IN Variant PolynomialS OF WEyL GROUPS AND
APPLICATIONS TO THE CENTREs OF UNIVERSal
ENVELOPING ALGEBRAS

C. Y. LEE

Introduction. An element in the centre of the universal enveloping algebra of a semisimple Lie algebra was first constructed by Casimir by means of the Killing form. By Schur’s lemma, in an irreducible finite-dimensional representation elements in the centre are represented by diagonal matrices of all whose eigenvalues are equal. In section 2 of this paper, we show the existence of a complete set of generators whose eigenvalues in an irreducible representation are closely related to polynomial invariants of the Weyl group $W$ of the Lie algebra (Theorem 1). Such a set of generators and their eigenvalues have found many applications in physics [5, p. 1314]. In section 3, we obtain the polynomial invariants of the Weyl groups of the classical Lie algebras as well as those of the five exceptional ones. These invariant polynomials can be used directly to compute eigenvalues of a complete set of generators of $\mathfrak{g}$. In the last section, we point out a possible way of the explicit constructions of these generators.

1. Preliminaries. All the basic results we need can be found in [2, part III]. We use the same notations from that part of [2] and will not define them. The dimension of $\mathfrak{g}$ is assumed to be $n$ instead of $l$ as in [2].

2. A set of generators of $\mathfrak{s}$ and their eigenvalues in $\pi_\lambda$. In this section, we will study the eigenvalues of a set of generators of $\mathfrak{s}$.

We will first consider some results in [2]. The function $\chi_x$ [2, p. 71] was extended to $\mathfrak{s}$ [2, p. 72]. It was also shown [2, p. 72] that if $\beta$ is the isomorphism from $\mathfrak{g}$ to the algebra of polynomials in $x_1, \ldots, x_n$ defined by $\beta(H_i) = x_i$, then every $Z \in \mathfrak{s}$ has a unique decomposition

$$Z = P + H, \quad P \in \mathfrak{s}, H \in \mathfrak{h};$$

furthermore,

$$H = \beta^{-1}(\chi_x(Z))$$

and $\beta$ is an isomorphism of $\mathfrak{s}$ into the algebra $C[x]$ of polynomials in $x_1, \ldots, x_n$.
Now if $\pi_\Lambda$ is the irreducible representation with highest weight $\Lambda$ and $v_\Lambda$ is a vector of weight $\Lambda$, then for any $Z \in \mathfrak{g}$, we have, by (2.1),

\begin{equation}
\pi_\Lambda(Z)v_\Lambda = \pi_\Lambda(P)v_\Lambda + \pi_\Lambda(H)v_\Lambda = \pi_\Lambda(H)v_\Lambda.
\end{equation}

But $\pi_\Lambda(H_i)v_\Lambda = \Lambda(H_i)v_\Lambda$, $1 \leq i \leq n$ and $H = \beta^{-1}(\chi_\beta(Z))$, thus (2.3) become

\begin{equation}
\pi_\Lambda(Z)v_\Lambda = \pi_\Lambda(H)v_\Lambda = (\chi_\beta(Z)|_{Z_i=\Lambda(H_i)})v_\Lambda
\end{equation}

and the eigenvalue of $Z$ is $\chi_\beta(Z)|_{Z_i=\Lambda(H_i)}$.

**Example.** We apply the above ideas to the Casimir operator. Let $H_1, \ldots, H_n$ be dual basis of $H_1, \ldots, H_n$ with respect to the Killing form and $\Sigma$ be the set of all roots of $\mathfrak{g}$. If for any $\alpha \in \Sigma$, $E_\alpha$ and $E_{-\alpha}$ satisfy $(E_\alpha, E_{-\alpha}) = 1$ and $[E_\alpha, E_{-\alpha}] = H_\alpha'$, then the Casimir operator $Z_c$ is defined by

\begin{equation}
Z_c = \sum_{I=1}^n H_iH_i' + \sum_{\alpha \in \Sigma} E_{-\alpha}E_\alpha.
\end{equation}

Using $[E_\alpha, E_{-\alpha}] = H_\alpha'$, we obtain a decomposition of $Z_c$ (2.1) as

\begin{equation}
Z_c = \left(2 \sum_{\alpha > 0} E_{-\alpha}E_\alpha\right) + \left(\sum_{i=1}^n H_iH_i' + \sum_{\alpha > 0} H_\alpha'\right).
\end{equation}

Thus in $\pi_\Lambda$ we have

\begin{equation}
\pi_\Lambda(Z_c)v_\Lambda = \pi_\Lambda\left(\sum_{i=1}^n H_iH_i' + \sum_{\alpha > 0} H_\alpha'\right)v_\Lambda.
\end{equation}

Now $\sum_{i=1}^n \Lambda(H_i)\Lambda(H_i') = (\Lambda, \Lambda)$ and $\sum_{\alpha > 0} H_\alpha' = 2\rho$, where $\rho = \frac{1}{2}(\sum_{\alpha > 0} H_\alpha)$ [1, p. 246-7]. Thus the eigenvalue of $Z_c$ is $(\Lambda, \Lambda + 2\rho)$.

If $s_{\alpha_i} \in W$ is a fundamental reflection, then $s_{\alpha_i}^{-1} = s_{\alpha_i}$ and from definition [2, p. 69], we have

\begin{equation}
\lambda(s_{\alpha_i}H) = s_{\alpha_i}\lambda(H) = \left(\lambda - \frac{2(\lambda, \alpha_i)}{\alpha_i(\alpha_i)} \alpha_i\right)H,
\end{equation}

thus,

\begin{equation}
s_{\alpha_i}H = H - \alpha_i(H)H_i.
\end{equation}

From (2.5) it follows that

\begin{equation}
s_{\alpha_i}x_j = x_n(s_{\alpha_i}H_j) = x_j - \alpha_i(H_j)x_i,
\end{equation}

where

\begin{equation}
\alpha_i(H_j) = \frac{2(\alpha_i, \alpha_j)}{\alpha_j(\alpha_j)}.
\end{equation}

**Definition.** A polynomial function $f(x_1, \ldots, x_n)$ is said to be invariant under $W$ if it is invariant under the transformation (2.6). We also write $sf(x) = f(x)$, for all $s \in W$. 

https://doi.org/10.4153/CJM-1974-055-x Published online by Cambridge University Press
It is known that each Weyl group \( W \) has \( n \) basic homogeneous invariant polynomials [3].

**Theorem 1.** If \( f_1(x), \ldots, f_n(x) \) is a basic set of homogeneous invariant polynomials of \( W \), then there exists \( Z_1, \ldots, Z_n \) of \( \mathfrak{g} \) such that

(i) \( x^e(Z_i) = f_i(x') \), \( 1 \leq i \leq n \);
(ii) \( Z_1, \ldots, Z_n \) is a complete set of generators of \( \mathfrak{g} \).

**Proof.** Let \( \sigma \) be the unique automorphism of \( C[x] \) which maps \( x_i \rightarrow x_i + \rho(H_i) \) (\( i = 1, \ldots, n \)). Let \( I \) be the \( W \) invariant polynomials of \( C[x] \). By [2, Lemma 36 p. 72], the function \( x^e \) is an injective homomorphism from \( \mathfrak{g} \) into \( C[x] \). It is clear from definition that \( \sigma^{-1}x^e \) maps \( \mathfrak{g} \) into \( I \). According to [2, Lemma 38 and Corollary to Lemma 39], \( \sigma^{-1}x^e \) maps \( \mathfrak{g} \) onto \( I \); thus \( \mathfrak{g} \cong I \) (under \( \sigma^{-1}x^e \)).

If \( f_1(x), \ldots, f_n(x) \) is a basic set of homogeneous polynomials in \( I \), then the elements \( Z_i = x^e^{-1}\sigma(f_i(x)) (i = 1, \ldots, n) \) clearly satisfies (i) and (ii).

### 3. Invariant polynomials

In this section, we study polynomials that are invariant under \( W \). Basic sets of homogeneous invariant polynomials will be constructed for the classical Lie algebras as well as the five exceptional ones. A list of the degrees of such a basic set for these algebras can be found in [4, p. 780].

If \( (a_{ij}) \) denotes the Cartan matrix [1, p. 121] of \( \mathfrak{g} \) relative to \( \mathfrak{h} \), then it follows from (2.6) and (2.7) that the transformation \( s_{ij} \) on \( (x_1, \ldots, x_n) \) corresponds to the \( i \)th column of \( (a_{ij}) \), i.e.,

\[
\begin{align*}
(i) & \quad A_n: \text{ The Cartan matrix is} \\
& \quad 
\begin{bmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
& & \ddots & -1 \\
& & & -1 & 2
\end{bmatrix}
\end{align*}
\]

Let \( s_{ai} (1 \leq i \leq n) \) be the transformation on \( (x_1, \ldots, x_n) \) corresponding to the \( i \)th column. Then

\[
\begin{align*}
(x_1, \ldots, x_n) & \xrightarrow{s_{ax}} (-x_1, x_1 + x_2, x_3, \ldots, x_n) \\
(x_1, x_2, \ldots, x_n) & \xrightarrow{s_{ax^2}} (x_1 + x_2, -x_2, x_3 + x_2, x_4, \ldots, x_n) \\
& \quad \vdots \\
(x_1, x_2, \ldots, x_n) & \xrightarrow{s_{ax^{n-1}}} (x_1, \ldots, x_{n-2} + x_{n-1}, -x_{n-1}, x_n + x_{n-1}) \\
(x_1, x_2, \ldots, x_n) & \xrightarrow{s_{ax^n}} (x_1, \ldots, x_{n-2}, x_{n-1} + x_n, -x_n).
\end{align*}
\]
If
\[ y_1 = nx_1 + (n - 1)x_2 + (n - 2)x_3 + \ldots + x_n, \]
\[ y_2 = -x_1 + (n - 1)x_2 + (n - 2)x_3 + \ldots + x_n, \]
\[ y_3 = -x_1 - 2x_2 + (n - 2)x_3 + \ldots + x_n, \]
\[ y_4 = -x_1 - 2x_2 - 3x_3 + (n - 3)x_4 + \ldots + x_n, \]
\[ \ldots \]
\[ y_n = -x_1 - 2x_2 - 3x_3 - \ldots - (n - 1)x_{n-1} + x_n \]
\[ y_{n+1} = -y_1 - \ldots - y_n, \]
then \( s_a \) becomes the transposition \((y_i, y_{i+1})\) and \( W \) becomes the permutation group on \( y_1, \ldots, y_{n+1} \). To show that \( y_1, \ldots, y_n \) are independent, we notice that the coefficients of \( x_a \) in all \( y_i \)s \((1 \leq i \leq n)\) are 1. Thus by multiplying the last column of the Jacobian \( \partial(y_1, \ldots, y_n)/\partial(x_1, \ldots, x_n) \) by \( i \) and adding it to the \( i \)th column, the Jacobian becomes an upper diagonal matrix whose determinant is \((n + 1)^n\). Thus, as basic invariant polynomials, we may choose the symmetric functions \((y_1, \ldots, y_{n+1})\)
\[ \psi_1 = \sum_{i<j} y_i y_j, \quad \psi_2 = \sum_{i<j<k} y_i y_j y_k, \quad \ldots, \quad \psi_n = y_1 \ldots y_{n+1}. \]

(ii) \( B_n \) \((n \geq 2)\): The Cartan matrix is
\[
\begin{bmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& & \ddots & & \\
& & & -1 & \\
& & & & 2 & -1 \\
& & & & -2 & 2
\end{bmatrix}
\]
Let
\[ y_1 = 2x_1 + 2x_2 + \ldots + 2x_{n-1} + x_n, \]
\[ y_2 = 2x_2 + 2x_3 + \ldots + 2x_n + x_n, \]
\[ \ldots \]
\[ \ldots \]
\[ y_{n-1} = 2x_{n-1} + x_n, \]
\[ y_n = x_n. \]
Then it can be verified that the transformations \( s_{ai}, (1 \leq i \leq n - 1) \) corresponding to the first \( n - 1 \) columns are \((y_i, y_{i+1})\) and that \( s_{an} \) changes the sign.
of $y_n$. The Jacobian $\frac{\partial (y_1, \ldots, y_n)}{\partial (x_1, \ldots, x_n)}$ is easily seen to be $2^{n-1}$.

Thus as basic invariant polynomials, we may choose

$$\psi_1 = \sum_i y_i^2, \quad \psi_2 = \sum_{i<j} y_i^2 y_j^2, \ldots, \quad \psi_n = y_1^2 \cdots y_n^2.$$  

(iii) $C_n(n \geq 3)$: The Cartan matrix is

$$\begin{bmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 2
\end{bmatrix}.$$  

Put

$$y_1 = x_1 + \ldots + x_n$$
$$y_2 = x_2 + \ldots + x_n$$
$$\vdots$$
$$y_n = x_n.$$  

Then it can be proved that we have a similar case as in $B_n$. As basic invariants, we may choose

$$\psi_1 = \sum_i y_i^2, \quad \psi_2 = \sum_{i<j} y_i^2 y_j^2, \ldots, \quad \psi_n = y_1^2 \cdots y_n^2.$$  

(iv) $D_n(n \geq 4)$: The Cartan matrix is

$$\begin{bmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 2
\end{bmatrix}.$$  

Let

$$y_1 = 2x_1 + \ldots + 2x_{n-2} + x_{n-1} + x_n,$$
$$y_2 = 2x_2 + \ldots + 2x_{n-2} + x_{n-1} + x_n,$$
$$\vdots$$
$$\vdots$$
$$y_{n-1} = x_{n-1} + x_n,$$
$$y_n = -x_{n-1} + x_n.$$
It can be verified that the first \( n - 1 \) columns correspond to \( (y_{i_1}, \ldots, y_{i_{n-1}}) \) and the last column transposes \( y_{n-1} \) and \( y_n \) and changes their signs. Also, the Jacobian \( \frac{\partial (y_1, \ldots, y_n)}{\partial (x_1, \ldots, x_n)} \) is not zero. Thus as basic invariant polynomials, we may choose

\[
\psi_1 = \sum_i y_i^2, \quad \psi_2 = \sum_{i<j} y_i^2 y_j^2, \ldots,
\]

\[
\psi_{n-1} = \sum_{i_1 < \ldots < i_{n-1}} y_{i_1}^2 \cdots y_{i_{n-1}}^2, \quad \psi_n = y_1 \ldots y_n.
\]

(v) \( G_2 \): The Cartan matrix is

\[
\begin{bmatrix}
2 & -1 \\
-3 & 2
\end{bmatrix}.
\]

Let

\[
y_1 = 3x_1 + x_2, \quad y_2 = x_2, \quad y_3 = -y_1 - y_2.
\]

Then the transformation corresponding to the first and second columns are

\[
(y_1, y_2, y_3) \rightarrow (y_2, y_1, y_3)
\]

\[
(y_1, y_2, y_3) \rightarrow (-y_3, -y_2, -y_1)
\]

respectively. Thus a basic set of invariants may be chosen as

\[
\psi_1 = \sum_{i<j} y_i y_j, \quad \psi_2 = (y_1 y_2 y_3)^2.
\]

The independence of \( \psi_1 \) and \( \psi_2 \) can be shown by computing the Jacobian \( \frac{\partial (\psi_1, \psi_2)}{\partial (y_1, y_2)} \).

(vi) \( F_4 \): A fundamental root system of \( F_4 \) can be chosen as

\[
\{\frac{1}{2}(e_1 - e_2 - e_3 - e_4), e_4, e_3 - e_4, e_2 - e_3\}.
\]

The Dynkin diagram is

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4
\end{array}
\]

and the Cartan matrix is

\[
\begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -2 & 2 & -1 \\
0 & 0 & -1 & 2
\end{bmatrix}.
\]

If \( u = \frac{1}{2}(r_1 - r_2 - r_3 - r_4) \), then in \( R^4 \) an arbitrary vector \( (r_1, r_2, r_3, r_4) \) is transformed by \( S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3}, S_{\alpha_4} \) to \( (r_1 - u, r_2 + u, r_3 + u, r_4 + u) \), \( (r_1, r_2, r_3, -r_4) \), \( (r_1, r_2, r_4, r_3) \), \( (r_1, r_3, r_2, r_4) \) respectively. Thus the 24 linear forms \( \pm r_i \pm r_j \) \( (i \neq j) \) are permuted under \( W \).
Now let
\[ y_1 = 2x_1 + 3x_2 + 4x_3 + 2x_4, \]
\[ y_2 = x_2 + 2x_3 + 2x_4, \]
\[ y_3 = x_2 + 2x_3, \]
\[ y_4 = x_2. \]
Then \( \frac{1}{4}(y_1 - y_2 - y_3 - y_4) = x_1. \) It can be verified that under the transformations generated by the four columns of the Cartan matrix, the images of \( (y_1, y_2, y_3, y_4) \) are \((y_1 - x_1, y_2 + x_1, y_3 + x_1, y_4 + x_1), \) \((y_1, y_2, y_3, -y_4), \) \((y_1, y_2, y_4, y_3)\) and \((y_1, y_3, y_2, y_4)\). For a basic set of invariant polynomials, we may choose the functions
\[ \psi_m = \sum_{i<j} [(y_i + y_j)^m + (y_i - y_j)^m], \quad m = 2, 6, 8, 12. \]

To show the independence of \( \psi_2, \psi_6, \psi_8, \psi_{12} \), we may either expand the Jacobian and show that it is not identically zero or show that it is not zero for particular values of \( y_1, y_2, y_3 \) and \( y_4 \) (e.g., \( y_1 = 1/\sqrt{2}, y_2 = i/\sqrt{2}, y_3 = -1/\sqrt{2}, y_4 = -i/\sqrt{2} \)).

(vii): \( E_6 \): The Cartan matrix is
\[
\begin{bmatrix}
2 & -1 & -1 & -1 & -1
-1 & 2 & -1 & -1
-1 & 2 & -1
-1 & 2 & 2
\end{bmatrix}.
\]

This case was studied in \([4, p. \text{776-9}]\). Let
\[ y_1 = 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \]
\[ y_2 = -x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \]
\[ y_3 = -x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 \]
\[ y_4 = -x_1 - 2x_2 - 3x_3 + 2x_4 + x_5 \]
\[ y_5 = -x_1 - 2x_2 - 3x_3 - 4x_4 + x_5 \]
\[ y_6 = -y_1 - \ldots - y_5 \]
and
\[ y = -3(x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 + 2x_6). \]
Then the first five columns are the transpositions \((y_1y_2), \ldots, (y_5y_6)\) and the last one is
\[ (y_1, \ldots, y_6; y) \rightarrow \]
\[ (y_1 - u, y_2 - u, y_3 - u, y_4 + u, y_5 + u, y_6 + u, y - u) \]
where \( u = \frac{1}{2} (y_1 + y_2 + y_3 + y) \). Let
\[
\begin{align*}
a_i &= y_i + y_i, \quad b_i = y_i - y_i \quad (i = 1, 2, \ldots, 6), \\
c_{ij} &= -y_i - y_j \quad (i < j).
\end{align*}
\]
As a basic set of invariant polynomials, we may choose
\[
\psi_m = \sum_i a_i^m + \sum_i b_i^m + \sum_{i<j} c_{ij}^m, \quad m = 2, 5, 6, 8, 9, 12.
\]
By the method used in (i), it is quite easy to show that \( y_1, \ldots, y_5 \) are independent. It is then clear that \( y_1, \ldots, y_5, y \) are independent. The independence of \( \psi_m \)'s then follows from [4].

(viii) \( E_7 \): In the seven dimensional space \( \{(r_1, \ldots, r_8)|r_1 + \cdots + r_8 = 0\} \), a fundamental root system can be chosen as
\[
\{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_6 = e_6 - e_7, \alpha_7 = \frac{1}{2} e^{(8)} - e_1 - e_2 - e_3 - e_4\}.
\]
Where \( e^{(8)} = \sum_{i=1}^8 e_i \). Reflections defined by \( \alpha_1, \alpha_2, \ldots, \alpha_6 \) are transpositions \((r_1 r_2), \ldots, (r_6 r_7)\) and the one defined by \( \alpha_7 \) is
\[
(r_1, \ldots, r_8) \rightarrow (r_1, \ldots, r_8) - \frac{1}{2} (-r_1 - r_2 - r_3 - r_4) \\
\times (-1, -1, -1, -1, 1, 1, 1, 1).
\]
The 56 linear forms \( r_i + r_j \) and \(-r_i - r_j (i \neq j)\) are permuted under \( W \). The Cartan matrix is
\[
\begin{bmatrix}
2 & -1 & -1 & -1 & -1 & -1 & 1 \\
-1 & 2 & -1 & -1 & -1 & -1 & 1 \\
-1 & 2 & -1 & -1 & -1 & -1 & 1 \\
-1 & 2 & -1 & -1 & -1 & -1 & 1 \\
-1 & 2 & -1 & -1 & -1 & -1 & 1 \\
-1 & 2 & -1 & -1 & -1 & -1 & 1 \\
-1 & 2 & -1 & -1 & -1 & -1 & 1
\end{bmatrix}
\]
Let
\[
\begin{align*}
y_1 &= 3x_1 + 2x_2 + x_3 - x_7 \\
y_2 &= -x_1 + 2x_2 + x_3 - x_7 \\
y_3 &= -x_1 - 2x_2 + x_3 - x_7 \\
y_4 &= -x_1 - 2x_2 - 3x_3 - x_7 \\
y_5 &= -x_1 - 2x_2 - 3x_3 - 4x_4 - x_7 \\
y_6 &= -x_1 - 2x_2 - 3x_3 - 4x_4 - 4x_5 - x_7 \\
y_7 &= -x_1 - 2x_2 - 3x_3 - 4x_4 - 4x_5 - 4x_6 - x_7 \\
y_8 &= -y_1 - y_2 - \ldots - y_7.
\end{align*}
\]
Then \((y_1 + y_2 + y_3 + y_4)/2 = -2x_7\). It can be verified that \(s_i\) acts as 
\((y_i y_{i+1})\), \((1 \leq i \leq 6)\) and 
\[
(y_1, \ldots, y_8) \xrightarrow{s_i} (y_1, \ldots, y_8) - \frac{1}{2} (-y_1 - y_2 - y_3 - y_4) \times (-1, -1, -1, -1, 1, 1, 1, 1).
\]

Let \(a_{ij} = y_i + y_j\); then as invariants we may choose 
\[
\psi_m = \sum_{i<j} a_{ij}^m, \ m = 2, 6, 8, 10, 12, 14, 18.
\]

To show the independence of these functions, we have computed the value of their Jacobian at particular values of the \(y_i\)'s. For the cases of \(E_7\) and \(E_8\), we have computed the values of the invariants at particular values on an IBM-370 and found them to be non-zero.

(ix) \(E_8\): In \(\mathbb{R}^8\), a fundamental root system of this algebra can be chosen as 
\[
\{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_4 - e_5, \alpha_5 = e_5 - e_6, \\
\alpha_6 = e_6 + e_7, \alpha_7 = e_8 - \frac{1}{2}e(8), \alpha_9 = e_6 - e_7\}.
\]

Then \(s_{a_1} = (r_1 r_2), s_{a_2} = (r_2 r_3), s_{a_3} = (r_3 r_4), s_{a_4} = (r_4 r_5), s_{a_5} = (r_5 r_6), s_{a_6} = (r_6 r_7), s_{a_7} = - (r_7 r_8)\) and 
\[
(r_1, \ldots, r_8) \xrightarrow{s_{a_7}} (r_1, \ldots, r_8) - \frac{1}{4} (-r_1 - \ldots - r_7 + r_8) \times (-1, \ldots, -1, 1).
\]

The 240 linear forms 
\[
r_i + r_j, r_i - r_j, -r_i - r_j \quad (i \neq j)
\]

\[
\sum_{i=1}^{8} \epsilon_i r_i, \quad (\epsilon_i = \pm 1, \prod_{i=1}^{8} \epsilon_i = -1) \text{ permute.}
\]

The Cartan matrix is 
\[
\begin{bmatrix}
2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & \\
& -1 & 2 & -1 & & & & \\
& & -1 & 2 & -1 & & & \\
& & & -1 & 2 & -1 & -1 & \\
& & & & -1 & 2 & -1 & \\
& & & & & -1 & 2 & \\
& & & & & & -1 & 2
\end{bmatrix}.
\]
Let
\[ y_1 = 2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_8 \]
\[ y_2 = 2x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_8 \]
\[ y_3 = 2x_3 + 2x_4 + 2x_5 + x_6 + x_8 \]
\[ y_4 = 2x_4 + 2x_5 + x_6 + x_8 \]
\[ y_5 = 2x_5 + x_6 + x_8 \]
\[ y_6 = x_8 + x_8 \]
\[ y_7 = x_6 + x_8 \]
\[ y_8 = 2x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 + 7x_6 + 4x_7 + 5x_8. \]

Then it can be verified that \( s_1 = (y_1y_2), s_2 = (y_2y_3), s_3 = (y_3y_4), s_4 = (y_4y_5), s_5 = (y_5y_6), s_6 = (y_6y_7), s_8 = -(y_8y_7), \) and
\[ (y_1, \ldots, y_8) \mapsto (y_1, \ldots, y_8) - \frac{1}{4} \left( -y_1 - y_2 - \ldots - y_7 + y_8 \right) \]
\[ \times (-1, \ldots, -1, 1). \]

Thus we may choose as invariants the functions
\[ \psi_m = 2 \sum_{i<j} \{ (y_i + y_j)^m + (y_i - y_j)^m \} + \sum_{\epsilon_{i_1} = -1, \epsilon_{i_i} = 1} \left( \sum \epsilon_{i} x_i \right)^m, \]
\[ m = 2, 8, 12, 14, 18, 20, 24, 30. \]

Independence of these functions can be shown as in the case of \( E_7. \)

4. Conclusion. For the explicit construction of complete set of generators of \( G, \) we mention a process of lifting invariant polynomials of \( W \) to invariants of the adjoint group as outlined in [6, p. 63-5]. It follows from a result in [7] that invariant polynomials of the adjoint group are closely related to \( G. \) In a future paper, we plan to follow these ideas and investigate the constructions of generators arising from invariant polynomials of \( W \) for all algebras discussed in this paper.

References

5. V. S. Poppov and A. M. Perelomov, Casimir operators for semisimple Lie groups, Math. USSR-Izv. 2 (1968), 1313–1335.

Simon Fraser University,
Burnaby, British Columbia