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ON STABLE DIFFEOMORPHISM OF EXOTIC SPHERES IN THE METASTABLE RANGE

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1. Introduction. Let Θ_n^{p+1} denote the subgroup of the Kervaire-Milnor group θ_n consisting of those n-spheres which imbed with trivial normal bundle in Euclidean $(n + p + 1)$-space, $n < 2p$. It is known that such imbeddings always exist [**6**], and that the normal bundle is independent of the imbedding [**10**]. Following [**2**], we write $\Omega_{n,p}$ for the quotient θ_n/Θ_n^{p+1}.

The order of $\Omega_{n,p}$, after identifying each element with its inverse, is equal to the number of diffeomorphically distinct (orientation preserved) $\Sigma^n \times S^p$ [**2**; **5**]. Indeed, $\Omega_{n,p}$ is closely linked to the problem of determining the number of smooth structures $\alpha(n, p)$ on $S^n \times S^p$. For instance, if $\Omega_{n,p} = 0$ then $\alpha(n, p)$ equals the order of θ_{n+p} [**5**]. Specific results are easily read off Table I and Theorem 2.1.

In the metastable range, computation of the order of $\Omega_{n,p}$ is reducible to an effectively computable homotopy question. Our results are stated in Section 2 along with preliminaries. The remaining sections of the paper deal with explicit computations.

2. Statement of results and preliminaries. From [**10**] it is immediate that $\Omega_{n,p} = 0$ for $p \geqq n - 3$ or $n \leqq 15$, $n < 2p$, as well as $\Omega_{16,12} = z_2$. The following theorem is an extension of these results.

THEOREM 2.1. *If* $\Omega_{n,p} \neq 0$, *then*

$$p \leqq \begin{cases} n - 4 & \text{if } n \equiv 0(8) \\ n - 7 & \text{if } n \equiv 1(8) \\ n - 8 & \text{if } n \equiv 2, 3, 6, 7(8) \\ n - 15 & \text{if } n \equiv 4, 5(8). \end{cases}$$

We compute the following table.

All groups not shown are trivial. Table I shows that Theorem 2.1 is best possible for $n \equiv 0, 1, 2, 5(8)$.

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Table I

n/p	9	10	11	12	$p \geqq 13$
16	z_2	z_2	z_2	z_2	0
17	z_2	z_2	0	0	0
18		z_2	0	0	0
19		0	0	0	0

n/p	17	18	·	·	26	27	28	$p \geqq 29$
32	z_2	z_2	·	·	z_2	z_2	z_2	0
33	z_2	z_2	·	·	z_2	0	0	0
34		z_2	·	·	z_2	0	0	0

n/p	19	20	21	22	$p \geqq 23$
37	z_2	z_2	z_2	z_2	0
38		z_2	0	0	0

$\Omega_{n,p}$ for $n \leqq 40$, $n < 2p$

Let $\phi_n^{p+1} : \theta_n \to \pi_{n-1}(SO(p+1))$ denote the homomorphism which assigns to each $\Sigma^n \in \theta_n$ the characteristic class of its (unique) normal bundle in codimension $p+1$, $n < 2p$. Then,

$$\Omega_{n,p} = \text{im } \phi_n^{p+1}. \tag{2.2}$$

Moreover, since normal bundles to homotopy spheres in Euclidean space are fibre-homotopy trivial [**18**] and stably trivial [**14**] we have

$$\Omega_{n,p} \subseteq \ker i_{n-1}^{p+1} \cap \ker J_{n-1}^{p+1} \tag{2.3}$$

where $i_{n-1}^{p+1} : \pi_{n-1}(SO(p+1)) \to \pi_{n-1}(SO)$ is induced by inclusion, and $J_{n-1}^{p+1} : \pi_{n-1}(SO(p+1)) \to \pi_{n+p}(S^{p+1})$ is the metastable J-homomorphism (see [**11**]). It follows from [**10**] that the inclusion of (2.3) can be improved to equality for $n \neq 2^a - 2$, a being a positive integer.

The main tools of our computations are the Barratt-Mahowald splitting theorem [**3**, Theorem 2], and from [**2**] the short exact sequence* ($n < 2p$; $n \neq 2^a - 2$)

$$0 \to bp_{n+1} \to \Theta_n^{p+1} \to \text{cok } J_n^{p+1} \to 0, \tag{2.4}$$

*Here bp_{n+1} denotes the subgroup of exotic spheres imbedding in R^{n+p+1} which bound parallelizable manifolds.

and the *PSH* diagram

$$(2.5)\quad \begin{array}{ccccccc} & & \pi_{n-1}(SO(p+1)) & \xrightarrow{\bar{\imath}_{n-1}{}^{p+1}} & \pi_{n-1}(SO(p+2)) & & \\ & \overset{\partial}{\nearrow} & \Big\downarrow J_{n-1}{}^{p+1} & & \Big\downarrow J_{n-1}{}^{p+2} & \overset{\rho_*}{\searrow} & \\ \pi_n(S^{p+1}) & & & & & & \pi_{n-1}(S^{p+1}) \\ & \underset{P}{\searrow} & & & & \underset{H}{\nearrow} & \\ & & \pi_{n+p}(S^{p+1}) & \xrightarrow[S]{} & \pi_{n+p+1}(S^{p+2}) & & \end{array}$$

Here, H is the Hopf homomorphism (see [**11**]); S is just suspension; the top sequence is a portion of the fibre-homotopy sequence of the fibering

$$SO(p+2) \overset{\rho}{\rightarrow} S^{p+1},$$

while the lower sequence is due to G. Whitehead and is exact for $n < 2p$; (2.5) commutes up to sign.

The following easily proved proposition is used throughout the paper.

PROPOSITION 2.6. *If* $\Omega_{n,p_0} = 0$, *then* $\Omega_{n,p} = 0$, *for all* $p \geqq p_0$, *for* $n < 2p$.

We shall also have occasion to use the following proposition.

PROPOSITION 2.7. $\Omega_{n,p}$ *is* 2-*primary in the metastable range*, $n < 2p$.

This follows directly from (2.3) and the well-known homotopy-theoretic fact that the finite part of ker $J_{n-1}{}^{p+1}$ is 2-primary in the metastable range.

3. Proof of (2.1). The proof falls naturally into four parts. We can suppose that $n \geqq 17$ throughout because results of [**10**] establish the theorem in the remaining cases $n \leqq 16$.

Part I. *The case* $n \equiv 0(8)$. This case follows directly from results stated previous to the statement of Theorem 2.1.

Part II. *The case* $n \equiv 1(8)$. From [**13**, p. 168] we have the short exact sequence

$$0 \rightarrow \pi_{8S+1}(V_{m,m-8S+i}) \rightarrow \pi_{8S}(SO(8S-i)) \rightarrow \pi_{8S}(SO(m)) \rightarrow 0$$

for large m and $i \leqq 6$, $S \geqq 2$. Let $i = 4$ and let $S \geqq 2$. Since

$$\Omega_{n,p} \subseteq \ker i_{n-1}{}^{p+1},$$

it follows from the above sequence that

$$\ker i_{8S}{}^{8S-4} = \pi_{8S+1}(V_{m,m-8S+4}).$$

But from [**9**], this group is trivial and thus $\Omega_{n,n-6} = 0$ for $n \equiv 1(8)$. This coupled with Proposition 2.6 concludes the proof of Part II.

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Part III. *The case* $n = 2, 3, 6, 7(8)$. The proof breaks into four cases.

(i) $n \equiv 2(8)$. From [**13**, p. 167] it follows that the sequence

$$0 \to \pi_{8S+2}(V_{m,m-8S+i}) \to \pi_{8S+1}(SO(8S - i)) \to \pi_{8S+1}(SO(m)) \to 0$$

is exact for m large and $i \leqq 4$, $S \geqq 2$. Set $i = 4$, and suppose that $S \geqq 2$. From [**9**], $\pi_{8S+2}(V_{m,m-8S+4}) = 0$. Hence, Bott periodicity [**4**] implies that the homomorphism

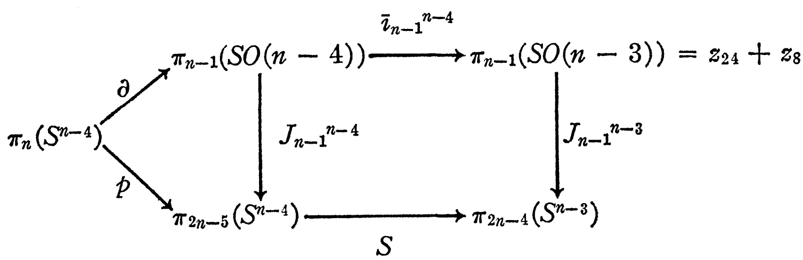
$$i_{8S+1}{}^{8S-4} : \pi_{8S+1}(SO(8S - 4)) \to \pi_{8S+1}(SO)$$

has trivial kernel. It follows that $\Omega_{n,n-7} = 0$ for $n \equiv 2(8)$. Proposition 2.6 completes this part of the proof.

(ii) $n \equiv 3(8)$. From Bott periodicity, $\pi_{n-1}(SO) = 0$, for $n \equiv 3(8)$. Therefore, using the isomorphism $\Omega_{n,p} = \ker i_{n-1}{}^{p+1} \cap \ker J_{n-1}{}^{p+1}$, $n \neq 2^a - 2$, $n < 2p$ it follows from [**10**] that $\ker J_{n-1}{}^{n-2} = 0$.

We wish to show that $J_{n-1}{}^{n-3}$ restricted to $\pi_{n-1}(SO(n - 3))/\mathrm{im}\,\partial$ is a monomorphism. In the *PSH* diagram (3.3), $\pi_n(S^{n-3}) = z_{24}$, $\pi_{n-1}(SO(n - 3)) = z_8 + z_{24}$, and $\pi_{n-1}(SO(n - 2)) = z_8$ [**13**] and neither $\mathrm{im}\,\partial$ nor $\mathrm{im}\,P$ vanishes. From exactness of the top sequence, the order of $\mathrm{im}\,(\partial)$ is greater than or equal to 24 so $\ker \partial = 0$. It follows that $\bar{\imath}_{n-1}{}^{n-3}$ is a monomorphism on $\pi_{n-1}(SO(n - 3))/\mathrm{im}\,\partial = z_8$ and since $\ker J_{n-1}{}^{n-2} = 0$, the desired result follows.

We wish to establish that $J_{n-1}{}^{n-4}$ is a monomorphism. Consider the *PSH* diagram

(3.4) 

Now, $\pi_n(S^{n-4}) = 0$ [**21**], and $\pi_{n-1}(SO(n - 4)) = z_8$ [**13**], so $\bar{\imath}_{n-1}{}^{n-4}$ is a monomorphism. Consider the fibre-homotopy sequence

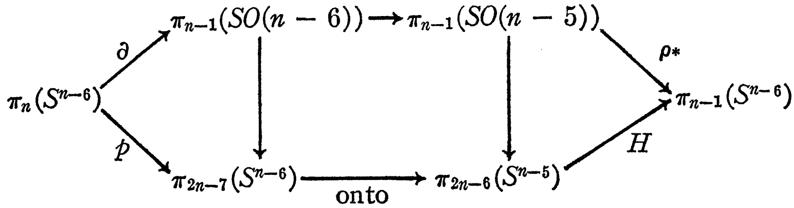
$$\to \pi_n(V_{n-2,2}) \to \pi_{n-1}(SO(n - 4)) \xrightarrow{j_*} \pi_{n-1}(SO(n - 2)) \to$$

associated with the inclusion $j : SO(n - 4) \to SO(n - 2)$. From [**8**], we have $\pi_n(V_{n-2,2}) = z_2$ for $n \equiv 3(8)$. It follows that $\mathrm{im}\,j_* \neq 0$. We know that $j_* = \bar{\imath}_{n-1}{}^{n-3} \circ \bar{\imath}_{n-1}{}^{n-4}$ and that $\bar{\imath}_{n-1}{}^{n-4}$ is a monomorphism. If

$$\mathrm{im}\,\bar{\imath}_{n-1}{}^{n-4} \subseteq \mathrm{im}\,\partial = z_{24} \quad \text{where} \quad \partial : \pi_n(S^{n-3}) \to \pi_{n-1}(SO(n - 3)),$$

then the exactness of (3.3) would give $\mathrm{im}\,j_* = 0$, hence $\mathrm{im}\,\bar{\imath}_{n-1}{}^{n-4}$ is isomorphic to $\pi_{n-1}(SO(n - 3))/\mathrm{im}\,\partial$. But, as we established above, $J_{n-1}{}^{n-3}$ restricted to this subgroup is a monomorphism and it follows that $\ker J_{n-1}{}^{n-4} = 0$.

Since $\pi_n(S^{n-5}) = 0$, the *PSH* diagram for $J_{n-1}{}^{n-5}$ and $J_{n-1}{}^{n-4}$ shows that $\ker J_{n-1}{}^{n-5} = 0$. Consider the diagram

(3.5) 

for $n \equiv 3(8)$. From [**3**; **9**; **22**] we have

$$\pi_{n-1}(SO(n-6)) = z_8, \quad \pi_{n-1}(SO(n-5)) = z_8 \quad \text{and} \quad \pi_{n-1}(S^{n-6}) = 0.$$

From exactness of the top sequence, we have $\ker J_{n-1}{}^{n-5} = 0$, implying that $\ker J_{n-1}{}^{n-6} = 0$ as desired. This completes the proof of (ii).

(iii) $n \equiv 6(8)$. From the tables [**9**] and metastable splitting we have $\pi_{n-1}(SO(n-6)) = z_2 + z_2$ and $\pi_{n-1}(SO(n-5)) = z_2$. Exactness implies that $\operatorname{im} \partial \neq 0$. It is known that $\operatorname{im} P = z_2$ in this case. It follows that $\ker J_{n-1}{}^{n-5} = 0$ implies $\ker J_{n-1}{}^{n-6} = 0$ which is the desired conclusion. In order to prove that $J_{n-1}{}^{n-5}$ is a monomorphism for $n \equiv 6(8)$ first recall that $\pi_{n-1}(SO) = 0$ and $\Omega_{n,n-3} = 0$ imply $\ker J_{n-1}{}^{n-2} = 0$ and then use [**9**], [**22**] and the three successive *PSH* diagrams to establish $\ker J_{n-1}{}^{n-5} = 0$. The arguments are particularly easy and we omit them.

(iv) $n \equiv 7(8)$. In the metastable range we have $n = 15, 23, \ldots$. The case $n = 15$ was settled in [**10**] while the case $n = 23$ is dealt with in the last part of this paper, where it is proved that $\Omega_{23,12} = 0$. For general n, $n \equiv 7(8)$, the result follows from 2.3 above and comparison of Table 4.2 and Table 4.1 in [**16**]. This last determines the appropriate J-homomorphism kernels.

Part IV. *The cases* $n \equiv 4, 5(8)$. From (2.3), $\Omega_{n,p} \subseteq \ker i_{n-1}{}^{p+1}$. But the Barratt-Mahowald splitting theorem [**3**] gives $\ker i_{n-1}{}^{p+1} = \pi_n(V_{2(p+1),p+1})$ for $p \geqq 12$. Therefore, $\Omega_{n,n-14} \subseteq \pi_n(V_{2(n-13),n-13})$ for $n \geqq 25$. But this group vanishes for $n \equiv 4(8)$ and furthermore the requirement $n < 2p$ becomes in this case $n \geqq 29$. For the case $n \equiv 5(8)$, $\pi_n(V_{2(n-13),n-13}) = z_3$ and we use Proposition 2.7 to obtain the result. The proof of Theorem 2.1 is therefore complete.

4. Calculation of Table I. From [**10**], $\Omega_{16,12} = z_2$ and $\Omega_{16,p} = 0$ for $p \geqq 13$. Since $\theta_{16} = z_2$, it follows from Proposition 2.6 that $\Omega_{16,p} = z_2$ for $9 \leqq p \leqq 12$. This establishes the results of the first row of Table I.

PROPOSITION 4.1. $\Omega_{17,10} = z_2$.

Proof. From [**13**, p. 168, II. 10] we have the short exact sequence

$$0 \to \pi_{8S+1}(V_{m,m-8S+i}) \to \pi_{8S}(SO(8S-i)) \to z_2 \to 0 \tag{4.2}$$

for $i \leqq 6$, $S \geqq 2$ and $m \geqq 8S + 2$, using Bott periodicity $\pi_{8S}(SO(m)) = \pi_{8S}(SO) = z_2$. Letting $S = 2$, $m = 19$, and $i = 5$ the sequence becomes

$$0 \to \pi_{17}(V_{19,8}) \to \pi_{16}(SO(11)) \to z_2 \to 0.$$

We are not able to obtain the middle group from tables of [**9**] directly. However, from [**8**], $\pi_{17}(V_{19,8}) = z_2$ and it follows from (4.2) that $\pi_{16}(SO(11))$ has order 4. On the other hand, $\pi_{16}(SO(12)) = z_2$ [**13**] and from [**22**] $\pi_{16}(S^{11}) = 0$, $\pi_{16+11}(S^{11}) = z_2$, and $\pi_{16+12}(S^{12}) = z_2$ evaluate groups in the *PSH* diagram below.

$$\begin{array}{ccccc} \pi_{16}(SO(11)) & \xrightarrow{\bar{\imath}_{16}{}^{11}} & \pi_{16}(SO(12)) & \xrightarrow{\rho_*} & \pi_{16}(S^{11}) \\ \downarrow J_{16}{}^{11} & & \downarrow J_{16}{}^{12} & \nearrow H & \\ \pi_{16+11}(S^{11}) & \longrightarrow & \pi_{16+12}(S^{12}) & & \end{array} \tag{4.3}$$

The proposition will be proved if we can show that $\ker J_{16}{}^{11} = \ker \bar{\imath}_{16}{}^{11} = z_2$, since $\Omega_{17,10} = \ker J_{16}{}^{11} \cap \ker i_{16}{}^{11}$ and $\ker \bar{\imath}_{16}{}^{11} \subseteq \ker i_{16}{}^{11}$.

From [**22**, p. 157], S is an isomorphism onto for the 2-primary parts. But, the odd primary groups vanish in our case, so S is an isomorphism in (4.3). Now, the sequence

$$0 \to bp_{17} \to \Theta_{16}{}^{12} \to \operatorname{cok} J_{16}{}^{12} \to 0$$

is exact and $\Theta_{16}{}^{12} = 0 = bp_{17}$. Therefore, $J_{16}{}^{12}$ is also an isomorphism onto. But $\pi_{16}(S^{11}) = 0$ [**21**], so commutativity of (4.3) gives the desired result.

PROPOSITION 4.4. $\Omega_{17,9} = z_2$.

Proof. It suffices to show that $\ker J_{16}{}^{10} = z_2$. Using $m = 19$, $S = 2$, $i = 6$ in the sequence (4.2) we obtain the exact sequence

$$0 \to \pi_{17}(V_{19,9}) \to \pi_{16}(SO(10)) \to z_2 \to 0.$$

From the table in [**8**], $\pi_{17}(V_{19,9}) = z_2 + z_3 + z_5 + z_{16}$, so $\pi_{16}(SO(10))$ has order $2^6 \cdot 3 \cdot 5$. Since $\Theta_{16}{}^{10} = 0$, the exactness of

$$0 \to bp_{17} \to \Theta_{16}{}^{10} \to \operatorname{cok} J_{16}{}^{10} \to 0$$

implies that $J_{16}{}^{10}$ is an epimorphism. But from [**21**], the order of $\pi_{16+10}(S^{10})$ is $2^5 \cdot 3 \cdot 5$ and hence $\ker J_{16}{}^{10} = z_2$, and the proof is complete.

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PROPOSITION 4.5. $\Omega_{18,10} = z_2$.

Proof. First note that the sequence

$$0 \to bp_{18} \to \Theta_{17}{}^{11} \to \operatorname{cok} J_{17}{}^{11} \to 0$$

is exact. Now the order of $\operatorname{cok} J_{17}{}^{11}$ is 4, by a simple calculation using $\Omega_{17,10} = z_2$, $\theta_{17} = z_2{}^{(4)}$, and $bp_{18} = z_2$ (see [**14**]). Since from [**21**], $\pi_{17+11}(S^{11}) = z_2{}^{(3)}$, it follows that $\operatorname{im} J_{17}{}^{11} = z_2$. It is known that

$$\pi_{17}(SO(11)) = \underset{\substack{\| \\ z_2}}{\pi_{17}(SO)} + \underset{\substack{\| \\ z_2}}{\pi_{18}(V_{22,11})}.$$

It follows that $\ker J_{17}{}^{11} = \pi_{18}(V_{22,11}) = z_2$ and thus that $\Omega_{18,10} = z_2$.

In order to complete the 4th row of Table I, it suffices to compute $\Omega_{19,10}$ and $\Omega_{19,11}$. The exactness of the sequence

$$0 \to bp_{19} \to \Theta_{18}{}^{12} \to \operatorname{cok} J_{18}{}^{12} \to 0$$

together with $\Omega_{18,11} = 0 = bp_{19}$ implies that

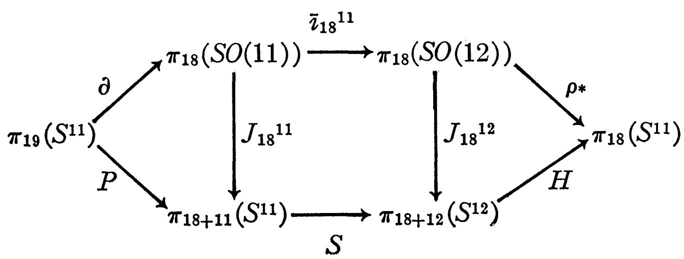
$$\Theta_{18}{}^{12} = \Theta_{18} = \operatorname{cok} J_{18} = z_2 + z_8$$

where J_{18} is the stable J-homomorphism, so $\operatorname{cok} J_{18}{}^{12} = z_2 + z_8$ (see [**14**]). But from [**9**], and Bott periodicity $\pi_{18}(SO(12)) = z_{240} + z_4$, while [**22**] gives

$$\pi_{18+12}(S^{12}) = z_{480} + z_4{}^{(2)} + z_2.$$

It follows that $\ker J_{18}{}^{12} = 0$ and thus that $\Omega_{19,11} = 0$.

It remains to show that $\Omega_{19,10} = 0$. Consider the *PSH* diagram



The pertinent groups are: $\pi_{19}(S^{11}) = z_2{}^{(2)}$, $\pi_{18+11}(S^{11}) = z_2 + z_4 + z_8$, $\pi_{18+12}(S^{12}) = z_2 + z_4{}^{(2)} + z_{480}$, and $\pi_{18}(S^{11}) = z_{240}$. The Whitehead product P vanishes [**21**, p. 165]. The short exact sequence

$$0 \to bp_{19} \to \Theta_{18}{}^{11} \to \operatorname{cok} J_{18}{}^{11} \to 0$$

implies that $\operatorname{im} J_{18}{}^{11}$ has order 8 since $\theta_{18} = z_2 + z_8$ and $\Omega_{18,10} = z_2$, $bp_{19} = 0$. Since $\pi_{18}(SO) = 0$, any element in $\ker J_{18}{}^{11}$ is stably trivial. Therefore, it will

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suffice to prove that the order of $\pi_{18}(SO(11))$ does not exceed 8. But this follows directly from the fibre-homotopy sequence of

$$SO(11) \xrightarrow{\rho} SO(22) \to V_{22,11},$$

noting that the order of $\pi_{19}(V_{22,11})$ is exactly 8 [**9**]. This completes computation of the 4th row of Table I.

We will compute $\Omega_{20,p}$ for $p \geqq 11$. We will show that

$$\ker J_{19}^{12} \cap \pi_{20}(V_{24,12}) = 0$$

from which it follows that $\Omega_{20,11} = 0$, which by Proposition 2.6 proves the desired result. Now, $\pi_{20}(V_{24,12}) = z_2^{(4)}$, $\pi_{19+12}(S^{12}) = z_{264} + z_2^{(5)}$, follows from [**9**] and [**22**], respectively. Since J_{19}^{12} restricted to $\pi_{19}(SO) = z$ has image z_{264} (see [**1**]), and $\Omega_{19,11} = 0$, it follows from 2.4, together with

$$bp_{20} = z_2, \ \theta_{19} = z_4,$$

that $\operatorname{cok} J_{19}^{12} = \operatorname{cok} J_{19} = z_2$, J_{19} being the stable J-homomorphism, and therefore that $\ker J_{19}^{12} \cap \pi_{20}(V_{24,12}) = 0$.

Consider the homomorphism $J_{20}^{12} : \pi_{20}(SO(12)) \to \pi_{20+12}(S^{12})$. One sees that $\pi_{20}(SO(12)) = z_2^{(5)}$ and $\pi_{20+12}(S^{12}) = z_{24} + z_2^{(5)}$ follows from [**9**; **22**]. The isomorphism $\pi_{20}(SO(12)) = \pi_{21}(V_{24,12})$ used results from the fibre-homotopy sequence of the fibering $SO(12) \to SO(24) \to V_{24,12}$ and Bott periodicity. From [**22**] we obtain $\theta_{20} = z_{24}$ and because $\Omega_{20,11} = 0$ implies that $\operatorname{cok} J_{20}^{12} = \operatorname{cok} J_{20} = z_{24}$ $(bp_{21} = 0)$, we obtain $\Omega_{21,11} = 0$, and hence $\Omega_{21,p} = 0$, $p \geqq 11$.

We will now show that $\Omega_{22,12} = 0$. First note that the order of θ_{21} is 8 and that $bp_{22} = z_2$. But, $\Omega_{21,12} = 0$ so that $\Theta_{21}^{13} = \theta_{21}$ and from the exactness of the sequence

$$0 \to bp_{22} \to \Theta_{21}^{13} \to \operatorname{cok} J_{21}^{13} \to 0$$

it follows that $\operatorname{cok} J_{21}^{13}$ has order 4. From [**9**], we have $\pi_{21}(SO(13)) = z_2 + z_4$, while $\pi_{21+13}(S^{13}) = z_4 + z_2^{(3)}$ comes from tables [**19**]. Clearly, $\ker J_{21}^{13} = 0$ follows, and we have proved the desired result, namely, $\Omega_{22,p} = 0$, $p \geqq 12$.

We wish now to show that $\Omega_{23,12} = 0$. First note that the order of θ_{22} is 4, that $bp_{23} = 0$, and that $\operatorname{cok} J_{22}^{13} = \Theta_{22}^{13} = \theta_{22}$. Since $\pi_{22}(SO(13)) = z_{16}$ [**9**], and $\pi_{22+13}(S^{13}) = z_{16} + z_2^{(2)}$ [**19**], it follows that $\ker J_{22}^{13} = 0$, which yields the desired result. Consequently, $\Omega_{23,p} = 0$, $p \geqq 12$.

The remainder of the results of Table I may be derived from [**16**] by comparing Table 4.1 and Table 4.2 in that paper.

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