x \in C \). For each non-negative integer \( i \), we can write

\[
\sum \mu(\{x_j\}) W(x - x_j, t) = \int_{\mathbb{R}^n} W(x - y, t) d\nu_i(y),
\]

where the summation is taken over all \( j \neq i \) and where, if \( \delta_j \) denotes the Dirac \( \delta \)-measure concentrated at \( x_j \),

\[
\nu_i = \sum \mu(\{x_j\}) \delta_j.
\]

Since \( \nu_i(\{x_j\}) = 0 \), [19, Example 1] shows that

\[
\lim_{t \to 0} \int_{\mathbb{R}^n} W(x - y, t) d\nu_i(y) = 0
\]

as \( t \to 0 \). It therefore follows from (27) that

\[
\mu(\{x_i\}) = (4\pi t)^{n/2} \mu(\{x_i\}) W(x_i - x_j, t) \to 0.
\]

Hence \( \mu(\{x_i\}) = 0 \) for each \( i \). It follows that \( u = h \), and hence that \( |u| \leq A V_\kappa \).

The representation (26) now follows by an argument similar to the one used to prove Corollary 1 of Theorem 12.
References


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