ON A THEOREM OF LE ROUX

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1. The Theorem of Le Roux. Let $U(x, y, \alpha)$ be any solution of the linear hyperbolic differential equation

$$(1) \qquad L(u) \equiv u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u = 0,$$

containing a parameter α. J. Le Roux **(5)** has shown that the function $u(x, y)$ defined by

$$(2) \qquad u(x, y) = \int_{\alpha_0}^{x} U(x, y, \alpha) f(\alpha)\, d\alpha \qquad \alpha_0 = \text{const.}$$

with f arbitrary, is also a solution of (1) provided that

$$(3) \qquad \frac{\partial U}{\partial y} + a(x, y)\, U = 0, \qquad \alpha = x.$$

In his proof Le Roux assumes $U(x, y, \alpha)$ is sufficiently regular in the region of interest, so that (2) may be differentiated under the integral sign. However, this last assumption is not fulfilled by the particular functions U used, for example, by Le Roux, Darboux **(**2, pp. 54ff.**)** and others in the theory of the Euler-Poisson equation $E(\beta, \beta')$:

$$(4) \qquad u_{xy} - \frac{\beta'}{x-y} u_x + \frac{\beta}{x-y} u_y = 0.$$

In this particular case of (1), as is easily verified, the function

$$U(x, y, \alpha) = (x-\alpha)^{-\beta} (\alpha-y)^{-\beta'},$$

which for $\beta > 0$ is singular on $x = \alpha$, is a solution of (4) satisfying (3) (in a limiting sense), and may be used, for $\beta < 1$, to generate other solutions of (4) by means of (2). Integrals (2) with "singular" U occur also in a discussion of Bergmann's integral operator method by Diaz and Ludford **(3)**. Integrals of a similar type again appear in a recent paper by Weinstein **(6)**, which is devoted to the solution of a problem which includes, as special cases, the radiation problem for the wave equation and a problem of Tricomi, Germain, and Bader.

It is therefore of interest to give a general theorem which completes that of Le Roux so as to cover certain cases, such as the one just mentioned, of singular functions $U(x, y, \alpha)$.

The theorem given below can also be considered as an extension of a theorem of Bergmann **(1)** to the case when the kernels involved become singular, but

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the integration is still taken along the real axis. It is possible to base a proof of the present theorem on Bergmann's theorem, but then the simplicity of Le Roux's original idea of employing the classical superposition principle of linear equations is obscured. The generalisations discussed at the conclusion of this paper apparently have no counterpart in Bergmann's theory.

2. An Extension of the theorem of Le Roux. *Let R be an open region of the x, y plane where the coefficients of* (1) *are continuous, and let $U(x, y, \alpha) = \tilde{U}(x, y, \alpha)/(x-\alpha)^{\lambda}$, with $0 < \lambda < 1$, be a solution of $L(u) = 0$ for all (x, y) in R, wherever $\alpha_0 \leqslant \alpha < x$. Also let U be such that $\tilde{U}$, $\tilde{U}_x$, $\tilde{U}_y$, $\tilde{U}_{xy}$, $\tilde{U}_\alpha$, $\tilde{U}_{\alpha x}$, $\tilde{U}_{\alpha y}$, $\tilde{U}_{\alpha xy}$ are continuous on the three dimensional region consisting of all (x, y, α) with (x, y) in R and $\alpha_0 \leqslant \alpha \leqslant x$. Then*

$$u(x, y) = \int_{\alpha_0}^{x} \frac{\tilde{U}(x, y, \alpha)}{(x-\alpha)^{\lambda}} f(\alpha)\, d\alpha \tag{5}$$

is, for an arbitrary (*once continuously differentiable*) *function $f(\alpha)$, a solution of* (1) *in R.*

Proof. It suffices to consider $f(\alpha) \equiv 1$, since otherwise $f(\alpha)$ may be absorbed into $\tilde{U}$. Notice first that the Le Roux condition (3) is apparently missing from the statement of the theorem. However, on substitution into (1), one obtains (for $\alpha \neq x$)

$$L(U) = L(\tilde{U}/(x-\alpha)^{\lambda}) = \frac{1}{(x-\alpha)^{\lambda+1}}[(x-\alpha)\, L(\tilde{U}) - \lambda(\tilde{U}_y + a\tilde{U})] = 0. \tag{6}$$

Therefore, since $\lambda \neq 0$,

$$\tilde{U}_y + a\tilde{U} = \frac{x-\alpha}{\lambda} L(\tilde{U}), \qquad \alpha \neq x. \tag{7}$$

But each side of this equation is, from the hypotheses made about $\tilde{U}$, continuous as $\alpha \to x$. Hence $\tilde{U}_y + a\tilde{U}$ vanishes at least as fast as $(x-\alpha)$ as $\alpha \to x$, which implies that $U_y + aU$ vanishes at least as fast as $(x-\alpha)^{1-\lambda}$, or that (3) is satisfied in the limit as $\alpha \to x$, i.e.,

$$\lim_{\alpha \to x} \left[\frac{\partial U}{\partial y}(x, y, \alpha) + a(x, y)\, U(x, y, \alpha)\right] = 0.$$

Before differentiating, it is necessary to integrate (5) by parts, to obtain, with $f(\alpha) \equiv 1$,

$$u(x, y) = \frac{(x-\alpha_0)^{1-\lambda}\, \tilde{U}(x, y, \alpha_0)}{1-\lambda} + \int_{\alpha_0}^{x} \frac{(x-\alpha)^{1-\lambda}}{1-\lambda} \tilde{U}_\alpha(x, y, \alpha)\, d\alpha.$$

With $u(x, y)$ in this form, the partial derivatives u_x, u_y, u_{xy} can be computed by direct differentiation under the integral sign, and

$$\begin{aligned} L(u) \equiv{} & \frac{(x-\alpha_0)^{1-\lambda}}{1-\lambda}\left[L(\tilde{U}) + \frac{1-\lambda}{x-\alpha_0}(\tilde{U}_y + a\tilde{U})\right]_{\alpha=\alpha_0} \\ & + \int_{\alpha_0}^{x} \frac{(x-\alpha)^{1-\lambda}}{1-\lambda}\left[\frac{\partial L(\tilde{U})}{\partial \alpha} + \frac{1-\lambda}{x-\alpha}\cdot\frac{\partial}{\partial \alpha}(\tilde{U}_y + a\tilde{U})\right] d\alpha\,. \end{aligned}$$

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But, from (7),

$$L(\tilde{U}) = \frac{\lambda}{x-\alpha}(\tilde{U}_y + a\tilde{U}), \qquad \alpha \neq x,$$

so that

$$\begin{aligned} L(u) &= \frac{(x-\alpha_0)^{-\lambda}}{1-\lambda}[\tilde{U}_y + a\tilde{U}]_{\alpha=\alpha_0} \\ &\quad + \int_{\alpha_0}^{x} \frac{(x-\alpha)^{1-\lambda}}{1-\lambda}\left[\lambda \frac{\partial}{\partial\alpha}\left(\frac{\tilde{U}_y + a\tilde{U}}{x-\alpha}\right) + \frac{1-\lambda}{x-\alpha}\frac{\partial}{\partial\alpha}(\tilde{U}_y + a\tilde{U})\right] d\alpha \\ &= \frac{(x-\alpha_0)^{-\lambda}}{1-\lambda}[\tilde{U}_y + a\tilde{U}]_{\alpha=\alpha_0} + \int_{\alpha_0}^{x}\frac{\partial}{\partial\alpha}\left[\frac{(x-\alpha)^{-\lambda}}{1-\lambda}(\tilde{U}_y + a\tilde{U})\right] d\alpha \\ &= 0, \end{aligned}$$

since $\lim\limits_{\alpha\to x}[U_y + aU] = 0$.

3. Generalisations. The theorem is more general than would appear from the special form of U employed. For, in any region where the coefficients a, b, c of (1) are regular in the sense of Hadamard (4, p. 11), a theorem of Le Roux and Delassus (4, p. 72) shows that any singular curve of a singular solution $U(x, y, \alpha)$ of (1)—certain assumptions being made as to the nature of the singularity—must be a characteristic of (1), whose position, in the present case, will depend upon α, that is, $x = c(\alpha)$ or $y = \bar{c}(\alpha)$. Assuming an algebraic type of singularity on such curves, a fairly general definite integral, with variable limits, to be considered as a possible solution of (1), is

$$(8) \qquad u(x,y) = \int_{\alpha_0}^{\theta(x,y)} \frac{\tilde{U}(x,y,\alpha) f(\alpha)}{[x-c(\alpha)]^{\lambda}[y-\bar{c}(\alpha)]^{\lambda'}} d\alpha, \qquad 0 < \lambda, \lambda' < 1,$$

where $\theta(x, y) > \alpha_0 =$ const. for (x, y) in R. Suppose that the curve $c^{-1}(x) = \bar{c}^{-1}(y)$ does not intersect[1] R, that the integrand in (8) is a one parameter family of solutions of (1) with $\tilde{U}$ regular, in a sense analogous to that in the theorem, and that (8) is to be a solution of (1). For (x, y) in any given sufficiently small neighborhood lying in R and not intersecting either $c^{-1}(x) = \theta(x, y)$ or $\bar{c}^{-1}(y) = \theta(x, y)$, the integral in (8) is easily seen to be the sum of integrals of the form

$$(9) \qquad \left\{ \begin{array}{l} \displaystyle\int_{\alpha_1}^{\theta(x,y)} \tilde{U}_1(x,y,\alpha) f(\alpha)\, d\alpha, \\ \displaystyle\int_{\alpha_2}^{x} \frac{\tilde{U}_2(x,y,\alpha)}{(x-\alpha)^{\lambda}} f(\alpha)\, d\alpha, \\ \displaystyle\int_{\alpha_3}^{y} \frac{\tilde{U}_3(x,y,\alpha)}{(y-\alpha)^{\lambda'}} f(\alpha)\, d\alpha, \end{array} \right.$$

where $\alpha_1, \alpha_2, \alpha_3$ are suitable constants, and $\tilde{U}_1, \tilde{U}_2, \tilde{U}_3$ are sufficiently regular (see theorem) in their respective domains of definition. On forming $L(u)$ from

[1]Although the curve may be part of the boundary of R.

(8) the integrals of the last two types in (9) give zero, according to the theorem. The remaining integral is of the form discussed by Le Roux, who showed that $\theta(x, y)$ must be either a function of x alone or of y alone if the operation L is to produce zero. Thus our theorem shows that if (8) is to be a solution of (1), then $\theta(x, y)$ must be a function of x or y alone, and the Le Roux condition corresponding to (3) is to be satisfied.

Similar considerations apply to definite integrals with both lower and upper limits of integration variable, such as were discussed by Le Roux; and also for λ or λ' zero in (8).

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