Glasgow Math. J. **54** (2012) 655–663. © Glasgow Mathematical Journal Trust 2012.
doi:10.1017/S0017089512000249.

BEST CONSTANTS IN THE WEAK-TYPE ESTIMATES FOR UNCENTERED MAXIMAL OPERATORS

ADAM OSĘKOWSKI
Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland
e-mail: ados@mimuw.edu.pl

(Received 20 November 2011; accepted 25 January 2012; first published online 30 March 2012)

Abstract. Let μ be a Borel measure on $\mathbb{R}$. The paper contains the proofs of the estimates

$$||\mathcal{M}_\mu f||_{L^{q,\infty}(A,\mu)} \leq c_{p,q}||f||_{L^p(\mathbb{R},\mu)}\,\mu(A)^{1/q-1/p}, \qquad 1 \leq p < \infty,\ q \in (0, p],$$

and

$$||\mathcal{M}_\mu f||_{L^{q,\infty}(A,\mu)} \leq C_{p,q}||f||_{L^{p,\infty}(\mathbb{R},\mu)}\,\mu(A)^{1/q-1/p}, \quad 1 < p < \infty, q \in (0, p].$$

Here A is a subset of $\mathbb{R}$, f is a μ-locally integrable function, $\mathcal{M}_\mu$ is the uncentred maximal operator with respect to μ and $c_{p,q}$, and $C_{p,q}$ are finite constants depending only on the parameters indicated. In the case when μ is the Lebesgue measure, the optimal choices for $c_{p,q}$ and $C_{p,q}$ are determined. As an application, we present some related tight bounds for the strong maximal operator on $\mathbb{R}^n$ with respect to a general product measure.

2000 *Mathematics Subject Classification.* Primary: 42B25. Secondary: 42B35, 46E30.

1. Introduction. Suppose μ is a non-negative Borel measure on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a μ-locally integrable function. The uncentred maximal function of f with respect to μ is given by the formula

$$(\mathcal{M}_\mu f)(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f|\mathrm{d}\mu,$$

where the supremum is taken over all closed balls B, which contain the point x. If μ is the Lebesgue measure, then $\mathcal{M}_\mu$ is the usual uncentred maximal operator of Hardy and Littlewood [**4**]. It is well known (see, e.g. Stein [**6**]) that if μ satisfies the doubling condition

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \qquad \text{for some } C < \infty \text{ and all } x \in \mathbb{R}^n,\ r > 0$$

(here $B(x, r)$ denotes the closed ball of centre x and radius r), then $\mathcal{M}_\mu$ maps $L^p(\mathbb{R}^n, \mu)$ into itself for $p > 1$, and $L^1(\mathbb{R}^n, \mu)$ into $L^{1,\infty}(\mathbb{R}^n, \mu)$. This is still true, without the doubling property if and only if $n = 1$ (see [**1**, **2**, **5**]).

The question about the precise evaluation of strong and weak norms of $\mathcal{M}_\mu$ has gained some interest in the literature, and the objective of this paper is to establish two new results of this type. We will be particularly interested in the one-dimensional case. We have the following L^p-estimates for $\mathcal{M}_\mu$: For any μ-locally integrable f and $1 < p < \infty$ we have

$$||\mathcal{M}_\mu f||_{L^p(\mathbb{R},\mu)} \leq c_p||f||_{L^p(\mathbb{R},\mu)}, \tag{1.1}$$

where c_p is the unique positive solution of the equation

$$(p-1)x^p - px^{p-1} - 1 = 0. \tag{1.2}$$

This statement, with μ being the Lebesgue measure, was proved by Grafakos and Montgomery-Smith in [**3**]; for the general case, consult Grafakos and Kinnunen [**2**]. In general, constant c_p in (1.1) cannot be replaced by a smaller number, see [**3**]. The L^1-inequality does not hold in general with any finite constant c_1, but we have the sharp weak-type estimate

$$||\mathcal{M}_\mu f||_{L^{1,\infty}(\mathbb{R},\mu)} \leq 2||f||_{L^1(\mathbb{R},\mu)},$$

as proved in [**2**]. Here, as usual, for any Borel subset A of $\mathbb{R}$ and any $0 < p < \infty$, we define the weak p-th norm of f on A by the formula

$$||f||_{L^{p,\infty}(A,\mu)} = \sup_{\lambda>0} \lambda\big[\mu(\{x \in A : |f(x)| > \lambda\})\big]^{1/p}.$$

There is a natural question about the best constants in the corresponding weak-type (p, p) estimates for $\mathcal{M}_\mu$, $1 < p < \infty$. In fact, we will study this question in a more general setting and compare the weak q-th norm of $\mathcal{M}_\mu f$ to the p-th norm of f, where $p \geq 1$ and $q \in (0, p]$. Introduce constant

$$C_p = \frac{(p-1)(2^{p/(p-1)} - 1)}{p}\Big((p-1)(2^{p/(p-1)} - 2)\Big)^{-1/p}$$

when $1 < p < \infty$, and put $C_1 = 2$. We will establish the following result.

Theorem 1.1. *For any μ-locally integrable function $f : \mathbb{R} \to \mathbb{R}$, any Borel subset A of $\mathbb{R}$ and any $1 \leq p < \infty$, $q \in (0, p]$, we have*

$$||\mathcal{M}_\mu f||_{L^{q,\infty}(A,\mu)} \leq C_p||f||_{L^p(\mathbb{R},\mu)}\mu(A)^{1/q-1/p}. \tag{1.3}$$

If μ is the Lebesgue measure, then the constant C_p is the best possible.

In particular, if $p = q$, then (1.3) yields the weak-type (p, p) estimate

$$||\mathcal{M}_\mu f||_{L^{p,\infty}(\mathbb{R},\mu)} \leq C_p||f||_{L^p(\mathbb{R},\mu)}, \tag{1.4}$$

which, as we will see, is also sharp, provided μ is the Lebesgue measure.

The next problem we will study concerns the sharp comparison of the weak norms of f and $\mathcal{M}_\mu f$. Here constants c_p of Grafakos and Montgomery-Smith [**3**] come into play; we will prove the following statement.

https://doi.org/10.1017/S0017089512000249 Published online by Cambridge University Press

THEOREM 1.2. *For any μ-locally integrable function $f : \mathbb{R} \to \mathbb{R}$, any Borel subset A of $\mathbb{R}$ and any $1 < p < \infty$, $q \in (0, p]$, we have*

$$||\mathcal{M}_\mu f||_{L^{q,\infty}(A,\mu)} \leq c_p ||f||_{L^{p,\infty}(\mathbb{R},\mu)} \mu(A)^{1/q-1/p}. \tag{1.5}$$

If μ is the Lebesgue measure, then the constant c_p is the best possible.

As previously, let us distinguish the choice $p = q \in (1, \infty)$. It gives the bound

$$||\mathcal{M}_\mu f||_{L^{p,\infty}(\mathbb{R},\mu)} \leq c_p ||f||_{L^{p,\infty}(\mathbb{R},\mu)}, \tag{1.6}$$

which will be proved to be sharp in the case when μ is the Lebesgue measure.

Theorems 1.1 and 1.2 will be established in the next section. In Section 3 we will apply these two theorems to obtain related results in the higher dimensional setting: more precisely, we will show tight weak-type estimates for the so-called strong maximal operator on $\mathbb{R}^n$, $n \geq 2$.

2. Proofs of theorems 1.1 and 1.2. We start with recalling the main lemma from [**2**] (see also [**3**] for the special case in which μ is the Lebesgue measure). This result can be regarded as an appropriate version of the weak-type estimate for $\mathcal{M}_\mu$. Here and below, we use the notation $\{f > \lambda\}$ for the set $\{x \in \mathbb{R} : f(x) > \lambda\}$.

LEMMA 2.1. *If f is a non-negative and μ-locally integrable function on $\mathbb{R}$, then for any $\lambda > 0$ we have*

$$\lambda\Big(\mu\left(\{\mathcal{M}_\mu f > \lambda\}\right) + \mu\left(\{f > \lambda\}\right)\Big) \leq \int_{\{\mathcal{M}_\mu f > \lambda\}} f d\mu + \int_{\{f > \lambda\}} f d\mu. \tag{2.1}$$

In other words, for any f, λ as in the statement above, we have

$$\int_{\mathbb{R}} u\big(f(x)/\lambda, \mathcal{M}_\mu f(x)/\lambda\big) \mathrm{d}\mu(x) \leq 0, \tag{2.2}$$

where $u : [0, \infty) \times [0, \infty] \to \mathbb{R}$ is the function given by the formula

$$u(x, y) = \big(\chi_{\{x>1\}} + \chi_{\{y>1\}}\big)(1 - x).$$

Introduce the parameters

$$r_p = \frac{p}{(p-1)(2^{p/(p-1)} - 1)}, \qquad s_p = \frac{p\, 2^{1/(p-1)}}{(p-1)(2^{p/(p-1)} - 1)}$$

and

$$\alpha_p = \frac{2^{p/(p-1)} - 1}{2^{p/(p-1)} - 2}.$$

LEMMA 2.2. *For any $0 \leq x \leq y$ and any $1 < p < \infty$, we have*

$$\alpha_p u(x, y) \geq \chi_{\{y>1\}} - C_p^p x^p. \tag{2.3}$$

https://doi.org/10.1017/S0017089512000249 Published online by Cambridge University Press

Proof. If $y \le 1$, then the estimate becomes $0 \ge -C_p^p x^p$, which is obvious. Suppose $y > 1$ and $x \le 1$. Then (2.3) is equivalent to

$$F(x) := \alpha_p(1-x) - 1 + C_p^p x^p \ge 0,$$

which holds true for all $x \ge 0$. This is the consequence of the fact that F is a convex function, combined with equalities $F(r_p) = F'(r_p) = 0$. Finally, if both x and y are larger than 1, inequality (2.3) can be rewritten in the form

$$G(x) := 2\alpha_p(1-x) - 1 + C_p^p x^p \ge 0,$$

which follows from the convexity of G and equalities $G(s_p) = G'(s_p) = 0$. □

Proof of (1.3) We may assume that f is a non-negative function which satisfies $||f||_{L^p(\mathbb{R},\mu)} < \infty$. Combining (2.2) and (2.3), we obtain that for $p > 1$,

$$\lambda^p \mu\big(\{\mathcal{M}_\mu f > \lambda\}\big) \le C_p^p ||f||^p_{L^p(\mathbb{R},\mu)}. \tag{2.4}$$

This bound is also true for $p = 1$, as we have already mentioned above. Thus, since $\mu\big(\{x \in A : \mathcal{M}_\mu f(x) > \lambda\}\big) \le \min\big\{\mu(A), \mu(\{\mathcal{M}_\mu f > \lambda\})\big\}$, we have

$$\begin{aligned}\lambda^q \mu(\{x \in A : \mathcal{M}_\mu f(x) > \lambda\}) &\le \lambda^q \mu(\{\mathcal{M}_\mu f \ge \lambda\})^{q/p} \mu(A)^{1-q/p} \\ &\le C_p^q ||f||^q_{L^p(\mathbb{R},\mu)} \mu(A)^{1-q/p},\end{aligned} \tag{2.5}$$

where the latter passage is due to (2.4). It remains to take supremum over λ in (2.5) to obtain (1.3). □

Sharpness for the Lebesgue measure. Let r_p and s_p be as above and introduce the parameter $\beta_p = 2(s_p - 1)/(1 - r_p)$. Consider the function

$$f = s_p \chi_{[-1,1]} + r_p \left(\chi_{[-\beta_p-1,-1)} + \chi_{(1,\beta_p+1]}\right)$$

and let $A = [-\beta_p - 1, \beta_p + 1]$. The identity

$$\frac{1}{\big|[-\beta_p-1,1]\big|} \int_{-\beta_p-1}^{1} f(x)\mathrm{d}x = \frac{1}{\big|[-1,\beta_p+1]\big|} \int_{-1}^{\beta_p+1} f(x)\mathrm{d}x = \frac{2s_p + \beta_p r_p}{2+\beta_p} = 1$$

and the definition of the maximal operator imply that $\mathcal{M}_{|\cdot|} f(x) \ge 1$ for $x \in A$. Therefore,

$$\frac{|\{x \in A : \mathcal{M}_{|\cdot|} f(x) \ge 1\}|}{||f||^q_{L^p(\mathbb{R},|\cdot|)} |A|^{1-q/p}} = \left(\frac{|A|}{||f||^p_{L^p(\mathbb{R},|\cdot|)}}\right)^{q/p} = \left(\frac{2(\beta_p+1)}{2\beta_p r_p^p + 2s_p^p}\right)^{q/p},$$

and the latter expression is easily checked to be equal to C_p^q. This proves the sharpness of (1.3). The same example yields the optimality of C_p in (1.4): we have

$$||\mathcal{M}_{|\cdot|} f||^p_{L^{p,\infty}(\mathbb{R},|\cdot|)} \ge |\{\mathcal{M}_{|\cdot|} f \ge 1\}| \ge |A| = C_p^p ||f||^p_{L^p(\mathbb{R},|\cdot|)}.$$

□

Proof of (1.5) It suffices to consider functions f, which are non-negative and satisfy $0 < ||f||_{L^{p,\infty}(\mathbb{R},\mu)} < \infty$. In addition, by homogeneity, we may and do assume

https://doi.org/10.1017/S0017089512000249 Published online by Cambridge University Press

that $||f||_{L^{p,\infty}(\mathbb{R},\mu)} = 1$. Rewrite (2.1) in the form

$$\lambda\mu(\{\mathcal{M}_\mu f > \lambda\}) \leq \int_{\{\mathcal{M}_\mu f>\lambda\}} f\mathrm{d}\mu + \int_{\{f>\lambda\}} (f-\lambda)\mathrm{d}\mu.$$

The well-known inequality of Hardy and Littlewood (see, e.g. [**4**]) states that if h is a non-negative function and A is a Borel subset of $\mathbb{R}$, then

$$\int_A h\mathrm{d}\mu \leq \int_0^{\mu(A)} h^*(t)\mathrm{d}t, \tag{2.6}$$

where $h^*(t) = \inf\{s > 0 : \mu(\{f > s\}) \leq t\}$ is the non-increasing rearrangement of h. Since $||f||_{L^{p,\infty}(\mathbb{R},\mu)} = 1$, we have $\mu(\{f > \lambda\}) \leq \lambda^{-p}$ for all $\lambda > 0$ and hence $f^*(t) \leq t^{-1/p}$ for all positive t. Putting all these facts together, we obtain

$$\begin{aligned}\lambda\,\mu(\{\mathcal{M}_\mu f > \lambda\}) &\leq \int_0^{\mu(\{\mathcal{M}_\mu f>\lambda\})} t^{-1/p}\mathrm{d}t + \int_0^{\lambda^{-p}} (t^{-1/p}-\lambda)\mathrm{d}t \\ &= \frac{p}{p-1}\mu(\{\mathcal{M}_\mu f > \lambda\})^{(p-1)/p} + \frac{\lambda^{1-p}}{p-1}.\end{aligned}$$

Multiplying both sides by $(p-1)\lambda^{p-1}$ yields

$$(p-1)\lambda^p\mu(\{\mathcal{M}_\mu f > \lambda\}) \leq p\big(\lambda^p\mu(\{\mathcal{M}_\mu f > \lambda\})\big)^{(p-1)/p} + 1.$$

In view of (1.2), this implies

$$\lambda^p\mu(\{\mathcal{M}_\mu f > \lambda\}) \leq c_p^p = c_p^p||f||_{L^{p,\infty}(\mathbb{R},\mu)}. \tag{2.7}$$

Indeed, we have $c_p \geq 1$ and the function $x \mapsto (p-1)x^p - px^{p-1}$ is increasing on $[1,\infty)$. Thus, we have established (1.6). Furthermore, (2.7) yields

$$\lambda^q\mu(\{x \in A : \mathcal{M}_\mu f(x) > \lambda\}) \leq c_p^q||f||^q_{L^{p,\infty}(\mathbb{R},\mu)}\mu(A)^{1-q/p},$$

which can be seen by repeating the argument leading from (2.4) to (2.5). The proof of (1.5) is complete. □

Sharpness for the Lebesgue measure. Fix $p > 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(t) = |2t|^{-1/p}$. It is easy to check that $||f||_{L^{p,\infty}(\mathbb{R})} = 1$. Furthermore, for any $x > 0$ we have

$$\frac{1}{|[-c_p^{-p}x, x]|}\int_{-c_p^p x}^{x} f(t)\mathrm{d}t = (2x)^{-1/p}\frac{p(1+c_p^{1-p})}{(p-1)(1+c_p^{-p})} = c_p(2x)^{-1/p}, \tag{2.8}$$

where the latter equality follows from (1.2). Thus, by the definition of the maximal operator, we have $\mathcal{M}_{|\cdot|}f(x) \geq c_p(2x)^{-1/p}$ for $x > 0$ and similarly $\mathcal{M}_{|\cdot|}f(x) \geq c_p(-2x)^{-1/p}$ for negative x. Consequently, $||\mathcal{M}_{|\cdot|}f||_{L^{p,\infty}(\mathbb{R},|\cdot|)} \geq c_p$ and the equality in (1.6) is attained. Next, putting $A = \{\mathcal{M}_{|\cdot|}f \geq 1\}$, we see that $[-c_p^p/2, c_p^p/2] \subseteq A$ and hence

$$||\mathcal{M}_{|\cdot|}f||^q_{L^{q,\infty}(A,|\cdot|)} \geq |A| \geq c_p^q|A|^{1-q/p} = c_p^q|A|^{1-q/p}||f||^q_{L^{p,\infty}(\mathbb{R},|\cdot|)}.$$

This yields the desired optimality of c_p in (1.5). □

https://doi.org/10.1017/S0017089512000249 Published online by Cambridge University Press

3. Estimates for the strong maximal function. This section contains applications of previous results to the study of maximal operators in higher dimensions. Let $n \geq 1$ be a fixed integer and let μ be a product measure on $\mathbb{R}^n$: $\mu = \mu_1 \otimes \mu_2 \otimes \ldots \otimes \mu_n$ for some Borel measures $\mu_1, \mu_2, \ldots, \mu_n$ on $\mathbb{R}$. The strong maximal operator M_μ is an operator that acts on μ-locally integrable functions f by the formula

$$M_\mu f(x) = \sup_{x \in D} \frac{1}{\mu(D)} \int_D |f| \mathrm{d}\mu,$$

where the supremum is taken over all closed rectangles D, with sides parallel to the axes, satisfying $x \in D$. Observe that for $n = 1$, operators M_μ and $\mathcal{M}_\mu$ coincide.

We will prove the following fact.

THEOREM 3.1. *Let μ and M_μ be as above.*

(i) *If $n \geq 2$, then in general M_μ* does not *map $L^1(\mathbb{R}^n, \mu)$ into $L^{1,\infty}(\mathbb{R}^n, \mu)$.*

(ii) *If $1 < p < \infty$, then for any $f : \mathbb{R}^n \to \mathbb{R}$ we have*

$$||M_\mu f||_{L^{p,\infty}(\mathbb{R}^n,\mu)} \leq C_p c_p^{n-1} ||f||_{L^p(\mathbb{R}^n\mu)}. \tag{3.1}$$

If μ is the Lebesgue measure on $\mathbb{R}^n$, then the constant has the optimal order $O((p-1)^{1-n})$ as $p \to 1$.

(iii) *If $1 < p < \infty$, then for any $f : \mathbb{R}^n \to \mathbb{R}$ we have*

$$||M_\mu f||_{L^{p,\infty}(\mathbb{R}^n,\mu)} \leq c_p^n ||f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}. \tag{3.2}$$

If μ is the Lebesgue measure on $\mathbb{R}^n$, then the constant is the best possible.

REMARK 3.2. By the argument from the previous section, (3.1) and (3.2) imply the estimates

$$||M_\mu f||_{L^{q,\infty}(A,\mu)} \leq C_p c_p^{n-1} ||f||_{L^p(\mathbb{R}^n\mu)} \mu(A)^{1/q-1/p}$$

and

$$||M_\mu f||_{L^{q,\infty}(A,\mu)} \leq c_p^n ||f||_{L^{p,\infty}(\mathbb{R}^n,\mu)} \mu(A)^{1/q-1/p} \tag{3.3}$$

for all μ-locally integrable functions $f : \mathbb{R}^n \to \mathbb{R}$, all Borel subsets A of $\mathbb{R}^n$ and all $1 < p < \infty$, $0 < q \leq p$. We will prove below that (3.3) is sharp, provided μ is the Lebesgue measure.

Proof of Theorem 3.1. (i) This will be shown in the proof of (ii) below.

(ii) The key observation is that

$$M_\mu \leq \mathcal{M}_{\mu_1}^{(1)} \circ \mathcal{M}_{\mu_2}^{(2)} \circ \ldots \circ \mathcal{M}_{\mu_n}^{(n)}, \tag{3.4}$$

where $\mathcal{M}_{\mu_k}^{(k)}$ denotes the maximal operator $\mathcal{M}_{\mu_k}$ applied to the k-th coordinate. Let f be a non-negative function on $\mathbb{R}^n$ satisfying $||f||_{L^p(\mathbb{R}^n,\mu)} < \infty$. Using (1.4) with respect

https://doi.org/10.1017/S0017089512000249 Published online by Cambridge University Press

to $\mathcal{M}_{\mu_1}$ and then (1.1) with respect to $\mathcal{M}_{\mu_2}, \mathcal{M}_{\mu_3}, \ldots, \mathcal{M}_{\mu_n}$, we obtain

$$\begin{aligned}
&\lambda^p \mu\big(\{\mathcal{M}^{(1)}_{\mu_1} \circ \mathcal{M}^{(2)}_{\mu_2} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f > \lambda\}\big)\\
&\quad = \int_{\mathbb{R}^{n-1}} \lambda^p \mu_1 \left(\{x_1 : \mathcal{M}^{(1)}_{\mu_1} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x_1, x_2, \ldots, x_n) > \lambda\}\right) \mathrm{d}\mu_2(x_2) \ldots \mathrm{d}\mu_n(x_n)\\
&\quad \le C_p^p \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} [\mathcal{M}^{(2)}_{\mu_2} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x_1, x_2, \ldots, x_n)]^p \mathrm{d}\mu_1(x_1)\right] \mathrm{d}\mu_2(x_2) \ldots \mathrm{d}\mu_n(x_n)\\
&\quad = C_p^p \int_{\mathbb{R}^n} [\mathcal{M}^{(2)}_{\mu_2} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x_1, x_2, \ldots, x_n)]^p \mathrm{d}\mu_1(x_1) \mathrm{d}\mu_2(x_2) \ldots \mathrm{d}\mu_n(x_n)\\
&\quad = C_p^p \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} [\mathcal{M}^{(2)}_{\mu_2} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x)]^p \mathrm{d}\mu_2(x_2)\right] \mathrm{d}\mu_1(x_1) \mathrm{d}\mu_3(x_3) \ldots \mathrm{d}\mu_n(x_n)\\
&\quad \le C_p^p c_p^p \int_{\mathbb{R}^n} [\mathcal{M}^{(3)}_{\mu_3} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n} f(x_1, x_2, \ldots, x_n)]^p \mathrm{d}\mu_1(x_1) \mathrm{d}\mu_2(x_2) \ldots \mathrm{d}\mu_n(x_n)\\
&\quad \le \ldots\\
&\quad \le C_p^p c_p^{(n-1)p} ||f||^p_{L^p(\mathbb{R}^n, \mu)}.
\end{aligned}$$

This yields (3.1). It is not difficult to check that $1 \le C_p \le 2$ and $\frac{p}{p-1} \le c_p \le \frac{2p}{p-1}$ for $1 < p < \infty$, so the constant $C_p c_p^{n-1}$ is of the order $O((p-1)^{1-n})$ when $p \to 1$. To see that this order is optimal when μ is the Lebesgue measure, take $p \in (1, 2)$, $n \ge 2$ and put $f = \chi_{[-1,1]^n}$. Then, for any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we have

$$M_\mu f(x) \ge \prod_{k=1}^n \min\left(\frac{2}{|x_k| + 1}, 1\right),$$

which can be verified by considering the smallest rectangle that contains x and the cube $[-1, 1]^n$. Thus, for any $\lambda \in (0, 1)$ we may write

$$\begin{aligned}
|\{M_\mu f > \lambda\}| &\ge 2^n \left|\left\{x \in [1, \infty)^n : \prod_{k=1}^n \frac{2}{x_k + 1} > \lambda\right\}\right|\\
&= 2^n \int_1^{a_1} \int_1^{a_2} \ldots \int_1^{a_n} \mathrm{d}x_n \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1, \qquad (3.5)
\end{aligned}$$

where $a_1 = 2/\lambda - 1$ and

$$a_k = \frac{2^k}{\lambda(x_1 + 1) \ldots (x_{k-1} + 1)} - 1, \qquad k = 2, 3, \ldots, n.$$

Denote the right-hand side of (3.5) by γ_n. Deriving the inner integral with respect to x_n gives the identity

$$\gamma_n = 2^n \int_1^{a_1} \int_1^{a_2} \ldots \int_1^{a_{n-1}} \frac{2^n}{\lambda(x_1 + 1) \ldots (x_{n-1} + 1)} \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1 - 4\gamma_{n-1},$$

valid for $n \ge 2$. By induction, we easily verify that

$$\int_1^{a_k} \ldots \int_1^{a_{n-1}} \frac{1}{(x_k + 1) \ldots (x_{n-1} + 1)} \mathrm{d}x_{n-1} \ldots \mathrm{d}x_k = \frac{1}{(n-k)!} \left(\log \frac{a_k + 1}{2}\right)^{n-k}$$

https://doi.org/10.1017/S0017089512000249 Published online by Cambridge University Press

and hence

$$\frac{\gamma_n}{4^n} = \frac{\left(\log \lambda^{-1}\right)^{n-1}}{\lambda(n-1)!} - \frac{\gamma_{n-1}}{4^{n-1}}. \tag{3.6}$$

This, in turn, implies that for $n \geq 3$,

$$\frac{\gamma_n}{4^n} = \frac{\left(\log \lambda^{-1}\right)^{n-1}}{\lambda(n-1)!} - \frac{\left(\log \lambda^{-1}\right)^{n-2}}{\lambda(n-2)!} + \frac{\gamma_{n-2}}{4^{n-2}} > \frac{\left(\log \lambda^{-1}\right)^{n-1}}{\lambda(n-1)!} - \frac{\left(\log \lambda^{-1}\right)^{n-2}}{\lambda(n-2)!}. \tag{3.7}$$

This is also true for $n = 2$: we have $\gamma_1 = 4(\lambda^{-1} - 1)$ and hence by (3.6),

$$\frac{\gamma_2}{4} = \frac{\log \lambda^{-1}}{\lambda} - \frac{1}{\lambda} + 1.$$

Consequently, we have $\lim_{\lambda \to 0} \lambda |\{M_\mu f > \lambda\}| = \infty$ and (i) is proved. Next, if we plug $\lambda = \exp(-(n-1)/(p-1))$ into (3.7), we obtain that

$$\begin{aligned} \frac{||M_\mu f||^p_{L^{p,\infty}(\mathbb{R}^n, |\cdot|)}}{||f||^p_{L^p(\mathbb{R}^n, |\cdot|)}} &\geq \frac{\lambda^p |\{M_\mu f > \lambda\}|}{2^n} \\ &> 2^n e^{1-n} \frac{(n-1)^{n-1}}{(n-1)!} \frac{2-p}{(p-1)^{n-1}} \\ &\geq \frac{\kappa_n}{(p-1)^{(n-1)p}}, \end{aligned}$$

for some constant κ_n depending only on n. This gives the optimality of the order.

(iii) Introduce the operators $T_k = \mathcal{M}^{(k)}_{\mu_k} \circ \mathcal{M}^{(k+1)}_{\mu_{k+1}} \circ \ldots \circ \mathcal{M}^{(n)}_{\mu_n}$, $k = 1, 2, \ldots, n$, and let $T_{n+1} = \mathrm{Id}$. We will prove that

$$||T_k f||_{L^{p,\infty}(\mathbb{R},\mu)} \leq c_p ||T_{k+1} f||_{L^{p,\infty}(\mathbb{R}^n \mu)} \tag{3.8}$$

for any f and any $k \in \{1, 2, \ldots, n\}$; this will immediately yield (3.2). To do this, fix $\lambda > 0$ and let $A_\lambda = \{T_k f > \lambda\}$ and $B_\lambda = \{T_{k+1} f > \lambda\}$. Let $\mu^{(k)}$ denote the product measure $\mu_1 \otimes \mu_2 \otimes \ldots \otimes \mu_{k-1} \otimes \mu_{k+1} \otimes \ldots \otimes \mu_n$ on $\mathbb{R}^{n-1}$. By (2.1), applied to $\mathcal{M}^{(k)}_{\mu_k}$, the measure μ_k and the function $t \mapsto T_{k+1} f(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)$, $t \in \mathbb{R}$,

$$\begin{aligned} &\lambda \mu_k\big(\{x_k \in \mathbb{R} : T_k f(x_1, x_2, \ldots, x_n) > \lambda\}\big) \\ &\leq \int_{\{x_k \in \mathbb{R} : T_k f(x) > \lambda\}} T_{k+1} f(x) \mathrm{d}\mu_k(x_k) + \int_{\{x_k \in \mathbb{R} : T_{k+1} f(x) > \lambda\}} \big(T_{k+1} f(x) - \lambda\big) \mathrm{d}\mu_k(x_k). \end{aligned}$$

Integrating this over $\mathbb{R}^{n-1}$ with respect to $\mathrm{d}\mu^{(k)}(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ and multiplying both sides by λ^{p-1}, we obtain

$$\lambda^p \mu(A_\lambda) \leq \lambda^{p-1} \left[\int_{A_\lambda} T_{k+1} f(x) \mathrm{d}\mu(x) + \int_{B_\lambda} \big(T_{k+1} f(x) - \lambda\big) \mathrm{d}\mu(x) \right].$$

Let $(T_{k+1} f)^*$ be the non-increasing rearrangement of $T_{k+1} f$ (the definition is analogous to that of one-dimensional setting). We have

$$\mu(B_\lambda) = \mu(\{T_{k+1} f > \lambda\}) \leq \lambda^{-p} ||T_{k+1} f||^p_{L^{p,\infty}(\mathbb{R}^n, \mu)}, \tag{3.9}$$

https://doi.org/10.1017/S0017089512000249 Published online by Cambridge University Press

so $(T_{k+1}f)^*(t) \le t^{-1/p}||T_{k+1}f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}$ for any $t > 0$. Therefore, using the version of inequality (2.6) in $\mathbb{R}^n$, we obtain

$$\lambda^p \mu(A_\lambda) \le \lambda^{p-1}\left[\int_0^{\mu(A_\lambda)} t^{-1/p}||T_{k+1}f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}\mathrm{d}t \right.$$
$$\left. + \int_0^{\mu(B_\lambda)} \left(t^{-1/p}||T_{k+1}f||_{L^{p,\infty}(\mathbb{R}^n,\mu)} - \lambda\right)\mathrm{d}t\right].$$

If we apply (3.9) and compute the integrals above, we obtain an inequality which can be rewritten in the equivalent form

$$(p-1)\frac{\lambda^p\mu(A_\lambda)}{||T_{k+1}f||^p_{L^{p,\infty}(\mathbb{R}^n,\mu)}} \le p\left(\frac{\lambda^p\mu(A_\lambda)}{||T_{k+1}f||^p_{L^{p,\infty}(\mathbb{R}^n,\mu)}}\right)^{1-1/p} + 1.$$

By virtue of (1.2), this yields $\lambda^p\mu(A_\lambda) \le c_p||T_{k+1}f||_{L^{p,\infty}(\mathbb{R}^n,\mu)}$ and (3.8) follows. We turn to the sharpness. Let $\mu = |\cdot|$ be the Lebesgue measure on $\mathbb{R}^n$, fix $p' > p$ and consider the function

$$f(x_1, x_2, \dots, x_n) = \prod_{k=1}^n |2x_k|^{-1/p'}\chi_{[-1,1]^n}(x).$$

It belongs to $L^p(\mathbb{R}^n, |\cdot|)$, so in particular $||f||_{L^{p,\infty}(\mathbb{R}^n,|\cdot|)} < \infty$. By (2.8), applied to each coordinate (here we use the product structure of f), we have $M_{|\cdot|}f \ge c^n_{p'}f$ on $\mathbb{R}^n$. Therefore, $||M_{|\cdot|}f||_{L^{p,\infty}(\mathbb{R}^n,|\cdot|)} \ge c^n_{p'}||f||_{L^{p,\infty}(\mathbb{R}^n,|\cdot|)}$ and it remains to let $p' \to p$ to see that c^n_p is optimal in (3.2). Finally, to prove the sharpness of (3.3), let f be as above. Fix $\kappa > 1$ and choose $\lambda > 0$ such that $\lambda^p|\{f > \lambda\}| \cdot \kappa > ||f||^p_{L^{p,\infty}(\mathbb{R}^n,\mu)}$. If we put $A = \{f > \lambda\}$, then $M_{|\cdot|}f > c^n_{p'}\lambda$ on A, so

$$\frac{||M_{|\cdot|}f||_{L^{q,\infty}(A,|\cdot|)}}{||f||_{L^{p,\infty}(\mathbb{R}^n,|\cdot|)}} \ge \frac{c^n_{p'}\lambda|A|^{1/q}}{\kappa^{1/p}\lambda|A|^{1/p}} = \frac{c^n_{p'}}{\kappa}|A|^{1/q-1/p}.$$

Since $\kappa > 1$ and $p' > p$ were arbitrary, constant c^n_p is the best in (3.3). □

ACKNOWLEDGEMENTS. The research was partially supported by MNiSW Grant N N201 397437.

REFERENCES

1. A. Bernal, *A note on the one-dimensional maximal function*, *Proc. Roy. Soc. Edinburgh* **111A** (1989), 325–328.

2. L. Grafakos and J. Kinnunen, *Sharp inequalities for maximal functions associated with general measures*, *Proc. Roy. Soc. Edinburgh* **128A** (1998), 717–723.

3. L. Grafakos and S. Montgomery-Smith, *Best constants for uncentered maximal functions*, *Bull. London Math. Soc.* **29** (1997), 60–64.

4. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed. (Cambridge University Press, Cambridge, UK, 1952).

5. P. Sjögren, *A remark on the maximal function for measures in* $\mathbb{R}^n$, *Amer. J. Math.* **105** (1983), 1231–1233.

6. E. M. Stein, *Harmonic analysis: Real variable theory, orthogonality, and oscillatory integrals* (Princeton University Press, Princeton, NJ, 1993).