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# INDUCTION AND RESTRICTION OF $\pi$-SPECIAL CHARACTERS

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**1. Introduction.** The character theory of solvable groups has undergone significant development during the last decade or so and it can now be seen to have quite a rich structure. In particular, there is an interesting interaction between characters and sets of prime numbers.

Let $G$ be solvable and let $\pi$ be a set of primes. The "$\pi$-special" characters of $G$ are certain irreducible complex characters (defined by D. Gajendragadkar [**1**]) which enjoy some remarkable properties, many of which were proved in [**1**]. (We shall review the definition and relevant facts in Section 3 of this paper.) Actually, we need not assume solvability: that $G$ is $\pi$-separable is sufficient, if we are willing to use the Feit-Thompson "odd order" theorem occasionally. We shall state and prove our results under this weaker hypothesis, but we stress that anything of interest in them is already interesting in the solvable case where, of course, the "odd order" theorem is irrelevant.

Gadjendragadkar's paper proves a number of results concerning the behavior of $\pi$-special characters with respect to induction and restriction, but it leaves some questions unanswered. For instance, suppose $G$ is $\pi$-separable, $H \subseteq G$ and $\chi \in \mathrm{Irr}(G)$ is $\pi$-special. Gadjendragadkar shows that if $|G:H|$ is a $\pi'$-number, then $\chi_H$ is necessarily irreducible and $\pi$-special. Without the assumption on the index of $H$, of course, there is no reason to expect $\chi_H$ to be irreducible. If we assume $\chi_H$ is irreducible, however, must it be $\pi$-special? The answer is "no" in general, but "yes" if $2 \notin \pi$.

THEOREM A. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and let $\chi \in \mathrm{Irr}(G)$ be $\pi$-special. If $H \subseteq G$ and $\psi = \chi_H \in \mathrm{Irr}(H)$, then $\psi$ is $\pi$-special.*

There are two difficulties which arise when considering the analogous question about induced characters. If $\chi = \psi^G \in \mathrm{Irr}(G)$ is to be $\pi$-special, where $\psi \in \mathrm{Irr}(H)$, the definition requires $\chi(1)$ be a $\pi$-number, and this forces $H$ to have $\pi$-index. The other problem is more subtle, since the definition requires $\chi$ to have $\pi$-determinantal order and yet extra minus signs occasionally appear in the computation of determinants of induced representations. These signs can introduce a factor of 2 into the determinantal order, and this causes difficulties if $2 \notin \pi$.

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In order to resolve this sign problem, Gadjendragadkar introduced a certain linear character of $H$ (with values $\pm 1$) and he called this somewhat mysterious sign character $\mathrm{csn}_{[H]}$ ($H$ on $G$). Section 2 of this paper serves two purposes. The first of these is to give an exposition of Gadjendragadkar's sign character which includes a description of its definition and key properties (with proofs). Also, at the suggestion of E. C. Dade, we have extended the definition of this character (which in our notation is written $\delta_{(G:H)}$) to include the case where $|G:H|$ is not necessarily a $\pi$-number. In fact, in all but Section 7 of this paper, this extension of the definition is irrelevant and we only use $\delta_{(G:H)}$ when $H$ has $\pi$-index. In Section 7, however, we give (our proof of) a theorem of Dade for which the extended definition is essential.

Suppose $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$ and let $H = I_G(\theta)$, the inertia group. In this situation, character induction defines a bijection (called the *Clifford correspondence*) from $\mathrm{Irr}(H|\theta)$ onto $\mathrm{Irr}(G|\theta)$. If $|G:H|$ is a $\pi$-number, Gadjendragadkar showed (in Theorem 5.10 of [**1**] ) that for $\psi \in \mathrm{Irr}(H|\theta)$, $\psi^G$ is $\pi$-special if and only if $\nu\psi$ is $\pi$-special, where

$$\nu = \delta_{(G:H)} \quad \text{if } 2 \notin \pi \quad \text{and} \quad \nu = 1_H \quad \text{if } 2 \in \pi.$$

Our Theorems B and C generalize Gadjendragadkar's result to the situation where $H$ is not necessarily an inertia group (except that we require the assumption $2 \notin \pi$ to make the "if" part of the result go through.)

THEOREM B. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and suppose $H \subseteq G$ has $\pi$-index. Suppose $\psi \in \mathrm{Irr}(H)$ and $\psi^G = \chi \in \mathrm{Irr}(G)$. Then $\chi$ is $\pi$-special if and only if $\delta_{(G:H)}\psi$ is $\pi$-special.*

The "only if" part of Gajendragadkar's theorem generalizes for all $\pi$ and we can even weaken the hypothesis that $\psi^G$ is irreducible.

THEOREM C. *Let $G$ be $\pi$-special and let $H \subseteq G$. Suppose $\psi \in \mathrm{Irr}(H)$ and that every irreducible constituent of $\psi^G$ is $\pi$-special. Then $|G:H|$ is a $\pi$-number and $\nu\psi$ is $\pi$-special, where*

$$\nu = 1_H \quad \textit{if } 2 \in \pi \quad \textit{and} \quad \nu = \delta_{(G:H)} \quad \textit{if } 2 \notin \pi.$$

We mention that the part of Theorem C which asserts that $|G:H|$ is a $\pi$-number was needed (and proved) in [**5**].

What is left of Theorem B if we drop the condition that $|G:H|$ is a $\pi$-number? In this case, E. C. Dade, in a private communication, pointed out that if we consider a certain larger set of irreducible characters of $G$ than the $\pi$-special characters, then Theorem B generalizes to this situation. We write $D_\pi(G)$ to denote the set of $\chi \in \mathrm{Irr}(G)$ of the form $\chi = \psi^G$ where $\psi \in \mathrm{Irr}(H)$ and $\delta_{(G:H)}\psi$ is $\pi$-special for some subgroup $H$ of $G$.

THEOREM D (Dade). *Let $G$ be $\pi$-separable with $2 \notin \pi$ and let $H \subseteq G$. Suppose $\psi \in \mathrm{Irr}(H)$ and $\psi^G = \chi \in \mathrm{Irr}(G)$. Then $\chi \in D_\pi(G)$ if and only if $\delta_{(G:H)}\psi \in D_\pi(H)$.*

With Dade's permission, we include our proof of this result. We also present a few related properties of the set $D_\pi(G)$.

The final question we consider is the following. Suppose $H \subseteq G$ and $\psi \in \mathrm{Irr}(H)$ is $\pi$-special. If $\psi$ is extendible to $G$, must it necessarily have a $\pi$-special extension? The answer is "yes" if $2 \notin \pi$ or if $|G:H|$ is a $\pi'$-number, but we have not been able to settle the general case.

THEOREM E. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and let $H \subseteq G$. Suppose $\psi \in \mathrm{Irr}(H)$ is $\pi$-special and extendible to $G$. Then $\psi$ has a $\pi$-special extension to $G$.*

For the case that $|G:H|$ is a $\pi'$-number, we get an even stronger result in that we do not have to assume that $\psi$ is extendible to $G$. It suffices that it be "invariant" on $\pi$-elements.

THEOREM F. *Let $G$ be $\pi$-separable and suppose $H \subseteq G$ has $\pi'$-index. Let $\psi \in \mathrm{Irr}(H)$ be $\pi$-special, and assume $\psi(x) = \psi(y)$ whenever $x$ and $y$ are $G$-conjugate $\pi$-elements of $H$. Then $\psi$ has a $\pi$-special extension to $G$.*

In particular, if $H$ is a Hall $\pi$-subgroup of $G$, then $\psi \in \mathrm{Irr}(H)$ is automatically $\pi$-special and we have the following.

THEOREM G. *Let $H$ be a Hall $\pi$-subgroup of the $\pi$-separable group $G$ and let $\psi \in \mathrm{Irr}(H)$. Then $\psi$ extends to $G$ if and only if it is invariant in the sense that $\psi(x) = \psi(y)$ whenever $x, y \in H$ are $G$-conjugate.*

Note that although Theorem G is proved using $\pi$-special characters, its statement is independent of Gajendragadkar's definition. In fact, this result is a generalization of Gallagher's theorem which is precisely the case where $H \lhd G$. (See Corollary 8.16 of **[4]** ).

We mention that using the deeper techniques of **[8]**, it is possible to prove Theorem G even without the assumption that $\psi$ is irreducible. A proof of this will appear in **[9]**.

We close this introduction by referring the reader to the papers **[1]**, **[2]**, **[5]** and **[8]** where further applications and properties of $\pi$-special characters can be found.

**2. Sign characters.** Suppose $H \subseteq G$ and $\psi \in \mathrm{Char}(H)$. We wish to explore the connection between the linear characters $\lambda = \det(\psi)$ of $H$ and $\mu = \det(\psi^G)$ of $G$. These linear characters are related via the transfer map

$$V_{(G:H)}:G \to H/H'$$

and in fact we shall see that

$$\mu(g) = \pm\lambda(V_{(G:H)}(g))$$

where the $\pm$ sign depends on the particular element $g \in G$. (For instance,

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if $\psi = 1_H$, then $\lambda = 1_H$ but $\mu = \det((1_H)^G)$ is the permutation sign character associated with the action of $G$ on the right cosets of $H$.)

Often, the sign problem can be corrected by introducing an appropriate sign character $\delta$ of the subgroup $H$. (By a *sign character*, we mean a linear character with values $\pm 1$.) It was for this purpose that Gadjendragadkar [**1**] introduced his sign character $\text{csn}_{[H]}$ ($H$ on $G$). The principal goal of this section is to give an exposition of, and to expand upon, Gadjendragadkar's sign character. (The treatment we present here was influenced by helpful comments of the referee and by the suggestion of E. C. Dade that the case where the index $|G:H|$ is not necessarily a $\pi$-number should be considered too.)

Suppose $H \subseteq G$ and let $\lambda$ be a linear character of $H$. We define the function $\mu$ on $G$ by setting

$$\mu(g) = \lambda(V_{(G:H)}(g)).$$

Since the transfer map is a homomorphism, $\mu$ is a linear character of $G$ and in fact, $\mu = \lambda^{\otimes G}$, the "tensor induced" character. (Readers unfamiliar with tensor induction may take this as the definition for linear $\lambda$.)

(2.1) LEMMA *Let $H \subseteq G$ and let $\psi \in \text{Char}(H)$. Then*

$$\det(\psi^G) = \epsilon \det(\psi)^{\otimes G}$$

*where $\epsilon = \det((1_H)^G)$ if $\psi(1)$ is odd and $\epsilon = 1_G$ otherwise. In any case, $\epsilon$ is a sign character of $G$.*

*Proof.* Let $\mathcal{Y}$ be a representation of $H$ affording $\psi$ and let $T$ be a right transversal for $H$ in $G$. Then $\psi^G$ is afforded by the "block monomial" representation $\mathcal{Y}^G$ with block positions indexed by pairs $(t, s) \in T \times T$. For $g \in G$, the $(t, s)$-block of $\mathcal{Y}^G(g)$ is zero unless $tg \in Hs$ in which case the block equals $\mathcal{Y}(tgs^{-1})$.

To compute $\det \mathcal{Y}^G(g)$, permute the rows of $\mathcal{Y}^G(g)$ so as to rearrange the blocks and put the matrix into block diagonal form. It follows that for some sign function $\epsilon(g) = \pm 1$, we have

$$\epsilon(g)\det(\mathcal{Y}^G(g)) = \prod_{t \in T} \det \mathcal{Y}(tgs^{-1})$$

where in each factor on the right, $s \in T$ is uniquely determined by $t \in T$ so that $tg \in Hs$.

This yields

$$\det(\psi^G)(g) = \epsilon(g)\det(\psi)(V_{(G:H)}(g))$$

and

$$\det(\psi^G) = \epsilon \det(\psi)^{\otimes G}$$

as required.

What remains is to compute $\epsilon$. If $\psi(1)$ is even, the blocks in the matrix $\mathscr{Y}^G(g)$ have even size and there is no sign penalty for their rearrangement. If $\psi(1)$ is odd, however, $\epsilon(g)$ is the sign of the permutation of $T$ induced by $g$. This equals $\det((1_H)^G)(g)$.

We can sometimes eliminate the sign character $\epsilon$ in 2.1 by multiplying $\psi$ by an appropriate sign character of $H$ before inducing to $G$. We say that a sign character $\delta$ of $H \subseteq G$ has the *induction determinant property* in $G$ if

$$\det((\delta\psi)^G) = \det(\psi)^{\otimes G}$$

for every $\psi \in \mathrm{Char}(H)$.

We mention that there does not always exist a sign character of $H$ having the induction determinant property nor is it necessarily unique if it does exist.

(2.2) LEMMA. *Let $H \subseteq G$ and let $\psi \in \mathrm{Char}(H)$. Then $o(\psi^G)$ divides $2o(\psi)$ and if $\delta$ is a sign character of H having the induction determinant property, then $o((\delta\psi)^G)$ divides $o(\psi)$.*

*Proof.* Write $\lambda = \det(\psi)$ and $m = o(\psi) = o(\lambda)$ so that $\lambda(h^m) = 1$ for all $h \in H$. We have

$$\det(\psi^G)(g^m) = \pm\lambda(V_{(G:H)}(g^m)) = \pm 1$$

since the transfer map $V_{(G:H)}$ is a homomorphism. It follows that

$$\det(\psi)(g^{2m}) = 1$$

as required.

If $\delta$ has the induction determinant property, then

$$\det((\delta\psi)^G)(g^m) = \lambda(V_{(G:H)}(g^m)) = 1$$

and the proof is complete.

Our next task is to produce sign characters which do, in fact, have the induction determinant property.

(2.3) LEMMA. *Suppose $H \subseteq G$ has odd index and let*

$$\delta = \det((1_H)^G)$$

*so that $\delta$ is the permutation sign character of the action of G on the right cosets of H. Then $\delta_H$ has the induction determinant property for H in G.*

*Proof.* We have $(\delta_H\psi)^G = \delta\psi^G$ and so

$$\det((\delta_H\psi)^G) = \delta^{\psi^G(1)}\det(\psi^G).$$

Since $|G:H|$ is odd, we have

$$\delta^{\psi^G(1)} = \delta^{\psi(1)} = \epsilon$$

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where $\epsilon$ is as in Lemma 2.1. We now have

$$\det((\delta_H\psi)^G) = \epsilon \det(\psi^G) = \det(\psi)^{\otimes G}$$

by 2.1.

Now suppose a group $H$ acts on some set $\Omega$. We shall write $\sigma_{[\Omega]}$ to denote the associated permutation sign character of $H$. In particular, if $H \subseteq G$ and an $H$-composition series for $G$ is chosen, then each composition factor $F$ in this series determines a sign character $\sigma_{[F]}$ of $H$.

(2.4) *Definition.* Let $H \subseteq G$ where $G$ is $\pi$-separable and fix an $H$-composition series for $G$. We define sign characters of $H$. Write

a) $$\sigma_{(G:H)} = \prod_F \sigma_{[F]}$$

where $F$ runs through the $H$-composition factors of $G$ which are $\pi$-groups. Also, write

b) $$\delta_{(G:H)} = \sigma_{(G:H)}\sigma_{(H:H)}.$$

We call $\delta_{(G:H)}$ the $\pi$-*standard* sign character of $H$ with respect to $G$.

Note that by the Jordan-Holder theorem, $\sigma_{(G:H)}$ and $\delta_{(G:H)}$ are independent of the choice of the composition series. In general, however, they do depend on $\pi$ and not just on the groups $H$ and $G$. In the case that $2 \notin \pi$ and $|G:H|$ is a $\pi$-number, we shall see that $\delta_{(G:H)}$ is exactly Gajendragadkar's sign character $\text{csn}_{[H]}$ ($H$ on $G$). Also, by Theorem 2.11, near the end of this section, we give another formula for $\delta_{(G:H)}$ in this case.

The next result lists some essential properties of the $\pi$-standard sign character $\delta_{(G:H)}$.

(2.5) THEOREM. *Let $G$ be $\pi$-separable with $2 \notin \pi$.*

a) *If $H \subseteq G$ and $|G:H|$ is a $\pi$-number, then $\delta_{(G:H)}$ has the induction determinant property in $G$.*

b) *If $H \subseteq K \subseteq G$, then*

$$\delta_{(G:H)} = (\delta_{(G:K)})_H\delta_{(K:H)}.$$

c) *If $H \subseteq G$ and $|G:H|$ is a $\pi'$-number, then*

$$\delta_{(G:H)} = 1_H.$$

d) *If $N \subseteq H \subseteq G$ and $N \lhd G$, then*

$$N \subseteq \ker(\delta_{(G:H)}).$$

e) *If $G = XY$ and either $X$ or $Y$ is of the form $N(X \cap Y)$ for some $N \lhd G$, then*

$$\delta_{(X:X\cap Y)} = (\delta_{(G:Y)})_{X\cap Y}.$$

We mention that in the situation of Theorem 2.5(a), Lemma 2.3 suggests that perhaps

$$\delta_{(G:H)} = \det( ( (1_H)^G)_H).$$

As we shall see at the end of this section, this is not necessarily true.

To prove Theorem 2.5 we accumulate a few preliminary results. The first of these is fairly well known and the others are essentially trivial.

(2.6) LEMMA. *Let $H$ act by automorphisms on a group $X$ and suppose $Y \subseteq X$ admits $H$.*

a) *If $( |H|, |X| ) = 1$, then each $H$-invariant right coset of $Y$ in $X$ contains exactly $|\mathbf{C}_Y(H)|$ $H$-fixed elements.*

b) *If $Y \lhd X$ and $|X|$ is odd, then*

$$\sigma_{[X]} = \sigma_{[Y]}\sigma_{[X/Y]}.$$

*Proof.* To prove (a), let $Yx$ be $H$-invariant. Then $Y$ acts transitively on $Yx$ via $z \cdot y = y^{-1}z$ for $z \in Yx$. This action is compatible with the action of $H$ on $Y$ (in the sense of Glauberman's lemma 13.8 of **[4]** ). Since one of $H$ or $Y$ is solvable by the "odd order" theorem, we conclude by Glauberman's lemma that $Yx$ contains $H$-fixed elements, and these constitute a single orbit under $\mathbf{C}_Y(H)$. It follows that these elements form one right coset of $\mathbf{C}_Y(H)$ and (a) is proved.

For (b), it suffices to check that the sign characters behave as stated when restricted to a Sylow 2-subgroup of $H$ and so, without loss of generality, we may assume that $H$ is a 2-group and hence $( |H|, |X| ) = 1$.

Let $\alpha$ be the permutation character of $H$ on the elements of $X$ and let $\beta$ and $\gamma$ be the permutation characters on $Y$ and $X/Y$ respectively. For $h \in H$, application of (a) to the group $\langle h \rangle$ yields that $\alpha(h) = \beta(h)\gamma(h)$ and so $\alpha = \beta\gamma$. We have

$$\sigma_{[X]} = \det(\alpha) = \det(\beta)^{\gamma(1)}\det(\gamma)^{\beta(1)} = \sigma_{[Y]}\sigma_{[X/Y]}$$

since $\gamma(1)$ and $\beta(1)$ are odd.

We can now replace the $H$-composition series in Definition 2.4 by an arbitrary $H$-invariant subnormal series for which each factor is either a $\pi$-group or a $\pi'$-group. We shall refer to such a series as a *good $H$-series* in $G$.

For the rest of this section we assume that $G$ is $\pi$-separable and $2 \notin \pi$.

(2.7) COROLLARY. *Let $H \subseteq G$ and choose any good $H$-series in $G$. Then*

$$\sigma_{(G:H)} = \prod_F \sigma_{[F]}$$

*where F runs over the $\pi$-factors of the given series.*

*Proof.* Refine the given good $H$-series to an $H$-composition series and apply Lemma 2.6(b) repeatedly.

(2.8) LEMMA. *Let $H \subseteq G$ and suppose a good $H$-series for $G$ is given with the property that each $\pi$-factor is either covered or avoided by $H$. Then*

$$\delta_{(G:H)} = \prod_F \sigma_{[F]}$$

*where $F$ runs over those $\pi$-factors (if any) avoided by $H$.*

*Proof.* We can write

$$\sigma_{(G:H)} = \prod_F \sigma_{[F]} \prod_E \sigma_{[E]}$$

where $E$ runs over the $\pi$-factors covered by $H$. These are $H$-isomorphic to all of the $\pi$-factors in a good $H$-series for $H$ and so the second product equals $\sigma_{(H:H)}$ and the result follows since $(\sigma_{(H:H)})^2$ is trivial.

Note that in an $H$-composition series for $G$, each $\pi$-factor is abelian by the "odd order" theorem and it follows that each is either covered or avoided by $H$. It follows via Lemma 2.8 that our $\pi$-standard sign character $\delta_{(G:H)}$ agrees with Gadjendragadkar's sign character $\mathrm{csn}_{[H]}(H \text{ on } G)$ whenever his is defined.

(2.9) COROLLARY. *Suppose $H \subseteq G$ is maximal and has $\pi$-index. Then*

a) $\delta_{(G:H)} = \det( ( (1_H)^G)_H)$.

b) $\delta_{(G:H)}$ *has the induction determinant property in G.*

*Proof.* Let $V = \mathrm{core}_G(H)$ and let $U/V$ be a chief factor of $G$. Then $U \not\subseteq H$ and so $HU = G$ and $U/V$ is a $\pi$-group. It follows that $U/V$ is an $H$-composition factor of $G$ which is avoided by $H$. (Using $2 \notin \pi$ and the "odd order" theorem.) In fact, $U/V$ is the only $H$-composition factor (in a suitable series) avoided by $H$ and so

$$\delta_{(G:H)} = \sigma_{[U/V]}.$$

To prove (a) it suffices to show that the actions of $H$ on its own set of right cosets by right multiplication and on $U/V$ by conjugation are permutation isomorphic. We have a natural bijection from the elements of $U/V$ onto the set of right cosets of $H$ in $G$ via the map which takes $Vu$ to $H(Vu) = Hu$. Since this carries $(Vu)^h$ to $H(Vu)^h = Huh$ for $h \in H$, part (a) follows.

Now (b) is immediate from (a) via Lemma 2.3.

*Proof of Theorem* 2.5. We start with (b) and so we assume $H \subseteq K \subseteq G$ and we fix a $K$-composition series for $G$ and observe that this is automatically a good $H$-series in $G$. It follows via Corollary 2.7 that

$$\sigma_{(G:H)} = (\sigma_{(G:K)})_H$$

and similarly

$$\sigma_{(K:H)} = (\sigma_{(K:K)})_H.$$

Therefore

$$\begin{aligned}(\delta_{(G:K)})_H\delta_{(K:H)} &= (\sigma_{(G:K)}\sigma_{(K:K)})_H\sigma_{(K:H)}\sigma_{(H:H)} \\ &= \sigma_{(G:H)}\sigma_{(K:H)}\sigma_{(K:H)}\sigma_{(H:H)} \\ &= \delta_{(G:H)}\end{aligned}$$

since $(\sigma_{(K:H)})^2$ is trivial. This proves (b).

We work by induction on $|G:H|$ to prove (a). The case where $H = G$ is trivial and the result follows by 2.9(b) when $H$ is maximal in $G$. In the remaining case, we choose $K$ with $H < K < G$ and observe that by the inductive hypothesis, $\delta_{(G:K)}$ and $\delta_{(K:H)}$ have the induction determinant property in $G$ and $K$ respectively. Let $\psi \in \mathrm{Char}(H)$ and compute

$$(\delta_{(G:H)}\psi)^G = (\,(\,(\delta_{(G:K)})_H\delta_{(K:H)}\psi)^K)^G = (\delta_{(G:K)}(\delta_{(K:H)}\psi)^K)^G.$$

Therefore

$$\det(\delta_{(G:H)}\psi)^G = (\,(\det(\psi)\,)^{\otimes K})^{\otimes G} = (\det(\psi)\,)^{\otimes G}$$

where the last equality follows, for instance, by the transivity of the transfer map. This proves (a).

Part (c) is immediate by Lemma 2.8 because $H$ covers every $H$-composition factor which is a $\pi$-group when $|G:H|$ is a $\pi'$-number.

For (d), choose an $H$-composition series for $G$ through $N$ and note that the only factors which $H$ avoids lie above $N$ and so $N$ acts trivially on each such factor. For all these factors $F$, we have $(\sigma_{[F]})_N = 1_N$ and hence

$$(\delta_{(G:H)})_N = 1_N$$

by 2.8.

Now for (e), assume first that $X = N(X \cap Y)$. Choose a $Y$-composition series for $G$ through $N$ and note that by intersecting the terms of this series with $X$, we obtain a good $(X \cap Y)$-series for $X$ through $N$. Since $N \subseteq X$, the terms of this series below $N$ are the terms of the original series below $N$ and since

$$Y \cap N = (X \cap Y) \cap N,$$

a $\pi$-factor of this series below $N$ is covered or avoided by $X \cap Y$. All factors of the original series above $N$ are covered by $Y$ and all factors above $N$ in the intersected series are covered by $X \cap Y$. Lemma 2.8 now yields that

$$(\delta_{(G:Y)})_{X\cap Y} = \delta_{(X:X\cap Y)}$$

as required.

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Finally, assume $Y = N(X \cap Y)$ in (e). By (b), we have

$$(\delta_{(G:Y)})_{X \cap Y}\delta_{(Y:X \cap Y)} = \delta_{(G:X \cap Y)} = (\delta_{(G:X)})_{X \cap Y}\delta_{(X:X \cap Y)}.$$

Since

$$(\delta_{(G:X)})_{X \cap Y} = \delta_{(Y:X \cap Y)}$$

by the part of (e) already done, substitution and cancellation yield the result.

We close this section with some further observations about the $\pi$-standard sign character which will not, in fact, be needed in the remainder of this paper. Although it is not true in general (or even under the assumption that $|G:H|$ is a $\pi$-number) that

$$\delta_{(G:H)} = \det(((1_H)^G)_H),$$

we have the following.

(2.10) LEMMA. *Let $S$ be a $\pi$-complement in $G$. Then*

$$\delta_{(G:S)} = \det(((1_S)^G)_S).$$

*Proof.* Let $\alpha$ be the permutation character of $S$ on its right cosets so that

$$\alpha = ((1_S)^G)_S.$$

By Theorem B of **[6]**, we can write

$$\alpha = \prod \alpha_{[F]}$$

where $F$ runs over the $\pi$-factors in an $S$-composition series for $G$ and $\alpha_{[F]}$ is the permutation character of $S$ in its action on $F$.

Since each $\alpha_{[F]}(1)$ is odd, we have

$$\sigma_{(G:S)} = \prod_F \sigma_{[F]} = \prod_F \det(\alpha_{[F]}) = \det(\alpha).$$

However, $\sigma_{(S:S)} = 1_S$ since $S$ has no $\pi$-factors, and thus

$$\delta_{(G:S)} = \sigma_{(G:S)} = \det(\alpha).$$

Lemma 2.10 enables us to produce a formula for $\delta_{(G:S)}$ (when $|G:H|$ is a $\pi$-number) which does not involve the choice of an $H$-series for $G$.

(2.11) THEOREM. *Let $H \subseteq G$ with $|G:H|$ a $\pi$-number. Let $S \subseteq H$ be a $\pi$-complement. Then*

$$\delta_{(G:H)} = \det(((1_S)^G)_H)\det((1_S)^H).$$

*Proof.* Since both sides of the asserted equality are sign characters of $H$ and $|H:S|$ is odd, it suffices to show that both sides have the same

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restriction to $S$. By 2.10, the right side yields $\delta_{(G:S)}\delta_{(H:S)}$ and the left side gives $(\delta_{(G:H)})_S$. By Theorem 2.5(b), we have

$$\delta_{(G:S)} = (\delta_{(G:H)})_S\delta_{(H:S)}$$

and the result follows.

We close by sketching an example which shows that the equation

$$\delta_{(G:H)} = \det(((1_H)^G)_H)$$

does not necessarily hold even when $|G:H|$ is a $\pi$-number.

Let $\pi$ be the set of all odd primes and let $G$ be solvable with $S \in \mathrm{Syl}_2(G)$ and $G = \mathbf{O}^2(G)$. Suppose we can find a subgroup $H \subseteq G$ with $S \subseteq H$, $|H:S| = 3$ and $S \ntriangleleft H$. Since $G = \mathbf{O}^2(G)$ can have no nontrivial sign character, we have

$$\det((1_S)^G) = 1_G$$

and so $\delta_{(G:H)} = \det((1_S)^H)$ by 2.11. Now $(1_S)^H$ is a permutation character of degree 3 and

$$|H/\mathrm{core}_H(S)| > 3$$

and it follows that $\delta_{(G:H)}$ is nontrivial. On the other hand,

$$\det((1_H)^G) = 1_G$$

and so we cannot have

$$\delta_{(G:H)} = \det(((1_H)^G)_H).$$

To construct an explicit example where all this happens, let

$$G = \mathbf{O}^2(\Sigma_4 \sim \mathbf{Z}_3)$$

so that $G = B \rtimes \mathbf{Z}_3$ and $B$ has index 2 in $\Sigma_4 \times \Sigma_4 \times \Sigma_4$. Take

$$H = (\Sigma_4 \times D \times D) \cap B$$

where $D \in \mathrm{Syl}_2(\Sigma_4)$. Note that $S = (D \times D \times D) \cap B$ is not normal in $H$ or else it would be normal in $\langle H, \mathbf{Z}_3\rangle = G$ and thus contained in $\mathbf{O}_2(\Sigma_4 \times \Sigma_4 \times \Sigma_4)$. This is not the case since

$$|\mathbf{O}_2(\Sigma_4 \times \Sigma_4 \times \Sigma_4)| = 2^6 < 2^8 = |S|.$$

**3. $\pi$-special characters.** In this section, we review the definition and some of the key properties of $\pi$-special characters.

(3.1) *Definition.* Let $G$ be $\pi$-separable and let $\chi \in \mathrm{Irr}(G)$. Then $\chi$ is *$\pi$-special* provided that its degree $\chi(1)$ is a $\pi$-number and that the determinantal order $o(\theta)$ is also a $\pi$-number for every irreducible constituent $\theta$ of the restriction of $\chi$ to every subnormal subgroup of $G$.

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Note that if $\chi$ is $\pi$-special, then every irreducible constituent of the restriction of $\chi$ to a normal subgroup is $\pi$-special.

One of the most striking properties of $\pi$-special characters is the following.

(3.2) THEOREM. (Gadjendragadkar) *Let $G$ be $\pi$-separable and let $\chi, \psi \in \mathrm{Irr}(G)$ be $\pi$-special and $\pi'$-special respectively. Then $\chi\psi$ is irreducible. Furthermore, if $\chi\psi = \chi'\psi'$ where $\chi'$ and $\psi'$ are also $\pi$-special and $\pi'$-special, then $\chi = \chi'$ and $\psi = \psi'$.*

*Proof.* This is Proposition 7.1 of [**1**].

In particular, if $2 \notin \pi$ and $\chi \in \mathrm{Irr}(G)$, then there is at most one character $\delta$ such that $\delta^2 = 1_G$ and $\delta\chi$ is $\pi$-special. This follows since $\chi = (\chi\delta) \cdot \delta$ is the $\pi - \pi'$ factorization of $\chi$. This proves a kind of uniqueness in our Theorems B and C.

How can one tell if $\chi \in \mathrm{Irr}(G)$ is $\pi$-special? Choose a maximal normal subgroup $N \lhd G$ and let $\theta$ be an irreducible constituent of $\chi_N$. If $\chi$ is $\pi$-special then so is $\theta$. Conversely, suppose we can establish somehow (perhaps via an inductive hypothesis) that $\theta$ is $\pi$-special. The following result is then useful.

(3.3) LEMMA. *Let $N \lhd G$ (where $G$ is $\pi$-separable) and let $\theta \in \mathrm{Irr}(N)$ be $\pi$-special.*

a) *If $G/N$ is a $\pi$-group, then every irreducible constituent of $\theta^G$ is $\pi$-special.*

b) *If $G/N$ is a $\pi'$-group and $\theta$ is invariant in $G$, then $\theta$ has a unique extension $\chi$ to $G$ with $o(\chi)$ a $\pi$-number. This extension is $\pi$-special.*

*Proof.* See Propositions 4.3 and 4.5 of [**1**].

An easy consequence of Lemma 3.3(b) which we shall need is the following, which appears as part of Lemma 2.10 of [**5**].

(3.4) LEMMA. *Let $N \subseteq H \subseteq G$ with $G/N$ a $\pi'$-group and let $\theta \in \mathrm{Irr}(H)$. Suppose every irreducible constituent of $\theta^G$ is $\pi$-special. Then $H = G$.*

Next, we mention the following result on restrictions.

(3.5) THEOREM. (Gadjendragadkar) *Let $G$ be $\pi$-separable and suppose $H \subseteq G$ has $\pi'$-index. Then restriction defines an injection of the set of $\pi$-special characters of $G$ into that of $H$.*

*Proof.* This is Proposition 1.6 of [**1**].

Finally, we give a useful criterion for establishing that a character has a $\pi - \pi'$ factorization in the sense of Theorem 3.2. In particular, this will show that every quasiprimitive character factors.

(3.6) THEOREM. *Let $G$ be $\pi$-separable and suppose $\chi \in \mathrm{Irr}(G)$. Assume there exists a normal series*

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$$1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_m = G$$

*such that each factor of the series is either a $\pi$-group or a $\pi'$-group and each restriction $\chi_{G_i}$ is a homogeneous character. Then $\chi = \alpha\beta$ for some $\pi$-special character $\alpha$ and $\pi'$-special character $\beta$.*

*Proof.* This follows by repeated application of Lemma 2.5 of [**5**].

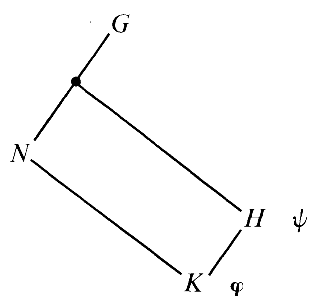
**4. Induced characters.** In this section, we prove Theorem C.

(4.1) THEOREM. *Let $G$ be $\pi$-separable. Suppose $H \subseteq G$ and $\psi \in \mathrm{Irr}(H)$ and assume that every irreducible constituent of $\psi^G$ is $\pi$-special. Then*

a) *$|G:H|$ is a $\pi$-number.*

b) *$\nu\psi$ is $\pi$-special, where $\nu = 1_H$ if $2 \in \pi$ and $\nu = \delta_{(G:H)}$ if $2 \notin \pi$.*

*Proof.* Working by induction on $|G|$ we may assume $|G| > 1$. Let $N \lhd G$ be a maximal normal subgroup and write $K = N \cap H$. Let $\varphi$ be an irreducible constituent of $\psi_K$ and observe that $\varphi^N$ is a constituent of $(\psi^G)_N$. Since $N \lhd G$, it follows that all irreducible constituents of $\varphi^N$ are $\pi$-special and the inductive hypothesis applies. We conclude that $|N:K|$ is a $\pi$-number and that $\mu\varphi$ is $\pi$-special, where $\mu = 1_K$ if $2 \in \pi$ and $\mu = \delta_{(N:K)}$ if $2 \notin \pi$.




Now consider the case where $G/N$ is a $\pi$-group. Here, (a) is clear since $|G:H|$ divides $|G:K| = |G:N|\,|N:K|$, a $\pi$-number.

We claim that $\nu_K = \mu$. This is clear if $2 \in \pi$ so assume $2 \notin \pi$. Then

$$\nu = \delta_{(G:H)} = \delta_{(NH:H)}(\delta_{(G:NH)})_H$$

by Theorem 2.5(b). However, $N \subseteq \ker(\delta_{(G:NH)})$ by 2.5(d) and so

$$\delta_{(G:NH)} = 1_{NH}$$

since $NH/N$ has odd order because $2 \notin \pi$. Therefore

$$\nu_K = (\delta_{(NH:H)})_K = \delta_{(N:K)} = \mu$$

by Theorem 2.5(e).

We now know that the $\pi$-special character $\mu\varphi$ is a constituent of $(\nu\psi)_K$. Since $H/K$ is a $\pi$-group, Lemma 3.3(a) implies that $\nu\psi$ is $\pi$-special, as desired.

In the remaining case, $G/N$ is a $\pi'$-group. Let $T = I_H(\varphi)$, the inertia group, and let $\eta \in \mathrm{Irr}(T)$ with $\eta^H = \psi$. Let $\xi$ be any irreducible constituent of $\eta^{NT}$. Then $\xi^G$ is a constituent of $\eta^G = \psi^G$ and so all irreducible constituents of $\xi^G$ are $\pi$-special. By Lemma 3.4, we conclude that $NT = G$ and it follows that $NH = G$ and $T = H$ so that $\varphi$ is invariant in $H$. In particular, $|G:H| = |N:K|$, a $\pi$-number, and (a) is proved.

In this case too, we have $\nu_K = \mu$. Again, this is trivial if $2 \in \pi$ and otherwise

$$\nu_K = (\delta_{(G:H)})_K = \delta_{(N:K)} = \mu$$

by Theorem 2.5(e). It follows that $\mu$ is invariant in $H$ and thus $\mu\varphi$ is an $H$-invariant $\pi$-special character of $K$. By Lemma 3.3(b), $\mu\varphi$ has a $\pi$-special extension $\gamma \in \mathrm{Irr}(H)$. Since $\nu\psi \in \mathrm{Irr}(H)$ lies over $\mu\varphi$, it follows by Gallagher's lemma (6.17 of **[4]**) that $\nu\psi = \beta\gamma$ for some $\beta \in \mathrm{Irr}(H/K)$. It now suffices to show that $\beta = 1_H$.

By Lemma 3.3(b), restriction defines an injection from the set of $\pi$-special characters of $G$ to that of $N$. Since all irreducible constituents of $\psi^G$ are $\pi$-special, it follows that if $\chi$ is any one of these we have

$$[\psi^G, \chi] = [(\psi^G)_N, \chi_N] = [(\psi_K)^N, \chi_N].$$

Now $\psi_K = \beta(1)\varphi$ and this yields

$$[\psi^G, \chi] = \beta(1)[\varphi^N, \chi_N] = \beta(1)[\varphi^G, \chi].$$

However, $\psi^G$ is a constituent of $\varphi^G$ and we have

$$[\psi^G, \chi] = \beta(1)[\varphi^G, \chi] \geqq \beta(1)[\psi^G, \chi].$$

Since $[\psi^G, \chi] > 0$, we conclude that $\beta(1) = 1$.

Now $\beta$ is a character of $H/K$ and so there exists a (linear) character $\lambda \in \mathrm{Irr}(G/N)$ with $\lambda_H = \beta$. We compute $o(\bar{\lambda}\psi^G)$ in two ways.

First,

$$\bar{\lambda}\psi^G = (\bar{\beta}\psi)^G = (\nu\gamma)^G$$

since $\nu\psi = \beta\gamma$ and $\beta(1) = 1$. If $2 \notin \pi$, then $\nu = \delta_{(G:H)}$ has the induction determinant property by Theorem 2.5(a) and so Lemma 2.2 yields that

$$o((\nu\gamma)^G) \text{ divides } o(\gamma).$$

Since $\gamma$ is $\pi$-special, we conclude that $o(\bar{\lambda}\psi^G)$ is a $\pi$-number. If $2 \in \pi$, however, then $\nu = 1_H$ and

$$o(\gamma^G) \text{ divides } 2 \cdot o(\gamma)$$

by Lemma 2.2. Thus, in any case, we have that $o(\bar{\lambda}\psi^G)$ is a $\pi$-number.

On the other hand,

$$\det(\bar{\lambda}\psi^G) = \bar{\lambda}^m \det(\psi^G)$$

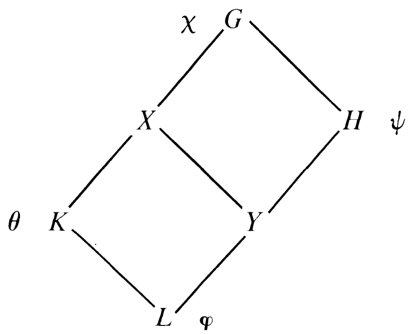
where

$$m = \psi^G(1) = |G:H|\psi(1) = |G:H|\varphi(1)$$

is a $\pi$-number. Also, $o(\psi^G)$ is a $\pi$-number since all of its irreducible constituents are $\pi$-special. Now $o(\bar{\lambda})$ is a $\pi'$-number since $\bar{\lambda} \in \mathrm{Irr}(G/N)$. It follows that $o(\bar{\lambda})$ divides $o(\bar{\lambda}\psi^G)$ which we have established is a $\pi$-number. Therefore $\lambda = 1_G$ and thus $\beta = 1_H$, as desired.

**5. Irreducible induction and restriction.** In order to prove Theorems A and B, we need the following result.

(5.1) PROPOSITION. *Let $G$ be $p$-solvable with $p \neq 2$, and let $K/L$ be a $p$-chief factor of $G$. Suppose $HK = G$ and $H \cap K = L$ and assume $\mathbf{C}_H(K/L) = L$. Let $K \subseteq X \subseteq G$ and write $Y = X \cap H$. Let $\theta \in \mathrm{Irr}(K)$ and $\varphi \in \mathrm{Irr}(L)$ with $[\theta_L, \varphi] \neq 0$.*




a) *If there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi_H$ is irreducible and $\chi_K$ is a multiple of $\theta$, then restriction defines a bijection*

$$\mathrm{Irr}(X|\theta) \rightarrow \mathrm{Irr}(Y|\varphi).$$

b) *If there exists $\psi \in \mathrm{Irr}(H)$ such that $\psi^G$ is irreducible and $\psi_L$ is a multiple of $\varphi$, then induction defines a bijection*

$$\mathrm{Irr}(Y|\varphi) \rightarrow \mathrm{Irr}(X|\theta).$$

*Proof.* In situation (a), it suffices to show that $\theta_L = \varphi$ since the result then follows by Lemma 10.5 of [**3**]. Similarly, in situation (b), it suffices to show that $\varphi^K = \theta$ since then $I_K(\varphi) = L$ and hence $I_X(\varphi) = Y$ and the result follows by the standard Clifford correspondence (Lemma 10.4 of [**3**] or Theorem 6.11 of [**4**] ).

Under the hypotheses of either (a) or (b), we claim that it is not possible for $\theta$ and $\varphi$ to be "fully ramified" with respect to $K/L$. This follows from Theorem 1.6 of [**7**], together with the observation that the hypotheses guarantee that every complement for $K/L$ in $G/L$ is conjugate to $H/L$. (We are using the $p$-solvability here and the assumption that $\mathbf{C}_H(K/L) = L$.)

Now assume we are in situation (a). Then $\theta$ is invariant in $G$ and since $\theta$

is not fully ramified with respect to $K/L$, there are two possibilities by the "going down" theorem (6.18 of [4]). Either $\theta_L = \varphi$ and we are done, or $\theta_L$ is a sum of $|K:L|$ distinct irreducible constituents. In the latter case, let $T = I_G(\varphi)$. Then $T \cap K = L$ and since $\theta$ is invariant in $G$, we have $TK = G$. Therefore, $T/L$ is a complement for $K/L$ in $G/L$ and thus $T$ is conjugate to $H$. It follows that $\chi_T$ is irreducible, and this is impossible since $T < G$ and $\chi$ is induced from $T$. This completes the proof of (a).

Under the hypotheses of (b), $\varphi$ is invariant in $H$ and the "going up" theorem (Problem 6.12 of [3]) applies. Since we know that $\varphi$ is not fully ramified with respect to $K/L$, there are just two possibilities. One is that $\varphi^K = \theta$, in which case we are done. What remains is to eliminate the possibility that $\varphi^K$ is a sum of $|K:L|$ distinct irreducible constituents.

In that case, $G$ permutes these irreducible characters transitively since $\psi^G$ is irreducible and $\varphi^K$ is a constituent of $(\psi^G)_K$. In particular, $K < G$ and we let $M/K$ be a chief factor of $G$. Since $\mathbf{C}_G(K/L) = K$ and $G$ is $p$-solvable, we conclude that $M/K$ is a $p'$-group. It follows that $M/K$ acts trivially on any set of $p$-power size on which $G/K$ acts transitively. We conclude that $M$ stabilizes all of the irreducible constituents of $\varphi^K$. In particular, $\theta$ is invariant in $M$.

Now

$$|K:L|\theta(1) = \varphi^K(1) = |K:L|\varphi(1)$$

and so $\theta(1) = \varphi(1)$ and $\theta_L = \varphi$. The other irreducible constituents of $\varphi^K$ are precisely the characters $\lambda\theta$ as $\lambda$ runs over $\mathrm{Irr}(K/L)$. Since these must be distinct for distinct $\lambda$, and all of them are invariant in $M$, it follows that $M$ stabilizes each $\lambda$ and so $M \subseteq \mathbf{C}(K/L)$. This is a contradiction, and completes the proof.

We remark that part (b) of this result has a nontrivial intersection with Proposition 3.6 of [5].

**6. More induced characters.** We are now ready to prove Theorem B which we restate here.

(6.1) THEOREM. *Let $G$ be $\pi$-separable with $2 \notin \pi$, and let $H \subseteq G$ with $|G:H|$ a $\pi$-number. Suppose $\psi \in \mathrm{Irr}(H)$ and assume $\psi^G = \chi \in \mathrm{Irr}(G)$. Let $\delta = \delta_{(G:H)}$. Then $\chi$ is $\pi$-special if and only if $\delta\psi$ is $\pi$-special.*

*Proof.* Since the "only if" part follows from Theorem 4.1, we need to prove "if". Work by double induction: first on $|G|$ and then on $|G:\mathrm{core}_G(H)|$. Note that if $H = G$, the result is trivial and so the inductions are suitably initialized and we may assume $H < G$. Let $L \lhd G$ be maximal with the property that $LH < G$. Write $U = LH$ and suppose $U > H$. Let

$$\lambda = \delta_{(U:H)} \quad \text{and} \quad \mu = \delta_{(G:U)}$$

and observe that $\delta = \mu_H\lambda$ by Theorem 2.5(b).

Write $\psi^U = \eta \in \mathrm{Irr}(U)$. We have

$$(\mu_H\psi)^U = \mu\eta \in \mathrm{Irr}(U)$$

and

$$\lambda(\mu_H\psi) = \delta\psi \text{ is } \pi\text{-special.}$$

Since $U < G$, the inductive hypothesis yields that $\mu\eta$ is $\pi$-special. Also, $\eta^G = \chi$ and a second application of the inductive hypothesis yields that $\chi$ is $\pi$-special. (Note that

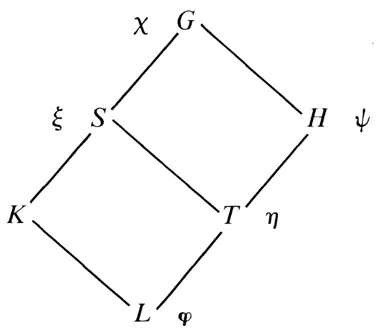
$$\mathrm{core}_G(U) = L > \mathrm{core}_G(H)$$

by the maximality of $L$ and the assumption that $U > H$.)

We may now assume $L \subseteq H$ (and thus $L = \mathrm{core}_G(H)$ ). Let $K/L$ be a chief factor of $G$ so that $KH = G$ by the maximality of $L$. Then $|G:H|$ divides $|K/L|$ which is therefore a $\pi$-group.

Since $2 \notin \pi$, the "odd order" theorem implies that $G$ is $\pi$-solvable and in particular, $K/L$ is an elementary abelian $p$-group and $p \neq 2$. Also, $G$ is $p$-solvable since $p \in \pi$. Furthermore, $H \cap K = L$ since $K/L$ is abelian, and $\mathbf{C}_H(K/L) = L$ since this centralizer is necessarily normal in $G$.

Let $\varphi$ be an irreducible constituent of $\psi_L$ and write $T = I_H(\varphi)$ and $S = KT$. Assume $T < H$ and let $\eta \in \mathrm{Irr}(T|\varphi)$ with $\eta^H = \psi$ and let $\xi = \eta^S$. Since $\xi^G = \eta^G = \psi^G = \chi$, we see that $\xi$ is irreducible.




Since $\delta\psi$ is $\pi$-special, $\psi(1)$ is a $\pi$-number and thus $|G:S| = |H:T|$ is a $\pi$-number. Write $\gamma = \delta_{(G:S)}$ and observe that $\gamma_T = \delta_{(H:T)}$ by Theorem 2.5(e). Now

$$(\delta_T\eta)^H = \delta\psi$$

is $\pi$-special, and thus $\gamma_T(\delta_T\eta)$ is $\pi$-special by Theorem 4.1. Also,

$$(\gamma_T\eta)^S = \gamma\xi \in \mathrm{Irr}(S)$$

and $\delta_T = \delta_{(S:T)}$ by Theorem 2.5(e). Since $\delta_T(\gamma_T\eta)$ is $\pi$-special, the inductive hypothesis yields that $\gamma\xi$ is $\pi$-special. (Since we are assuming

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$T < H$, we have $S < G$ and the inductive hypothesis does apply.) Finally

$$\text{core}_G(S) \supseteq K > L = \text{core}_G(H)$$

and the inductive hypothesis implies that $\chi$ is $\pi$-special, as desired.

We are now reduced to the situation where $\varphi$ is invariant in $H$. If $K = G$, then $H = L \lhd G$. In this case, $\delta = 1_H$ by Theorem 2.5(d), and thus $\psi$ is $\pi$-special and it follows that $\chi$ is $\pi$-special by Lemma 3.3(a). We may now assume $K < G$ and we let $K \subseteq N \lhd G$ where $N$ is a maximal normal subgroup. Let $M = N \cap H$.

Let $\eta$ be an irreducible constituent of $\psi_M$ and write $\xi = \eta^N$. Observe that $\xi \in \text{Irr}(N)$ by Proposition 5.1(b) and that

$$\chi_N = (\psi^G)_N = (\psi_M)^N$$

so that $\xi$ is a constituent of $\chi_N$. Also note that

$$\delta_{(N:M)} = \delta_M$$

by Theorem 2.5(e).

Since $\delta\psi$ is $\pi$-special and $M \lhd H$, we have $\delta_M\eta$ is $\pi$-special and thus $\xi$ is $\pi$-special by the inductive hypothesis applied in $N$. If $G/N$ is a $\pi$-group, then $\chi$ is $\pi$-special by Lemma 3.3(a).

Finally, suppose $G/N$ is a $\pi'$-group. Since $\chi(1) = |G:H|\psi(1)$ is a $\pi$-number, we conclude that $\chi_N$ is irreducible and thus $\chi_N = \xi$. By Lemma 3.3(b), it suffices to show that $o(\chi)$ is a $\pi$-number. We have, however,

$$\chi = (\delta(\delta\psi))^G$$

and thus

$$o(\chi) \text{ divides } o(\delta\psi)$$

by Theorem 2.5(a) and Lemma 2.2. Since $\delta\psi$ is $\pi$-special, the result follows.

That $2 \notin \pi$ was used in the above proof in order to apply the "odd order" theorem and conclude that $G$ is $\pi$-solvable. Even if we assume $G$ is solvable, however, there seems to be no analogous result when $2 \in \pi$ as the following example will show. It is in Proposition 5.1 where oddness is used in a fundamental way.

(6.2) *Example.* There exists a solvable group $G$ with subgroup $H$ and $\psi \in \text{Irr}(H)$ such that

a) $|G:H|$ is a $\{2\}$-number.

b) $\psi^G = \chi \in \text{Irr}(G)$.

c) $\lambda\psi$ is $\{2\}$-special for every linear character $\lambda$ of $H$.

d) $\chi$ is not $\{2\}$-special.

*Proof.* Take $G = GL(2, 3)$ and $H$ of index 4. Then $H = Z \times K$ where $Z = \mathbf{Z}(G)$ and we take $\psi = \tau \times 1_K$, where $\tau$ is the nontrivial linear character of $Z$. The stated properties are easy to establish.

**7. $\pi$-induction and $D_\pi(G)$.** In this section we prove a result of E. C. Dade [private communication] which extends Theorem B to the case where $|G:H|$ is not necessarily a $\pi$-number. We present this material here with his permission.

Suppose $G$ is $\pi$-separable and $2 \notin \pi$. Theorems B and C can be conveniently restated if we define a twisted form of character induction which we shall call *$\pi$-induction.* If $H \subseteq G$ and $\psi$ is any class function of $H$, we write

$$\psi^{\pi G} = (\delta_{(G:H)}\psi)^G.$$

Note that if $|H|$ is odd, then $\psi^{\pi G} = \psi^G$.

(7.1) THEOREM. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and let $H \subseteq G$ and $\psi \in \mathrm{Irr}(H)$.*

a) *Suppose $\psi^{\pi G} = \chi \in \mathrm{Irr}(G)$ and $|G:H|$ is a $\pi$-number. Then $\psi$ is $\pi$-special if and only if $\chi$ is $\pi$-special.*

b) *Suppose that every irreducible constituent of $\psi^{\pi G}$ is $\pi$-special. Then $\psi$ is $\pi$-special and $|G:H|$ is a $\pi$-number.*

*Proof.* To obtain (a) apply Theorem B to the character

$$\delta_{(G:H)}\psi \in \mathrm{Irr}(H)$$

and for (b), apply Theorem C to this character.

Our object will be to consider the situation of Theorem 7.1(a) if we drop the hypothesis that $|G:H|$ is a $\pi$-number. We define

$$D_\pi(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi = \psi^{\pi G} \text{ where } \psi \text{ is a } \pi\text{-special character of some subgroup of } G\}.$$

This definition, essentially due to Dade, of course only applies when $G$ is $\pi$-separable with $2 \notin \pi$.

As a preliminary result, we mention the following.

(7.2) COROLLARY. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and let $\chi \in \mathrm{Irr}(G)$. Then $\chi$ is $\pi$-special if and only if $\chi \in D_\pi(G)$ and $\chi(1)$ is a $\pi$-number.*

*Proof.* Since $\delta_{(G:G)} = 1_G$ by Theorem 2.5(d), we have $\chi^{\pi G} = \chi$ and so if $\chi$ is $\pi$-special, then $\chi \in D_\pi(G)$ and certainly $\chi(1)$ is a $\pi$-number.

Conversely, suppose $\chi \in D_\pi(G)$ and write $\chi = \psi^{\pi G}$ where $\psi \in \mathrm{Irr}(H)$ is $\pi$-special and $H \subseteq G$. Then

$$\chi(1) = |G:H|\psi(1)$$

and if this is a $\pi$-number, then $\chi$ is $\pi$-special by Theorem 7.1(a).

Our principal result is the following, which, in view of 7.2, generalizes 7.1(a).

(7.3) THEOREM (Dade). *Let $G$ be $\pi$-separable with $2 \notin \pi$ and suppose $\psi^{\pi G} = \chi$ where $\psi \in \text{Irr}(H)$ and $\chi \in \text{Irr}(G)$. Then $\psi \in D_\pi(H)$ if and only if $\chi \in D_\pi(G)$.*

We present some preliminary results which are the analogs for $\pi$-induction of well known facts about ordinary induction.

(7.4) LEMMA. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and let $H \subseteq G$. Suppose $\psi$ is a class function of $H$.*

a) *If $H \subseteq K \subseteq G$, then $(\psi^{\pi K})^{\pi G} = \psi^{\pi G}$.*

b) *If $M \lhd G$, then $(\psi^{\pi MH})_M = (\psi_{M \cap H})^{\pi M}$.*

c) *If $\psi \in \text{Char}(H)$ and $N \subseteq H$ with $N \lhd G$, then $\psi_N$ is a (not necessarily irreducible) constituent of $(\psi^{\pi G})_N$.*

*Proof.* For (a), we have

$$\begin{aligned}((\psi)^{\pi K})^{\pi G} &= (\delta_{(G:K)}(\delta_{(K:H)}\psi)^K)^G \\ &= (((\delta_{(G:K)})_H\delta_{(K:H)}\psi)^K)^G \\ &= (\delta_{(G:H)}\psi)^G = \psi^{\pi G}\end{aligned}$$

using Theorem 2.5(b).

To prove (b), use 2.5(e) to compute

$$\begin{aligned}(\psi^{\pi MH})_M &= ((\delta_{(MH:H)}\psi)^{MH})_M = ((\delta_{(MH:H)}\psi)_{M \cap H})^M \\ &= (\delta_{(M:M \cap H)}\psi_{M \cap H})^M = (\psi_{M \cap H})^{\pi M}.\end{aligned}$$

Finally, for (c), note that

$$(\delta_{(G:H)})_N = 1_N$$

by 2.5(d) and so

$$\psi_N = (\delta_{(G:H)}\psi)_N.$$

Since $\delta_{(G:H)}\psi$ is a constituent of $(\psi^{\pi G})_H$, we conclude that $\psi_N$ is a constituent of $(\psi^{\pi G})_N$.

Next, we have the $\pi$-induction analog of the Clifford correspondence.

(7.5) LEMMA. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and let $N \lhd G$ and $\theta \in \text{Irr}(N)$. Let $T = I_G(\theta)$, the inertia group. Then the map $\psi \mapsto \psi^{\pi G}$ defines a bijection*

$$\text{Irr}(T|\theta) \to \text{Irr}(G|\theta).$$

*Proof.* Since $N \subseteq \ker\delta_{(G:T)}$ by 2.5(d), the map

$$\psi \mapsto \delta_{(G:T)}\psi$$

defines a bijection of $\mathrm{Irr}(T|\theta)$ onto itself. (In fact, this map is self-inverse.) The map $\psi \mapsto \psi^{\pi G}$ is the composition of this bijection with the standard Clifford bijection

$$\mathrm{Irr}(T|\theta) \to \mathrm{Irr}(G|\theta)$$

defined by character induction.

We now begin work towards the proof of Theorem 7.3.

(7.6) LEMMA. *Let $G$ be $\pi$-separable with $2 \notin \pi$. Suppose $\chi \in D_\pi(G)$ and $N \lhd G$ and that the irreducible constituents of $\chi_N$ are $\pi$-special. Then there exists $H \supseteq N$ and $\pi$-special $\psi \in \mathrm{Irr}(H)$ such that $\psi^{\pi G} = \chi$.*

*Proof.* Since $\chi \in D_\pi(G)$, we have $\chi = \theta^{\pi G}$ for some $\pi$-special $\theta \in \mathrm{Irr}(K)$, where $K \subseteq G$. Let $H = NK$ and put $\psi = \theta^{\pi(NK)}$. By 7.4(a), $\psi^{\pi G} = \chi$ and so $\psi$ is irreducible and we will show that $\psi$ is $\pi$-special.

Since $\psi = \theta^{\pi H}$, we have $\psi \in D_\pi(H)$ and by Corollary 7.2 it suffices to show that $\psi(1)$ is a $\pi$-number. Now

$$\psi(1) = \theta(1)\,|H:K|$$

and $\theta(1)$ is a $\pi$-number since $\theta$ is $\pi$-special. What remains, is to show that $|H:K||H:K| = |N:N \cap K|$ is a $\pi$-number.

Now $(\theta_{N\cap K})^{\pi N} = \psi_N$ by 7.4(b). By 7.4(c), all of its irreducible constituents are constituents of $\chi_N$ and so are $\pi$-special. By Theorem 7.1(b), it follows that $|N:N \cap K|$ is a $\pi$-number.

(7.7) LEMMA. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and suppose $\chi \in D_\pi(G)$ but that $\chi$ is not $\pi$-special. Then there exists $N \lhd G$ such that the number of irreducible constituents of $X_N$ is not a $\pi$-number and all of these constituents are $\pi$-special.*

*Proof.* Let $N \lhd G$ be maximal with the property that the irreducible constituents of $\chi_N$ are $\pi$-special. By hypothesis, $N < G$ and we can choose a chief factor $M/N$ of $G$. Since the irreducible constituents of $\chi_M$ are not $\pi$-special, it follows by Lemma 3.3(a) that $M/N$ is not a $\pi$-group and so it is a $\pi'$-group.

By Lemma 7.6, choose $H \supseteq N$ and $\pi$-special $\psi \in \mathrm{Irr}(H)$ so that $\psi^{\pi G} = \chi$. Let $\theta$ be an irreducible constituent of $\psi_N$ so that by 7.4(c), $\theta$ is also a constituent of $\chi_N$. Let $T = I_G(\theta)$. Our object is to show that $|G:T|$ is not a $\pi$-number and so we assume that it is and work toward a contradiction.

Since $M/N \lhd G/N$ is a $\pi'$-group, we have $M \subseteq T$ and $\theta$ has a unique $\pi$-special extension $\hat{\theta} \in \mathrm{Irr}(M)$ by Lemma 3.3(b). Also, $(\hat{\theta})_{M\cap H}$ is the unique $\pi$-special character of $M \cap H$ over $\theta$ and so $(\hat{\theta})_{M\cap H}$ is a constituent of $\psi_{M\cap H}$ and therefore $\hat{\theta}$ is a constituent of $(\psi_{M\cap H})^M$.

Since $|M:M \cap H|$ is a $\pi'$-number, we have

$$\delta_{(M:M\cap H)} = 1_{M\cap H}$$

by Theorem 2.5(c) and thus $\hat{\theta}$ is a constituent of

$$(\psi_{M\cap H})^M = (\psi_{M\cap H})^{\pi M} = (\psi^{\pi(MH)})_M$$

by 7.4(b). However,

$$\chi = \psi^{\pi G} = (\psi^{\pi(MH)})^{\pi G}$$

and it follows that $\hat{\theta}$ is a constituent of $\chi_M$ by 7.4(c) applied to $\psi^{\pi(MH)}$. This contradicts the choice of $N$ since $M > N$.

*Proof of Theorem* 7.3. We have $\psi^{\pi G} = \chi$ where $\psi \in \text{Irr}(H)$. Suppose first that $\psi \in D_\pi(H)$. Then $\psi = \theta^{\pi H}$ for some $\pi$-special $\theta \in \text{Irr}(K)$ with $K \subseteq H$. By 7.4(a), we have

$$\chi = \psi^{\pi G} = (\theta^{\pi H})^{\pi G} = \theta^{\pi G}$$

and so $\chi \in D_\pi(G)$, as required.

Now assume $\chi \in D_\pi(G)$. We prove that $\psi \in D_\pi(H)$ by induction on $|G|$. If $\chi$ is $\pi$-special, then $|G:H|$ is a $\pi$-number and by Theorem 7.1(a) we conclude that $\psi$ is $\pi$-special and so $\psi \in D_\pi(G)$. We assume, therefore, that $\chi$ is not $\pi$-special so that by Lemma 7.7 we can fix a subgroup $N \lhd G$ such that the irreducible constituents of $\chi_N$*such that the irreducible constituents of* $\chi_N$ are $\pi$-special and the number of such constituents is not a $\pi$-number.

First we consider the case where $N \subseteq H$. Let $\theta$ be an irreducible constituent of $\psi_N$ and let $T = I_G(\theta)$ and $\xi \in \text{Irr}(T|\theta)$ with $\xi^{\pi G} = \chi$ (by Lemma 7.5). By 7.6, we can choose $U \supseteq N$ and $\gamma \in \text{Irr}(U)$ such that $\gamma^{\pi G} = \chi$ and $\gamma$ is $\pi$-special. By 7.4(c), the irreducible constituents of $\gamma_N$ and of $\psi_N$ lie under $\chi$ and so we may, if necessary, replace the pair $(U, \gamma)$ by a $G$-conjugate pair and assume that $\gamma$ lies over $\theta$. Therefore, $\gamma = \eta^{\pi u}$ for some

$$\eta \in \text{Irr}(U \cap T|\theta)$$

and since $\gamma$ is $\pi$-special, so is $\eta$ by Theorem 7.1(b). Also,

$$(\eta^{\pi U})^{\pi G} = \gamma^{\pi G} = \chi$$

and thus

$$(\,(\eta)^{\pi T})^{\pi G} = \chi$$

and so $\eta^{\pi T} = \xi$. Since $\eta$ is $\pi$-special, we conclude that $\xi \in D_\pi(T)$.

Now $\psi = \beta^{\pi H}$ for some $\beta \in \text{Irr}(T \cap H|\theta)$ and

$$(\beta^{\pi H})^{\pi G} = \psi^{\pi G} = \chi.$$

Therefore $(\beta^{\pi T})^{\pi G} = \chi$ and so

$$\beta^{\pi T} = \xi \in D_\pi(T).$$

However, $T < G$ by the choice of $N$ and so by the inductive hypothesis applied to $T$, since $\xi \in D_\pi(T)$, we conclude that

$$\beta \in D_\pi(T \cap H).$$

Since $\beta^{\pi H} = \psi$, we have $\psi \in D_\pi(H)$ as required (by the first part of the proof).

Now suppose $N \not\subseteq H$ but that $NH < G$. We have

$$(\psi^{\pi NH})^{\pi G} = \chi \in D_\pi(G)$$

and so since $NH \supseteq N$, we have

$$\psi^{\pi NH} \in D_\pi(NH)$$

by the part of the proof we have already done. Since we are assuming $NH < G$, the inductive hypothesis yields that $\psi \in D_\pi(H)$.

Finally, we assume $NH = G$ and we write $M = N \cap H$. Let $\varphi$ be an irreducible constituent of $\psi_M$ and write $S = I_H(\varphi)$. By 7.5, choose

$$\eta \in \mathrm{Irr}(S|\varphi) \quad \text{with } \eta^{\pi H} = \psi.$$

Then $\eta^{\pi G} = \chi$ and hence $\eta^{\pi SN}$ is irreducible. We have

$$(\eta^{\pi SN})^{\pi G} = \chi \in D_\pi(G)$$

and since $SN \supseteq N$, the part of the proof already completed yields that

$$\eta^{\pi SN} \in D_\pi(SN).$$

If $SN < G$, the inductive hypothesis yields that $\eta \in D_\pi(S)$ and so it follows that $\psi = \eta^{\pi H}$ lies in $D_\pi(H)$ as required.

We may therefore assume that $SN = G$ and hence $S = H$ and $\varphi$ is invariant in $H$. We can thus write $\psi_M = e\varphi$ for some integer $e$ and we have

$$\chi_N = (\psi^{\pi G})_N = (\psi_M)^{\pi N} = e\varphi^{\pi N}.$$

Also,

$$\chi_N = f \sum_{i=1}^{t} \theta_i$$

where the $\theta_i$ are distinct $\pi$-special characters of $N$ and $t$ is not a $\pi$-number. It follows that

$$\varphi^{\pi N} = (f/e) \sum_{i=1}^{t} \theta_i$$

and, in particular, $f/e$ is an integer. Since the $\theta_i$ are all $\pi$-special, Theorem 7.1(b) yields that $\varphi$ is $\pi$-special and $|N:M|$ is a $\pi$-number and so

$$\varphi^{\pi N}(1) = |N:M|\varphi(1)$$

is a $\pi$-number. However,

$$\varphi^{\pi N}(1) = (f/e)t\theta_1(1)$$

and thus $t$ divides $\varphi^{\pi N}(1)$. This is a contradiction and completes the proof.

An interesting application of Theorem 7.3 is the following.

(7.8) COROLLARY. *Let $G$ be solvable of odd order and let $\chi \in \mathrm{Irr}(G)$. Suppose $\chi = \lambda^G = \mu^G$ where $\lambda$ and $\mu$ are linear characters of subgroups. Then the multiplicative orders of $\lambda$ and $\mu$ involve exactly the same sets of primes.*

*Proof.* Write $\pi(\lambda)$ and $\pi(\mu)$ to denote the respective sets of primes and note that $\lambda$ is $\pi(\lambda)$-special. Since $|G|$ is odd, $\pi(\lambda)$-induction is just ordinary induction and hence $\chi \in D_{\pi(\lambda)}(G)$ since $\chi = \lambda^G$.

Since $\chi = \mu^G$, Theorem 7.3 yields that $\mu$ is a $D_\pi$-character and by 7.2, for instance, $\mu$ is $\pi(\lambda)$-special. This implies that $\pi(\mu) \subseteq \pi(\lambda)$. The reverse inclusion follows symmetrically.

We mention another easy consequence of Lemma 7.7.

(7.9) COROLLARY. *Let $G$ be $\pi$-separable with $2 \notin \pi$. Let $\chi \in D_\pi(G)$ and suppose $\chi$ factors as a product of a $\pi$-special character and a $\pi'$-special character. Then $\chi$ is itself $\pi$-special.*

*Proof.* If $\chi$ is not $\pi$-special, use Lemma 7.7 to choose $N \lhd G$ such that $\chi_N$ has $t$ distinct $\pi$-special irreducible constituents, where $t$ is not a $\pi$-number. Write $\chi = \alpha\beta$ where $\alpha$ is $\pi$-special and $\beta$ is $\pi'$-special and choose an irreducible constituent $\theta$ of $\chi_N = \alpha_N\beta_N$. Then $\theta$ is a constituent of $\alpha_0\beta_0$ where $\alpha_0$ and $\beta_0$ are irreducible constituents of $\alpha_N$ and $\beta_N$ and so are $\pi$-special and $\pi'$-special respectively. It follows by Gajendragadkar's theorem (3.2) that $\alpha_0\beta_0 \in \mathrm{Irr}(N)$ and so $\theta = \alpha_0\beta_0$. By 3.2 again, since $\theta$ is $\pi$-special and $\theta = \theta \cdot 1_N$, we have $\theta = \alpha_0$ and so the number $t$ of distinct $G$-conjugates of $\theta$ must divide $\alpha(1)$. Since $\alpha(1)$ is a $\pi$-number, we have a contradiction.

Our final result on $D_\pi(G)$ concerns restrictions to normal subgroups.

(7.10) THEOREM. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and let $\chi \in D_\pi(G)$. If $N \lhd G$, then the irreducible constituents of $\chi_N$ lie in $D_\pi(N)$.*

*Proof.* It is clearly no loss to assume that $N$ is a maximal normal subgroup of $G$. If for every $M \lhd G$ with $M \subseteq N$ we have that $\chi_M$ is homogeneous, then by Theorem 3.6, $\chi$ factors as a product of a $\pi$-special and a $\pi'$-special character and so by 7.9, $\chi$ is $\pi$-special and the result follows in this case.

Now let $M \subseteq N$ with $M \lhd G$ such that $\chi_M$ is not homogeneous and choose an irreducible constituent $\theta$ of $\chi_M$. Let $T = I_G(\theta) < G$ and take $\psi \in \mathrm{Irr}(T|\theta)$ with $\psi^{\pi G} = \chi$. By Theorem 7.3, $\psi \in D_\pi(T)$ and so, working by induction on $|G|$, we know that the irreducible constituents of $\psi_{N \cap T}$ lie in $D_\pi(N \cap T)$. One of these, say $\xi$, lies over $\theta$ and thus $\xi^{\pi N} \in \mathrm{Irr}(N)$ and, in fact, $\xi^{\pi N} \in D_\pi(N)$. It suffices to show that $\xi^{\pi N}$ is a constituent of $\chi_N$.

Now $\xi^{\pi N}$ is a constituent of

$$(\psi_{N \cap T})^{\pi N} = (\psi^{\pi NT})_N$$

and since

$$(\psi^{\pi NT})^{\pi G} = \psi^{\pi G} = \chi,$$

and it follows by 7.4(c) that $\xi^{\pi N}$ is a constituent of $\chi_N$ as desired.

Readers familiar with **[8]** may suspect the existence of a connection between $D_\pi(G)$ and the set $B_\pi(G) \subseteq \mathrm{Irr}(G)$ defined there. There are indeed connections, and for groups of odd order, the sets are in fact identical. This is not the place, however, to explore this further.

**8. Restricted characters.** We can now prove Theorem A which we restate below. Although this result does not mention the sign character $\delta_{(G:H)}$, the proof depends on our previous theorems.

(8.1) THEOREM. *Let G be $\pi$-separable with $2 \notin \pi$ and let $\chi \in \mathrm{Irr}(G)$ be $\pi$-special. Suppose $H \subseteq G$ and $\chi_H = \psi \in \mathrm{Irr}(H)$. Then $\psi$ is $\pi$-special.*

*Proof.* Observe that it is no loss to assume that $H$ is a maximal subgroup of $G$. Let $L = \mathrm{core}_G(H)$ and let $K/L$ be a chief factor of $G$ so that $KH = G$. Now $|G:H| = |K:K \cap H|$ divides $|K:L|$ and so if $K/L$ is a $\pi'$-group, then $H$ has $\pi'$-index and the result follows by Theorem 3.5.

We may assume that $K/L$ is a $\pi$-group and since $2 \notin \pi$, it follows from the "odd order" theorem that $K/L$ is an elementary abelian $p$-group for some odd prime $p$. Also, $L = K \cap H$.

Let $\theta$ be an irreducible constituent of $\chi_K$ and let $T = I_G(\theta)$. Take $\xi \in \mathrm{Irr}(T)$ with $\xi^G = \chi$ and write $S = T \cap H$ and $\eta = \xi_S$. Then

$$\psi = (\xi^G)_H = (\xi_S)^H = \eta^H$$

and $\eta$ is irreducible. Note that $|G:T|$ is a $\pi$-number.

By Theorem 4.1, we conclude that $\delta\xi$ is $\pi$-special where $\delta = \delta_{(G:T)}$. If $T < G$, then working by induction on $|G|$, we can conclude that $\delta_S\eta$ is $\pi$-special. However, $\delta_S = \delta_{(H:S)}$ by Theorem 2.5(e) and so by Theorem 6.1, we have that $\psi$ is $\pi$-special, as required. We may therefore assume that $T = G$ and so $\theta$ is invariant in $G$.

If $H \lhd G$, then certainly $\psi$ is $\pi$-special and so we can assume $L < H$ and thus $K < G$. Let $K \subseteq X \lhd G$ with $X$ maximal normal, and write

$Y = X \cap H$. Let $\xi$ be any irreducible constituent of $\chi_X$ and let $\eta = \xi_Y$. Then $\xi$ is $\pi$-special and by Proposition 5.1, $\eta$ is irreducible. By the inductive hypothesis, we have that $\eta$ is $\pi$-special.

If $G/X$ is a $\pi$-group, then so is $H/Y$ and thus $\psi$ is $\pi$-special by Lemma 3.3(a). The only remaining possibility is that $G/X$ is a $\pi'$-group. In that case, $\psi_Y$ is irreducible since

$$(\psi(1), |H:Y|) = (\chi(1), |G:X|) = 1.$$

Also, $o(\psi)$ divides $o(\chi)$ and so is a $\pi$-number and the result follows by Lemma 3.3(b).

As was the case in Section 6, the principal significance of the hypothesis that $2 \notin \pi$ in Theorem 8.1 is so that Proposition 5.1 could be applied.

(8.2) *Example.* There exists a solvable group $G$ with subgroup $H$ and $\chi \in \mathrm{Irr}(G)$ such that

a) $\chi$ is $\{2\}$-special.

b) $\chi_H = \psi \in \mathrm{Irr}(H)$.

c) $\psi$ is not $\{2\}$-special.

*Proof.* As in Example 6.2, take $G = GL(2, 3)$ and $H$ of index 4. Let $\chi$ be either of the two faithful characters of $G$ of degree 2.

**9. $\pi$-special extensions.** If $H \subseteq G$ and $\psi \in \mathrm{Char}(H)$, we shall say $\psi$ is *invariant* in $G$ if $\psi(x) = \psi(y)$ whenever $x, y \in H$ are conjugate in $G$. This generalizes the usual context of this word, where $H \lhd G$. It is obviously necessary that $\psi$ be invariant in $G$ if we hope to extend $\psi$ to a character of $G$.

If $\psi \in \mathrm{Char}(H)$ and $g \in G$, then $\psi^g \in \mathrm{Char}(H^g)$ is the function defined by

$$\psi^g(x^g) = \psi(x) \quad \text{for } x \in H.$$

If $\psi$ is invariant in $G$, then

$$(\psi^g)_{H \cap H^g} = (\psi)_{H \cap H^g}.$$

(9.1) THEOREM. *Let $H$ be a Hall $\pi$-subgroup of the $\pi$-separable group $G$, and let $\psi \in \mathrm{Irr}(H)$ be invariant in $G$. Then $\psi$ extends to some $\pi$-special character of $G$.*

Note that by Theorem 3.5, there can be only one $\pi$-special extension of $\psi$.

*Proof of Theorem* 9.1. We use induction on $|G|$. Let

$$N = \mathbf{O}^{\pi}(G) \quad \text{and} \quad M = \mathbf{O}^{\pi'}(N) \lhd G.$$

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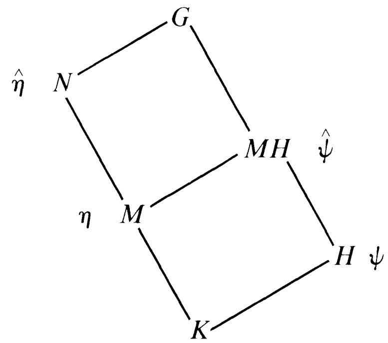
If $M = N$, then $N = 1$ and $H = G$ and there is nothing to prove and we may therefore assume that $M < N$. Let $K = N \cap H$ and observe that $K$ is a Hall $\pi$-subgroup of $N$ so that $K \subseteq M$. Also, since $G/N$ is a $\pi$-group, we have $NH = G$. (Note that we are not assuming that $N \lhd G$.)




Since $MH \cap N = M < N$, we have $MH < G$ and by the inductive hypothesis, $\psi$ extends to $\pi$-special $\hat{\psi} \in \mathrm{Irr}(MH)$. Let $\eta \in \mathrm{Irr}(M)$ be a constituent of $(\hat{\psi})_M$ so that $\eta$ is $\pi$-special (since $M \lhd MH$).

We will show that $\eta$ is invariant in $N$. Assuming this for now, we can extend $\eta$ to $\pi$-special $\hat{\eta} \in \mathrm{Irr}(N)$ by Lemma 3.3(b). Now

$$((\hat{\eta})^G)_{MH} = \eta^{MH}$$

and $\hat{\psi}$ is a constituent of this character. It follows that $(\hat{\eta})^G$ has some irreducible constituent $\chi \in \mathrm{Irr}(G)$ which lies over $\hat{\psi}$. By Lemma 3.3(a), $\chi$ is $\pi$-special since $\hat{\eta}$ is and thus $\chi_H$ is irreducible by Theorem 3.5. It follows that $\chi_H = \psi$, as required.

What remains, is to show that $\eta$ is invariant in $N$. Let $U = \mathbf{N}_G(K)$ so that $N = M(N \cap U)$ by the Frattini argument (using the "odd order" theorem). It therefore suffices to show that $N \cap U$ stabilizes $\eta$.

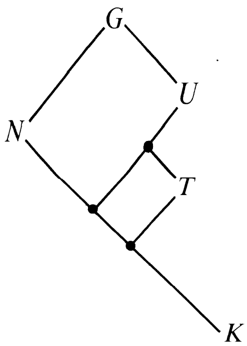
Let $\varphi = \eta_K$ and observe that $\varphi$ is irreducible by Theorem 3.5 and also, the pair $(K, \varphi)$ uniquely determines $\eta$. We will show that $N \cap U$ stabilizes $\varphi$ in order to complete the proof.

We have $H \subseteq U$ and we claim that $H$ acts transitively on the $U$-orbit containing $\varphi$ in the action on $\mathrm{Irr}(K)$. Note that $\varphi$ is a constituent of $\psi_K$ and by hypothesis,

$$\psi_K = (\psi^u)_K$$

for all $u \in U$. It follows that $\varphi^u$ is also a constituent of $\psi_K$ and thus, since $\psi$ is irreducible, $\varphi^u$ is conjugate to $\varphi$ in $H$, as claimed.

Let $T = I_U(\varphi)$. Then $|U:T| = |H:H \cap T|$ is a $\pi$-number and it follows (since $N \cap U \lhd U$) that $|N \cap U:N \cap T|$ is a $\pi$-number. However, $K \subseteq N \cap T$ and $K$ is a Hall $\pi$-subgroup of $N$. This forces $N \cap U = N \cap T$ and so $N \cap U$ does stabilize $\varphi$. The proof is now complete.




In particular, Theorem G of the introduction is now proved. The next result is a restatement of Theorem F.

(9.2) COROLLARY. *Let $G$ be $\pi$-separable and suppose $H \subseteq G$ has $\pi'$-index. Let $\psi \in \mathrm{Irr}(H)$ be $\pi$-special and assume that $\psi(x) = \psi(y)$ whenever $x$ and $y$ are $G$-conjugate $\pi$-elements of $H$. Then $\psi$ extends to some $\pi$-special character of $G$.*

*Proof.* Let $U \subseteq H$ be a Hall $\pi$-subgroup and observe that $\psi_U$ is irreducible by Theorem 3.5 and $\psi_U$ is invariant in $G$ by hypothesis. By Theorem 9.1, $\psi_U$ has a $\pi$-special extension $\chi \in \mathrm{Irr}(G)$ and by Theorem 3.5 $\chi_H \in \mathrm{Irr}(H)$ is $\pi$-special.

Since $\chi_H$ and $\psi$ are two $\pi$-special characters of $H$ with equal restrictions to $U$, Theorem 3.5 implies that $\chi_H = \psi$.

Finally, we prove Theorem E.

(9.3) COROLLARY. *Let $G$ be $\pi$-separable with $2 \notin \pi$ and let $H \subseteq G$. Suppose $\psi \in \mathrm{Irr}(H)$ is $\pi$-special and extends to $G$. Then $\psi$ has a $\pi$-special extension to $G$.*

*Proof.* Let $V \subseteq H$ be a Hall $\pi$-subgroup of $H$ and let $V \subseteq U$ where $U$ is a Hall $\pi$-subgroup of $G$. By 3.5, $\psi_V$ is an irreducible character of $V$ and it extends to $\xi \in \mathrm{Irr}(G)$. Let $\eta = \xi_U$ so that $\eta$ is invariant in $G$ and thus has a $\pi$-special extension $\chi \in \mathrm{Irr}(G)$. Since

$$\chi_V = \eta_V = \xi_V = \psi_V$$

is irreducible, certainly $\chi_H$ is irreducible and thus $\chi_H$ is $\pi$-special by Theorem 8.1. Now $\psi$ and $\chi_H$ are $\pi$-special characters of $H$ with equal restrictions to $V$ and so $\chi_H = \psi$ by 3.5.

The author has been unable to settle whether or not Corollary 9.3 would remain true if the hypothesis that $2 \notin \pi$ were dropped.

We mention that at the end of [**1**] there is a result of Dade which characterizes the image of the restriction map from the set of $\pi$-special characters of $G$ into $\mathrm{Irr}(H)$, where $H$ is a Hall subgroup of $G$. Our Theorem 9.1 provides a simpler characterization: the image is the set of invariant irreducible characters.

Theorem 9.1 is also somewhat related to Theorem 1 of **[10]** where the hypothesis is that any two $G$-conjugate elements of the Hall $\pi$-subgroup $H$ are already $H$-conjugate. The conclusion (for $\pi$-separable $G$) is that there is a normal $\pi$-complement. Our 9.1 at least proves that all $\psi \in \mathrm{Irr}(H)$ extend to $G$ in this situation.

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