Fourth Meeting, 9th February 1906.

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On the relations of certain conics to a triangle.

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If we join the angular points ABC of a triangle to any point O the locus of the centres of conics passing through A, B, C, O is a conic bisecting the six joins of the four points and passing through the intersections of OA, BC; OB, AC; and OC, AB. This conic is analogous to the nine-point circle, and at last meeting of the Society Mr Pinkerton showed that its centre lies on the line joining O to the centroid. In what follows an attempt is made still further to generalise this conception.

Figure 8.

Take any two points O and O′, and let OA, OB, OC meet BC, CA, AB in P, Q, R, with a corresponding construction with regard to O′. Then the six points P, Q, R, P′, Q′, R′ lie on a conic.

If ξ, η, ζ and ξ', η', ζ' be the areal coordinates of O and O′, referred to the triangle ABC, the equation of the conic is

$$x^2/\xi\xi' + y^2/\eta\eta' + z^2/\zeta\zeta' - yz(1/\eta\zeta' + 1/\eta'\zeta) - zx(1/\zeta\xi' + 1/\zeta'\xi)$$
$$- xy(1/\xi\eta' + 1/\xi'\eta) = 0.$$

If this conic cut OA, OB, etc., in L, M, etc., then at L we have $y/\eta = z/\zeta$, whence

$$x^2/\xi\xi' - x\{\zeta(1/\zeta\xi' + 1/\zeta'\xi) + \eta(1/\xi\eta' + 1/\xi'\eta)\} = 0.$$

The root $x = 0$ corresponds to P; hence, at L

$$x = 2\xi + \xi'(\eta/\eta' + \zeta/\zeta')$$
$$= \xi + \xi'(\xi/\xi' + \eta/\eta' + \zeta/\zeta'), \text{ with } y = \eta,\ z = \zeta.$$

Now the coordinates of O are (ξ, η, ζ), and of A, $(1, 0, 0)$. Hence, if $\mathrm{OL}/\mathrm{LA} = \kappa_1$, the coordinates of L are proportional to $\xi + \kappa_1, \eta, \zeta$, so that

$$\kappa_1 = \xi'(\xi/\xi' + \eta/\eta' + \zeta/\zeta') \qquad - \quad - \quad (1)$$

with similar results for M, N, L′, M′, N′. This may be thrown into the form

$$\frac{\mathrm{OL}}{\mathrm{LA}} = \frac{\Delta \mathrm{BO'C}}{\Delta \mathrm{ABC}}\left\{\frac{\Delta \mathrm{BOC}}{\Delta \mathrm{BO'C}} + \frac{\Delta \mathrm{COA}}{\Delta \mathrm{CO'A}} + \frac{\Delta \mathrm{AOB}}{\Delta \mathrm{AO'B}}\right\}.$$

If we write p, q, r for $1/\xi, 1/\eta, 1/\zeta$, the equation of the conic becomes

$$pp'x^2 + qq'y^2 + rr'z^2 - (qr' + q'r)yz - (rp' + r'p)zx - (pq' + p'q)xy = 0. \quad (2)$$

This conic, the self-conjugate conic

$$pp'x^2 + qq'y^2 + rr'z^2 = 0 \qquad - \quad - \quad (3)$$

and the circum-conic

$$(qr' + q'r)yz + (rp' + r'p)zx + (pq' + p'q)xy = 0 \quad (4)$$

have four points in common. Another conic through the four points is

$$pp'x^2 + qq'y^2 + rr'z^2 + (qr' + q'r)yz + (rp' + r'p)zx + (pq' + p'q)xy = 0$$
$$\text{or} \quad (px + qy + rz)(p'x + q'y + r'z) = 0$$

which represents the "trilinear polars" of O and O′.

If we say that the conic (2) is a conic "concurrently connected" with the triangle, and that it is "determined" by the points O and O′, we see that

If a concurrently-connected conic meet the trilinear polars of its determining points in X, Y, X′, Y′, then the family of conics passing through X, Y, X′, Y′ will include a self-polar conic and a circumconic.

These three conics are generalisations of the nine-point, self-polar, and circum-circles, for which two of the common points are the circular points, and the other two lie on the "orthic axis," which is their radical axis.

We may notice that the other two pairs of lines through X, Y, X′, Y′ will, through their trilinear poles, determine other two concurrently-connected conics of the same family.

The four-point family have a common self-conjugate triangle, one vertex of which is V, the intersection of XY and X′Y′. This point has therefore the same polar with respect to all the conics; now the coordinates of V are

$$qr' - q'r, \; rp' - r'p, \; pq' - p'q$$

The polar of this for the self-conjugate conic (3) is

$$pp'(qr'-q'r)x+qq'(rp'-r'p)y+rr'(pq'-p'q)z=0$$

or
$$(\eta\zeta'-\eta'\zeta)x+(\zeta\xi'-\zeta'\xi)y+(\xi\eta'-\xi'\eta)z=0$$

i.e., the line OO′. Thus OO′ is the polar of V for every conic of the family, and reciprocally, the locus of the poles of any line through V is OO′ (together with the harmonic conjugate of the given line with respect to VX and VX′).

Now if V be at infinity we deduce that

If the trilinear polars of O and O′ be parallel, the centres of all the conics through X, Y, X′, Y′ lie on the line OO′ (together with the line midway between XY and X′Y′).

Furthermore, XY and X′Y′ are parallel to the diameters conjugate to OO′, so that OO′ bisects XY and X′Y′.

The analytical condition that XY and X′Y′ be parallel is

$$\begin{vmatrix} p & p' & 1 \\ q & q' & 1 \\ r & r' & 1 \end{vmatrix}=0$$

or
$$\begin{vmatrix} \dfrac{1}{\xi} & \dfrac{1}{\xi'} & 1 \\ \dfrac{1}{\eta} & \dfrac{1}{\eta'} & 1 \\ \dfrac{1}{\zeta} & \dfrac{1}{\zeta'} & 1 \end{vmatrix}=0$$

showing that O, O′, and the centroid lie on a circum-conic.

Again, let X′Y′ be altogether at infinity. Then O′ coincides with the centroid G. By (1), the original concurrently-connected conic bisects OA, OB, OC, and becomes a nine-point conic; two of the common points of the family are at infinity, so that the conics are all homothetic; and their centres lie on OG. The equations of the nine-point, self-conjugate, and circum-conics of the family become

$$px^2+qy^2+rz^2-(q+r)yz-(r+p)zx-(p+q)xy=0 \quad \text{- (2′)}$$

$$px^2+qy^2+rz^2=0 \quad \text{- - - - - - - (3′)}$$

and
$$(q+r)yz+(r+p)zx+(p+q)xy=0 \quad \text{- - - - (4′)}$$

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The centre, K, of (2′) is given by

$$2px_0-(p+q)y_0-(p+r)z_0=-(q+p)x_0+2qy_0-(q+r)z_0$$
$$=-(r+p)x_0-(r+q)y_0+2rz_0$$

whence
$$\frac{x_0}{2\xi+\eta+\zeta}=\frac{y_0}{\xi+2\eta+\zeta}=\frac{z_0}{\xi+\eta+2\zeta}.$$

Now if x_0, y_0, z_0 and ξ, η, ζ be the *actual* coordinates of K, and of O,

$$x_0+y_0+z_0=\xi+\eta+\zeta=1$$

whence
$$\frac{x_0}{\xi+1}=\frac{y_0}{\eta+1}=\frac{z_0}{\zeta+1}=\frac{1}{4};$$

$$\therefore\quad x_0=\frac{\xi+1}{4}=\frac{\xi+3\cdot\frac{1}{3}}{4},\quad y_0=\frac{\eta+3\cdot\frac{1}{3}}{4},\quad z_0=\frac{\zeta+3\cdot\frac{1}{3}}{4}.$$

But $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ are the coordinates of G. Hence K divides OG internally in the ratio of 3 to 1.

The centre of (3′) is clearly at O; and the centre of (4′) may be shown to divide OG externally in the ratio of 3 to 1. The various centres are therefore relatively situated exactly as those of the circles which the conics become when O is the orthocentre.