The Absolute Summability of Fourier Series.

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§ 1. *Absolute summability* (A).

An elementary proposition states that an absolutely convergent series is convergent, *i.e.* that if

$$|s_1 - s_0| + |s_2 - s_1| + \ldots + |s_n - s_{n-1}| < K, \text{ (all } n)$$

then

$$s_n \to \text{a limit, as } n \to \infty .$$

This is the analogue for series of the theorem on functions that if a function $f(x)$ is of bounded variation in an interval, the limits $f(x \pm 0)$ exist at every point. Consider, in particular, the function

$$(1) \qquad f(x) = \sum_0^\infty a_n x^n$$

the series being supposed convergent in $(0 \leqslant x < 1)$. Then the theorem states that the Abel limit

$$\lim_{x \to 1-0} f(x)$$

exists provided that $f(x)$ is of bounded variation in $(0, 1)$ so that

$$(2) \qquad \sum_{r=1}^{m} |f(x_r) - f(x_{r-1})| < K$$

for all subdivisions $0 = x_0 < x_1 < x_2 < \ldots . < x_m < 1$.

If the Abel limit exists Σa_n is said to be *summable* (A). It is therefore natural to say that Σa_n is *absolutely summable* (A) if $f(x)$ is of bounded variation in $(0, 1)$. It will then be true that a series which is *absolutely summable* (A) is also *summable* (A).

By Abel's classical theorem, every convergent series is summable (A). It is easy to prove that every absolutely convergent series is absolutely summable (A). For the condition (2) is equivalent to

$$(3) \qquad \int_0^1 |f'(x)|\, dx \qquad \text{exists}$$

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and if Σa_n is absolutely convergent

$$\int_0^{x_1} |f'(x)|\, dx \leqslant \int_0^{x_1} \{\sum_1^\infty n\, |a_n|\, x^{n-1}\}\, dx, \qquad (0 \leqslant x_1 < 1)$$
$$= \sum_1^\infty |a_n|\, x_1^n \leqslant \sum_1^\infty |a_n|$$

and (3) is satisfied.

§ 2. *Absolute summability* (A) *of Fourier series.*

The convergence of a Fourier series at a point depends only on the values of the function in the immediate neighbourhood of the point, but this is not true of absolute convergence. Thus it is known[1] that unless a function is continuous in the whole interval $(-\pi, \pi)$, its Fourier series can only converge absolutely at the points of a set of measure zero. It is however true of absolute summability (A), and it will be shown that the Fourier series of an integrable function $f(\theta)$ is absolutely summable (A) at every point at which Dini's condition is satisfied, in particular at every point at which $f'(\theta)$ exists. Thus, let

$$\tag{4} \tfrac{1}{2} a_0 + \sum_{n=1}^\infty (a_n \cos n\theta + b_n \sin n\theta)$$

be the Fourier series of a function $f(\theta)$ which has a Lebesgue integral in $(-\pi, \pi)$ and let

$$\phi(t) = \frac{f(\theta + 2t) + f(\theta - 2t) - 2l}{2}$$

Then, (4) *is absolutely summable* (A) *to sum* l *if*

$$\int_0^\delta \left| \frac{\phi(t)}{t} \right| dt$$

exists.

In other words, every Fourier series which converges in virtue of Dini's condition is absolutely summable (A).

The Poisson series[2] (convergent for $0 \leqslant x < 1$, since $a_n, b_n \to 0$) is

$$P(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^\infty x^n (a_n \cos n\theta + b_n \sin n\theta)$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a) \frac{1 - x^2}{1 - 2x \cos(\theta - a) + x^2}\, da$$

so that, writing $a = \theta + 2t$,

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[1] Hobson. *Functions of a Real Variable.* **2** (1926), 550.

[2] *ibid.*, 629.

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$$Q(x) = P(x) - l = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \phi(t)\, \frac{1 - x^2}{1 - 2x\cos 2t + x^2}\, dt.$$

Then the total variation of $Q(x)$ in $(0, x_1)$ is

$$\int_0^{x_1} |Q'(x)|\, dx = \frac{2}{\pi}\int_0^{x_1} \left| \int_0^{\frac{\pi}{2}} \phi(t) \frac{d}{dx}\left\{ \frac{1 - x^2}{1 - 2x\cos 2t + x^2} \right\} dt \right| dx$$

$$\leqslant \frac{4}{\pi}\int_0^{x_1} dx \int_0^{\frac{\pi}{2}} |\phi(t)| \left| \frac{(1 + x^2)\cos 2t - 2x}{(1 - 2x\cos 2t + x^2)^2} \right| dt$$

$$\leqslant \frac{2}{\pi}\int_0^{\frac{\pi}{2}} |\phi(t)|\, V(x_1, t)\, dt$$

inverting the order of integration[1], where

$$V(x_1, t) = 2\int_0^{x_1} \left| \frac{(1 + x^2)\cos 2t - 2x}{(1 - 2x\cos 2t + x^2)^2} \right| dx.$$

Let $t_1 = \frac{1}{2}\arccos \frac{2x_1}{1 + x_1^{\,2}} = \operatorname{arccot} x_1 - \frac{\pi}{4}, \qquad \left(0 < t_1 < \frac{\pi}{4}\right).$

Then if $0 \leqslant t \leqslant t_1$, and $p(x) = \frac{1 - x^2}{1 - 2x\cos 2t + x^2}$,

$$V(x_1, t) = 2\int_0^{x_1} \frac{(1 + x^2)\cos 2t - 2x}{(1 - 2x\cos 2t + x^2)^2}\, dx$$

$$= \int_0^{x_1} p'(x)\, dx,$$

$$= p(x_1) - 1 < p(x_1)$$

$$\leqslant \frac{\sin 2t_1}{1 - \cos 2t_1} = \cot t_1 < \frac{\pi}{2t_1} \leqslant \frac{\pi}{2t};$$

while if $t_1 \leqslant t \leqslant \frac{\pi}{4}$,

$$V(x_1, t) = \int_0^{\cot(t + \frac{\pi}{4})} p'(x)\, dx - \int_{\cot(t + \frac{\pi}{4})}^{x_1} p'(x)\, dx$$

$$= \frac{2}{\sin 2t} - 1 - p(x_1) < \frac{2}{\sin 2t}$$

$$\leqslant \frac{\pi}{2t}.$$

[1] *Op. cit.*, **1** (1921), 577.

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Finally, if $\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{2}$,

$$V(x_1, t) = -\int_0^{x_1} p'(x)\,dx < 1.$$

Thus

$$\int_0^{x_1} |Q'(x)|\,dx \leqslant \frac{2}{\pi}\left(\int_0^{t_1} + \int_{t_1}^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right) |\phi(t)|\, V(x_1, t)\,dt$$

$$< \int_0^{\frac{\pi}{4}} \frac{|\phi(t)|}{t}\,dt + \frac{2}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} |\phi(t)|\,dt$$

so that $Q(x)$ is of bounded variation in $(0, 1)$.

Dini's condition[1] is not necessary for absolute summability (A), nor even for absolute convergence. For the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos 2^n x}{n^{\frac{3}{2}}}$$

converges absolutely for all real values of x, but does not satisfy Dini's condition at $x = 0$. To prove this, take $\delta = \pi\, 2^{-m}$, where m is an integer. Then

$$\int_\delta^\pi \left|\frac{\phi(t)}{t}\right| dt = \int_\delta^\pi \left\{\sum_{n=1}^{\infty} \frac{1 - \cos 2^{n+1} t}{n^{\frac{3}{2}}}\right\} \frac{dt}{t}$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \int_\delta^\pi \frac{\sin^2 2^n t}{t}\,dt$$

$$> 2 \sum_{n=m+1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \int_{\pi 2^{n-m}}^{\pi 2^n} \frac{\sin^2 u}{u}\,du$$

$$> K \sum_{n=m+1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \log 2^m > K\sqrt{m}$$

where K is an absolute constant.

Again, a Fourier series may converge at a point without being absolutely summable (A). This is shown by the following example, suggested to me by Professor Littlewood. Let

$$\frac{z(1-z)^i}{\log \dfrac{1}{1-z}} = \sum_{n=0}^{\infty} c_n z^n .$$

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[1] The remainder of the paper was added, 14th August 1929, to answer two questions raised by Dr Copson.

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It is clear that the series converges for $|z| < 1$. Now let Γ_n denote the contour in the z-plane consisting of the straight lines joining successive points $1 + n^{-1}$, n, $n + ni$, $-n + ni$, $-n - ni$, $n - ni$, n, $1 + n^{-1}$, together with a circle of radius n^{-1} round $z = 1$. Then

$$c_n = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{(1-z)^i}{z^n \log \dfrac{1}{1-z}} \, dz$$

and a straightforward calculation shows that

$$n\, c_n \to 0.$$

Moreover

$$\lim_{r \to 1-0} \frac{r\, e^{i\theta} (1 - r\, e^{i\theta})^i}{\log \dfrac{1}{1 - r\, e^{i\theta}}}$$

exists for each θ. Thus, by Tauber's theorem,

$$\sum_{n=0}^{\infty} c_n\, e^{ni\theta}$$

converges for each θ. It is easy to prove that the series is not absolutely summable (A) for $\theta = 0$.

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