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ON THE INVERSION OF FOURIER TRANSFORMS

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Let G be a locally compact abelian group, with character group $\hat{G}$. Let ψ be an arbitrary continuous real-valued homomorphism defined on $\hat{G}$. For f in $\mathcal{L}_p(G)$, $1 < p \leqslant 2$, let

$$M^{\#}f = \left|\sup_{\nu \in I}\left(\hat{f}1_{]-\nu,\nu[} \circ \psi\right)^{\check{}}\right|,$$

where $1_{]-\nu,\nu[}$ is the indicator function of the interval $]-\nu, \nu[$, and I is an unbounded increasing sequence of positive real numbers. Then there is a constant M_p, independent of f, such that $\left\|M^{\#}f\right\|_p \leqslant M_p \|f\|_p$. Consequently, the pointwise limit of the function $\left(\hat{f}1_{]-\nu,\nu[} \circ \psi\right)^{\check{}}$ exists, almost everywhere on G, as ν tends to infinity. Using this result and a generalised version of Riesz's theorem on conjugate functions, we obtain a pointwise inversion for Fourier transforms of functions on $\mathbb{R}^a \times \mathbb{T}^b$, where a and b are nonnegative integers, and on various other locally compact abelian groups.

1. Introduction

Notation. Throughout this paper we adhere to the following notation. The symbol G will denote a locally compact abelian group with character group $\hat{G}$. the Haar measure on G will be denoted by μ_G. When no confusion may arise, we will simply write μ. Lebesgue measure on $\mathbb{R}$ will be denoted by λ.

If A is a set and B is a subset of A, the complement of B in A will be denoted by $A \setminus B$. The indicator function of B will be denoted by 1_B; it is the function with values 1 on B, and 0 on $A \setminus B$.

Other standard notation used here without explanation is as in [**8**] and [**9**].

Theme of this paper. We wish to transfer known results on the real line to a new setting in higher dimensional or otherwise different spaces. The principal result that we wish to transfer is the fact that partial sums for the inversion of the Fourier transform of a function f in $\mathcal{L}_p(\mathbb{R})$, $1 < p \leqslant 2$, converge pointwise λ-almost everywhere. This is of course the celebrated Carleson–Hunt theorem on the pointwise convergence of Fourier series on $\mathbb{R}$.

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Our transference result is achieved in the following set-up. Let ψ be an arbitrary continuous real-valued homomorphism defined on $\hat{G}$. For f in $\mathcal{L}_p(G)$, $1 < p \leqslant 2$, let

$$M^{\#} f = \left| \sup_{\nu \in I} \left(\hat{f} 1_{]-\nu,\nu[} \circ \psi \right)^{\check{}} \right|,$$

where I is an unbounded and increasing sequence of positive real numbers. We show that there is a constant M_p, independent of f, such that $\left\| M^{\#} f \right\|_p \leqslant M_p \left\| f \right\|_p$. Consequently, the pointwise limit of the function $\left(\hat{f} 1_{]-\nu,\nu[} \circ \psi \right)^{\check{}}$ exists almost everywhere on G as ν tends to infinity. Using this result and a generalised version of Riesz's theorem on conjugate functions, we obtain a pointwise inversion for Fourier transforms of functions on $\mathbf{R}^a \times \mathbf{T}^b$, where a and b are nonnegative integers, and on various other locally compact abelian groups.

2. The Fourier transform on R

2.1. Given p in $[1, 2]$, f in $\mathcal{L}_p(\mathbf{R})$, and for λ-almost all $s \in \mathbf{R}$, we write:

$$\left\| f \right\|_p = \left(\int_{\mathbf{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}};$$
$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(y) \exp(-isy) dy,$$

where the integral converges in the sense of [**11**, Vol. 2, Theorem (3.14), p.257].

For every positive $\nu \in \mathbf{R}$, the function $s \mapsto (\sin \nu s)/s$ is in $\mathcal{L}_q(\mathbf{R})$, for all $q > 1$. Thus for f in $\mathcal{L}_p(\mathbf{R})$, $1 < p \leqslant 2$, the integral

$$\frac{1}{\pi} \int_{\mathbf{R}} f(x - y) \frac{\sin \nu y}{y} dy$$

is finite for all $x \in \mathbf{R}$. We denote this integral by $f^{\nu}(x)$.

The following are well-known facts about the function f:

(2.1.1) for f in $\mathcal{L}_p(\mathbf{R}) \cap \mathcal{L}_1(\mathbf{R})$ where $1 \leqslant p \leqslant 2, (f^{\nu})^{\wedge}(s) = \hat{f}(s) 1^{*}_{[-\nu,\nu]}(s)$,

where $1^{*}_{[-\nu,\nu]}(s) = 1_{[-\nu,\nu]}(s)$ for all $|s| \neq \nu$, and $1^{*}_{[-\nu,\nu]}(s) = 1/2$ for $|s| = \nu$;

(2.1.2) for f in $\mathcal{L}_p(\mathbf{R}) \cap \mathcal{L}_1(\mathbf{R})$ where $1 < p < \infty$, the inequality
$$\left\| f^{\nu} \right\|_p \leqslant A_p \left\| f \right\|_p$$

holds, where A_p depends only on p.

The following version of the Carleson–Hunt theorem follows from the classical version on the circle and [**10**, Theorem 1].

THEOREM 2.2. *Suppose that* f *is in* $\mathcal{L}_p(\mathbf{R})$, *where* $1 < p \leqslant 2$. *Let*

$$Mf(x) = \sup\{|f^{\nu}(x)| : \nu > 0\}$$

for all $x \in \mathbf{R}$. *Then*

$$\|Mf\|_p \leqslant M_p \|f\|_p,$$

where M_p *depends only on* p.

3. THE PARTIAL SUM OPERATOR ON G

A particular case 3.1. Throughout this section we suppose that ϕ is a topological isomorphism of $\mathbf{R}$ into G. For each $\nu \in \mathbf{R}^+$, and every continuous function f with compact support on G, let

$$S_\nu f(x) = \frac{1}{\pi}\int_{\mathbf{R}} f(x - \phi(t))\frac{\sin \nu t}{t}dt. \tag{3.1.1}$$

Note that, for all x in G, the function $t \mapsto f(x - \phi(t))$ is continuous on $\mathbf{R}$, with compact support. Hence the integral in (1) is finite for all x in G. The operator S_ν is well-defined, so far, on a dense subset of $\mathcal{L}_p(G)$, for all p in $[1, \infty[$. To extend this operator to all $\mathcal{L}_p(G)$ and study its properties, we will introduce the adjoint homomorphism ψ of ϕ. The homomorphism ψ maps $\hat{G}$ into $\mathbf{R}$ and satisfies the identity:

$$\exp(i\psi(\chi)(t)) = \chi \circ \phi(t) \tag{3.1.2}$$

for all χ in $\hat{G}$, and all t in $\mathbf{R}$. See [**8**, Definition (24.37), p.392].

In the present case of a topological isomorphism ϕ, the properties of the operator S_ν and the maximal operator associated with it are easily obtained using well-known properties of the partial sum operator on $\mathbf{R}$ and a judicious application of the Weil formula. We now set the stage for this formula.

The subgroup $\phi(\mathbf{R})$ is locally compact in its relative topology. Hence [**8**, Theorem 5.11, p.35] implies that $\phi(\mathbf{R})$ is closed in G. Let w denote the Haar measure on $\phi(\mathbf{R})$ normalised so that the equality

$$\int_{\phi(\mathbf{R})} g(x)dw(x) = \int_{\mathbf{R}} g(\phi(s))ds$$

obtains for all g in $\mathcal{L}_1(\phi(\mathbf{R}))$. (Simply define $w(A) = \lambda(\phi^{-1}(A))$, for all Borel subsets A of $\phi(\mathbf{R})$.) Now normalise the Haar measure $\mu_{G/\phi(\mathbf{R})}$ so that the Weil formula, [**9**, (28.54.iii), p.91] holds. We have

$$\begin{aligned}\int_G f(x)d\mu(x) &= \int_{G/\phi(\mathbf{R})}\int_{\phi(\mathbf{R})} f(x+y)dw(y)d\mu_{G/\phi(\mathbf{R})}(x)\\ &= \int_{G/\phi(\mathbf{R})}\int_{\mathbf{R}} f(x+\phi(y))dyd\mu_{G/\phi(\mathbf{R})}(x)\end{aligned}$$

for all f in $\mathcal{L}_1(G)$.

We start now deriving some properties of the transferred operator.

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LEMMA 3.2. *Suppose that f is continuous with compact support on G, and let $p \in \,]1, \infty[$. We have*

$$\|S_\nu f\|_p \leqslant A_p \|f\|_p,$$

where A_p is as in (2.1.2).

PROOF: We compute the norm using the Weil formula, and (2.1.2). We have

$$\begin{aligned}
\|S_\nu f\|_p^p &= \int_{G/\phi(\mathbf{R})} \int_{\mathbf{R}} \left| \frac{1}{\pi} \int_{\mathbf{R}} f(x + \phi(s-t)) \frac{\sin \nu t}{t} dt \right|^p ds d\mu_{G/\phi(\mathbf{R})}(x) \\
&\leqslant A_p^p \int_{G/\phi(\mathbf{R})} \int_{\mathbf{R}} |f(x + \phi(s))|^p ds d\mu_{G/\phi(\mathbf{R})}(x) \\
&= A_p^p \|f\|_p^p .
\end{aligned}$$

∎

LEMMA 3.3. *The notation is as above. Suppose that f is continuous with compact support on G and such that $\hat{f}$ is in $\mathcal{L}_1\left(\hat{G}\right)$. For every positive number ν, and all χ in $\hat{G}$, we have*

(i) $$(S_\nu f)\hat{}(\chi) 1^*_{[-\nu,\nu]}(\psi(\chi)),$$

where $1^*_{[-\nu,\nu]}(x) = 1_{[-\nu,\nu]}(x)$ for all $|x| \neq \nu$, and $1^*_{[-\nu,\nu]}(x) = 1/2$ for $|x| = \nu$.

PROOF: We compute (i) using Lebesgue's dominated convergence theorem, some well-known facts about Fourier integrals, Fubini's theorem, (3.1.1) and (3.1.2).

$$\begin{aligned}
\int_{\hat{G}} \hat{f}(\chi) 1^*_{[-\nu,\nu]}(\psi(\chi)) \overline{\chi}(x) d\chi &= \int_{\hat{G}} \hat{f}(\chi) \overline{\chi}(x) \lim_{n\to\infty} \frac{1}{\pi} \int_{[-n,n]} \exp(i\psi(\chi)t) \frac{\sin \nu t}{t} dt d\chi \\
&= \lim_{n\to\infty} \int_{\hat{G}} \hat{f}(\chi) \overline{\chi}(x) \frac{1}{\pi} \int_{[-n,n]} \exp(i\psi(\chi)t) \frac{\sin \nu t}{t} dt d\chi \\
&= \lim_{n\to\infty} \frac{1}{\pi} \int_{[-n,n]} \int_{\hat{G}} \hat{f}(\chi) \overline{\chi}(x - \phi(t)) d\chi \frac{\sin \nu t}{t} dt \\
&= \lim_{n\to\infty} \frac{1}{\pi} \int_{[-n,n]} f(x - \phi(t)) \frac{\sin \nu t}{t} dt \\
&= S_\nu f(x).
\end{aligned}$$

The identity (i) follows now from the uniqueness of Fourier transforms. ∎

Remark 3.4. Combining Lemmas 3.2 and 3.3, we see that, for all positive numbers ν, the function $1^*_{[-\nu,\nu]} \circ \psi$ is an $\mathcal{L}_p(G)$-multiplier, for $1 < p \leqslant 2$, with norm $\leqslant A_p$, where A_p is as in (2.1.2).

THEOREM 3.5. *For all positive numbers ν, the function $1_{]-\nu,\nu[} \circ \psi$ is an $\mathcal{L}_p(G)$-multiplier, for $1 < p \leqslant 2$, with norm $\leqslant A_p$, where A_p is as in (2.1.2).*

PROOF: Let f be in $\mathcal{L}_p(G)$. By Remark (3.4), it is enough to show that $\psi^{-1}(\{-\nu, \nu\}) \cap \operatorname{supp} \hat{f}$ has measure zero. Since $\operatorname{supp} \hat{f}$ is σ-compact, it is enough to show that $\psi^{-1}(\{0\})$ is locally null. If not, then $\psi^{-1}(\{0\})$ is open, and the annihilator H in G of $\psi^{-1}(\{0\})$ is compact. We also have $\phi(\mathbf{R}) \subseteq H$. Clearly this is impossible since ϕ is a topological isomorphism, so that $\phi(\mathbf{R})$ cannot be compact. ∎

A maximal operator on G 3.6. For p in $]1, 2]$, and all f in $\mathcal{L}_p(G)$, let

$$M^{\#} f(x) = \sup_{\nu \in I} \left| \left(\hat{f} 1_{]-\nu,\nu[} \circ \psi \right)^{\vee}(x) \right|$$

where I is a fixed, unbounded, increasing sequence of positive numbers. The function $M^{\#} f$ is defined almost everywhere on G. When f has compact support and $\hat{f}$ is in $\mathcal{L}_1\left(\hat{G}\right)$, Lemma 3.3 and Theorem 3.5 show that

$$M^{\#} f(x) = \sup_{\nu \in I} |S_\nu f(x)|, \tag{3.6.1}$$

for almost all x in G.

THEOREM 3.7. *The operator $M^{\#}$ is a bounded sublinear operator on $\mathcal{L}_p(G)$, for all p in $]1, 2]$, with norm less than or equal to M_p, where M_p is as in Theorem 2.2.*

PROOF: The operator $M^{\#}$ is clearly sublinear. To prove that it is bounded, it is enough to consider functions in a dense subset of $\mathcal{L}_p(G)$. (See [**7**, Theorem (3.1.1), p.36]), Suppose that f has compact support on G and is such that $\hat{f}$ is in $\mathcal{L}_1\left(\hat{G}\right)$. Now we use (3.6.1), Theorem 2.2, and the Weil formula, as we did in the proof of Lemma 3.2. We have

$$\begin{aligned}
\int_G \left(M^{\#} f(x)\right)^p d\mu(x) &= \int_{G/\phi(\mathbf{R})} \int_{\mathbf{R}} \left(M^{\#} f(x + \phi(y))\right)^p dy\, d\mu_{G/\phi(\mathbf{R})}(x) \\
&\leqslant \int_{G/\phi(\mathbf{R})} \int_{\mathbf{R}} \left(\sup_{0<\nu} \left| \frac{1}{\pi} \int_{\mathbf{R}} f(x + \phi(y - t)) \frac{\sin \nu t}{t} dt \right| \right)^p dy\, d\mu_{G/\phi(\mathbf{R})}(x) \\
&\leqslant M_p^p \int_{G/\phi(\mathbf{R})} \int_{\mathbf{R}} |f(x + \phi(y))|^p \, dy\, d\mu_{G/\phi(\mathbf{R})}(x) \\
&= M_p^p \, \|f\|_p^p .
\end{aligned}$$

This completes the proof of the theorem. ∎

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4.1 The general case

Unwinding the solenoid 4.1. We continue with the notation of Section 3, with the exception that ϕ is a continuous homomorphism of $\mathbf{R}$ into G which is not a topological isomorphism. Thus, according to [8, Theorem (9.1), p.84] $\varphi(\mathbf{R})^-$, the closure of $\phi(\mathbf{R})$ in G, is compact. The subgroup $\phi(\mathbf{R})$ may be thought of as a solenoid wrapped densely inside $\varphi(\mathbf{R})^-$.

To be able to use the results of Section 3, we will undo the solenoid by embedding G in $G \times \mathbf{R}$, the direct product of G and $\mathbf{R}$.

Define the continuous homomorphism Φ from $\mathbf{R}$ into $G \times \mathbf{R}$ by $\Phi(r) = (\phi(r), r)$. It is clear that Φ is a topological isomorphism of $\mathbf{R}$ onto $\Phi(\mathbf{R})$. Moreover, it is easy to verify that the adjoint homomorphism Ψ of Φ is given by $\Psi(\chi, s) = \psi(\chi) + s$, where ψ is the adjoint homomorphism of ϕ.

For F in $\mathcal{L}_p(G \times \mathbf{R})$, where $1 < p \leqslant 2$, let $T_\nu F$ be the function whose Fourier transform is given by

$$(4.1.1) \qquad (T_\nu F)\hat{}(\chi, s) = \hat{F}(\chi, s) 1_{]-\nu,\nu[} \circ \Psi(\chi, s) = \hat{F}(\chi, s) 1_{]-\nu,\nu[}(\psi(\chi) + s).$$

(Note that the operator T_ν is *the* partial sum operator S_ν corresponding to the homomorphism Ψ. However, since we will have the occasion to use both operators in the same proof, we have introduced a new notation to avoid confusion. For the same reason, we will introduce a new notation for the maximal operator $M^{\#}$.) Let

$$M^{\dagger} F(x, t) = \sup_{\nu \in I} |T_\nu F(x, t)|.$$

Since Φ is a topological isomorphism onto $\Phi(\mathbf{R})$, Theorem 3.7 applies and yields the inequality

$$(4.1.2) \qquad \left\|M^{\dagger} F\right\|_{p, G\times\mathbf{R}} \leqslant M_p \|F\|_{p, G\times\mathbf{R}}$$

where M_p is independent of F, and is the same as in Theorem 2.2. (Whenever confusion may arise, the symbol $\| \|_{p,X}$ will be used to denote the usual norm in $\mathcal{L}_p(X)$.)

The following remarks will simplify the proof of our main theorem.

Remarks 4.2. (a) Suppose that $\varphi(\mathbf{R})^-$ is a compact subgroup of G. Using the structure theorem for locally compact abelian groups, write G as $\mathbf{R}^a \times \Omega$, where a is a nonnegative integer and Ω contains a compact open subgroup. It is clear, in this case, that $\varphi(\mathbf{R})^-$ is contained in $\{0\} \times \Omega$. The character group $\hat{G}$ is topologically isomorphic to $\mathbf{R}^a \times \hat{\Omega}$, where $\hat{\Omega}$ is the character group of Ω. The annihilator of $\varphi(\mathbf{R})^-$ in $\hat{G}$ is an open subgroup, and hence is of the form $\mathbf{R}^a \times Y_0$, for some open subgroup Y_0 of

$\hat{\Omega}$. Since the adjoint homomorphism ψ of ϕ maps the annihilator of $\varphi(\mathbf{R})^-$ to zero, it follows that $\psi(\mathbf{R}^a \times Y_0) = \{0\}$. Now, given a compact nonvoid subset K of $\hat{G}$, cover K with finitely many cosets of the open subgroup $\mathbf{R}^a \times Y_0$. It follows that $\psi(K)$ is a finite subset of $\mathbf{R}$. We will say, in this case, that $\psi\left(\hat{G}\right)$ is a *discrete* subgroup of $\mathbf{R}$, meaning that $\psi(K)$ is finite for every compact subset K of $\hat{G}$.

(b) Suppose that $\varphi(\mathbf{R})^-$ is a compact subgroup of G. Let f be in $\mathcal{L}_p(G)$, where $1 \leqslant p \leqslant 2$, such that $\hat{f}$ is compactly supported. By (a), $\psi\left(\operatorname{supp} \hat{f}\right)$ is a finite subset of $\mathbf{R}$. Write $\psi\left(\operatorname{supp} \hat{f}\right) = \{a_j\}_{j=1}^n$. Let $\alpha > 0$ be such that: for all $j = 1, \ldots, n$, if $\nu_{j-1} \leqslant a_j \leqslant \nu_j$ for a unique couple (ν_{j-1}, ν_j), where ν_{j-1}, and ν_j are in $I \cup (-I)$, then $\nu_{j-1} < \alpha a_j < \nu_j$. (To find α, simply let α' be such that: $|\alpha' a_j| < \nu_j - \nu_{j-1}$, for all $j = 1, \ldots, n$; and take $\alpha = 1 - \alpha'$.) Let ψ' denote the homomorphism $\alpha\psi$. We clearly have: $\hat{f} 1_{]-\nu_j, \nu_j[} \circ \psi = \hat{f} 1_{]-\nu_j, \nu_j[} \circ \psi'$, for all $j = 1, \ldots, n$, where the equality holds everywhere on $\hat{G}$. Hence, from the uniqueness of the Fourier transform, it follows that the equality

$$\sup_{\nu \in I} \left| \left(\hat{f} 1_{]-\nu, \nu[} \circ \psi\right)^{\check{}} \right| = \sup_{\nu \in I} \left| \left(\hat{f} 1_{]-\nu, \nu[} \circ \psi'\right)^{\check{}} \right|$$

holds almost everywhere on G.

We can now state and prove our main theorem.

THEOREM 4.3. *Let ψ be an arbitrary continuous real-valued homomorphism on $\hat{G}$. Let f be in $\mathcal{L}_p(G)$, where $1 < p \leqslant 2$; then*

$$\left\| M^{\#} f \right\|_{p,G} \leqslant M_p \left\| f \right\|_{p,G} .$$

where M_p is independent of f, and is the same as for the case $G = \mathbf{R}$.

PROOF: We distinguish two cases.

Case 1. ϕ is a topological isomorphism. This is the case treated in Theorem 3.7.

Case 2. ϕ is not a topological isomorphism. By Remark 4.2 (a), $\psi\left(\hat{G}\right)$ is a discrete subgroup of $\mathbf{R}$. Also, to prove the theorem it is enough to consider f in a dense subset of $\mathcal{L}_p(G)$. So suppose that f has a compactly supported Fourier transform $\hat{f}$, and denote its support by K. By Remark 4.2, $\psi(K)$ is finite, and we may suppose that $\psi(K) \cap I' = \emptyset$, where $I' = I \cup \{-\nu \colon \nu \in I\}$, since the constants in our proof do not depend on the choice of the homomorphism ψ. Let $\delta = \operatorname{dist}(\psi(K), I') = \min\{|\psi(\chi) - \nu| : \chi \in K,\, \nu \in I'\}$; then $\delta > 0$. Let g be an arbitrary function in $\mathcal{L}_p(\mathbf{R})$ such that $\|g\|_{p,\mathbf{R}} > 0$, and $\operatorname{supp} \hat{g} \subseteq\,]0, \delta[$.

Note that, for all χ in $\hat{G}$, all s in $\mathbf{R}$, and all ν in I, we have

$$\hat{f}(\chi)\hat{g}(s) 1_{]-\nu,\nu[}(\psi(\chi) + s) = \hat{f}(\chi)\hat{g}(s) 1_{]-\nu,\nu[}(\psi(\chi)). \tag{4.3.1}$$

Let F in $\mathcal{L}_p(G \times \mathbf{R})$ be defined by: $F(x, t) = f(x)g(t)$. We have

$$\hat{F}(\chi, s) = \hat{f}(\chi)\hat{g}(s). \tag{4.3.2}$$

From (4.3.1), (4.3.2), and (4.1.1), it follows that

$$(T_\nu F)\widehat{}(\chi, s) = \hat{g}(s)\hat{f}(\chi)1_{]-\nu,\nu[}(\psi(\chi)); \quad \text{or}$$

$$T_\nu F(x, t) = g(t)\left(\hat{f}1_{]-\nu,\nu[} \circ \psi\right)\check{}(x) \tag{4.3.3}$$

where the equality holds $\mu \times \lambda$-almost everywhere $G \times \mathbf{R}$. We thus have, from (4.3.3)

$$M^\dagger F(x, t) = \sup_{\nu \in I} |g(t)| \left| \left(\hat{f}1_{]-\nu,\nu[} \circ \psi\right)\check{}(x)\right| = |g(t)|\, M^\# f(x). \tag{4.3.4}$$

Using (4.3.4), and applying (4.1.2) to $M^\dagger F$, we obtain

$$\left\|M^\dagger F(x, t)\right\|_{p, G\times \mathbf{R}} = \|g\|_{p,\mathbf{R}} \left\|M^\# f\right\|_{p,G} \leqslant M_p \|g\|_{p,\mathbf{R}} \|f\|_{p,G}. \tag{4.3.5}$$

The theorem follows now by dividing both sides of the inequality in (4.3.5) by $\|g\|_{p,\mathbf{R}}$. ∎

An immediate application of Theorem 4.3 is the following pointwise inversion result.

THEOREM 4.4. *Suppose that ψ is an arbitrary continuous real-valued homomorphism on $\hat{G}$. For every f in $\mathcal{L}_p(G)$, where $p \in]1, 2]$, we have*

$$\lim_{\substack{\nu \to \infty \\ \nu \in I}} \left(\hat{f}1_{]-\nu,\nu[} \circ \psi\right)\check{}(x) = f(x)$$

for almost all x in G.

PROOF: The theorem is clearly true if $\hat{f}$ is compactly supported. The set of all such functions is dense in $\mathcal{L}_p(G)$. The theorem follows now from Theorem 4.3 and [**7**, Theorem (1.2.1), p.11]. ∎

5. INVERSION OF THE FOURIER TRANSFORM ON $\mathbf{R}^a \times \mathbf{T}^b$

In this section we describe a method for recapturing pointwise a function f in $\mathcal{L}_p(\mathbf{R}^a \times \mathbf{T}^b)$ from its Fourier transform, where a and b are arbitrary nonnegative integers, and p is in $]1, 2]$. By using an abstract verion of Riesz's theorem on conjugate functions, we will reduce our problem to the one-dimensional case, then use Theorem 4.4 above. Several versions of Riesz's theorem appeared recently. We refer the interested reader to [**1**, Theorem (7.2)] and [**2**, Section 7].

Fix an integer m such that $\mathsf{R}^a \times \mathsf{Z}^b = S_1 \cup \ldots \cup S_m$, where each S_j is the intersection of finitely many half-spaces; then for each $j = 1, \ldots, m$, there is a continuous real-valued homomorphism ψ_j on $\mathsf{R}^a \times \mathsf{Z}^b$ such that $S_j \cap \psi_j^{-1}([-s, s])$ is relatively compact for all s in R. For example, write R^2 as the union of the four quadrants. Let $\psi_1(x_1, x_2) = \psi_3(x_1, x_2) = x_1 + x_2$; and $\psi_2(x_1, x_2) = \psi_4(x_1, x_2) = x_1 - x_2$.

By repeatedly using Riesz's theorem, we can write an arbitrary f in $\mathcal{L}_p(\mathsf{R}^a \times \mathsf{T}^b)$ as: $f = f_1 + f_2 + \ldots + f_m$, where $\|f_j\| \leqslant \beta_p |f|_p$, for all j in $\{1, \ldots, m\}$, where β_p depends only on p, a, and b; and $\hat{f}_j = \hat{f} 1_{S_j}$. Apply Theorem 4.4 to each f_j and ψ_j separately to see that

$$\lim_{\nu \to \infty} \left(\hat{f}_j 1_{\psi_j^{-1}(]-\nu, \nu[)} \right)\check{}(x) = f_j(x),$$

for almost all x in $\mathsf{R}^a \times \mathsf{T}^b$. Consequently, we obtain our inversion theorem for Fourier transforms of functions on $\mathsf{R}^a \times \mathsf{T}^b$.

THEOREM 5.1. *Let p be a number in $]1, 2]$. For each $\nu = 1, 2, \ldots$, let*

$$B_\nu = \bigcup_{j=1}^{m} \left(S_j \cap \psi_j^{-1}(]-\nu, \nu[) \right).$$

Then each B_ν is relatively compact; and for all f in $\mathcal{L}_p(\mathsf{R}^a \times \mathsf{T}^b)$, we have

$$\lim_{\nu \to \infty} \left(\hat{f} 1_{B_\nu} \right)\check{}(x) = f(x)$$

for almost all x in $\mathsf{R}^a \times \mathsf{T}^b$.

PROOF: The theorem follows from the preceding observations, and the equality

$$\hat{f} 1_{B_\nu} = \sum_{j=1}^{m} \hat{f}_j 1_{\psi_j^{-1}(]-\nu, \nu[)}.$$

$\square$

Remarks 5.2. (a) Take $a = 0$ in Theorem (5.1) to obtain the pointwise convergence results of Fefferman [5] concerning multiple Fourier series.

(b) Let T^∞ denote the countable product of the circle T. We denote the character group of T^∞ by Z^w, which is the weak direct product of countably many copies of Z. A generic element χ of Z^w is represented by a sequence of integers $(x_1, x_2, \ldots)$ all but finitely many of its terms are zero. Define a real-valued homomorphism ψ on Z^w by: $\psi((x_1, x_2, \ldots, x_j, \ldots)) = \sum_{j=1}^{\infty} j x_j$. Call a function f in $\mathcal{L}_1(\mathsf{T}^\infty)$ *analytic* if $\hat{f}$ vanishes

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outside the first orthant; that is $\hat{f}((x_1, x_2, \ldots)) = 0$ if $x_j < 0$, for some j. In this case, the set $\psi^{-1}([-\nu, \nu]) \cap \operatorname{supp} \hat{f}$ is finite for all ν. So, for analytic functions in $\mathcal{L}_p(\mathsf{T}^\infty)$, where $1 < p < \infty$, Theorem 4.4 gives a pointwise inversion for the Fourier transform analogous to the one given by Theorem 5.1.

(c) Finally we note that, in view of the negative results concerning the convergence of the restricted rectangular partial sums of multiple Fourier series in [**6**], one may not expect to get pointwise convergence using arbitrary rectangular blocks in Theorem 5.1.

As we mentioned in the introduction, our methods consist in transfering to a new set-up a certain operator and its properties. This idea has been explored extensively by various other writers. For example, in [**5**], the pointwise convergence of multiple Fourier series, and the boundedness of the maximal operator associated with them, are obtained by reducing to the one-dimensional case and using the Carleson-Hunt theorem. Also the idea of decomposing a function as a sum of functions each having a Fourier transform vanishing outside a sector is due to [**5**]. The main results of [**1**] were obtained by transferring the properties of the Hilbert transform. A general transference method is presented in [**3**], where the transference of maximal operators is also studied. Generalisations of [**3**] are taken-up in [**4**]. The novelty in our proof is that, unlike the results that we just mentioned, we succeeded in transferring properties of a maximal operator associated with convolution operators with kernels which are not compactly supported; or, for that matter, not even integrable. It is clear, however, that our proofs depend vitally on the fact that the action of R on $\mathcal{L}_p(G)$ is of a very special kind. More precisely, to each $r \in \mathsf{R}$, corresponds the translation operator on $\mathcal{L}_p(G)$ defined by translating by $\phi(r)$, where ϕ is a continuous homomorphism of R into G.

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