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Mixed Norm Type Hardy Inequalities

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Abstract. Higher dimensional mixed norm type inequalities involving certain integral operators are characterized in terms of the corresponding lower dimensional inequalities.

1 Introduction

Consider the reverse mixed norm type Hardy inequality

$$(1.1)\quad \left(\int_0^\infty V_1(x)\left(\int_0^\infty V_2(y) f^{p_2}(x,y)dy\right)^{\frac{p_1}{p_2}} dx\right)^{\frac{1}{p_1}} \leq C\left(\int_0^\infty U_1(x)\left(\int_0^\infty U_2(y)(H_2 f)^{q_2}(x,y)dy\right)^{\frac{q_1}{q_2}} dx\right)^{\frac{1}{q_1}},$$

where $0 < p_i, q_i < 1$, $i = 1, 2$, and H_2 is the two dimensional Hardy operator

$$(H_2 f)(x,y) = \int_0^x \int_0^y f(s,t)dt\,ds, \qquad x, y \in (0,\infty), \quad f \geq 0.$$

First of all, in this paper, we characterize (1.1) in terms of two one dimensional reverse Hardy inequalities. The precise weight conditions for these one dimensional inequalities for various choices of indices are known, *e.g.*, [2, 4, 12, 16, 19], and consequently the conditions for (1.1) are obtained.

In [1], Appell and Kufner studied the reverse inequality of (1.1) for the case $1 < p_i < \infty$, $0 < q_i < \infty$, $i = 1, 2$, *i.e.*, they studied the inequality

$$(1.2)\quad \left(\int_0^\infty U_1(x)\left(\int_0^\infty U_2(y)(H_2 f)^{q_2}(x,y)dy\right)^{\frac{q_1}{q_2}} dx\right)^{\frac{1}{q_1}} \leq C\left(\int_0^\infty V_1(x)\left(\int_0^\infty V_2(y) f^{p_2}(x,y)dy\right)^{\frac{p_1}{p_2}} dx\right)^{\frac{1}{p_1}}$$

in terms of two one dimensional Hardy inequalities. However, their result was not a characterization. Their necessary condition differs from that of the sufficient one.

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We show, using the techniques used to characterize (1.1), that the result of Appell and Kufner can be strengthened to give a characterization for (1.2) for the case $1 < p_i < \infty$, $0 < q_i < \infty$, $i = 1, 2$ also. Inequalities (1.1) and (1.2) are studied in Section 2.

Next, consider the operator

$$(H_{E,F}f)(x,y) = \int_{S_{M_x}} \int_{S_{N_y}} f(s,t)\,dt\,ds, \qquad x \in E,\ y \in F$$

and the corresponding reverse mixed norm inequality

$$\text{(1.3)} \quad \left(\int_E v_1(x) \left(\int_F v_2(y) f^{p_2}(x,y)\,dy \right)^{\frac{p_1}{p_2}} dx \right)^{\frac{1}{p_1}} \leq \\ C \left(\int_E u_1(x) \left(\int_F u_2(y) (H_{E,F}f)^{q_2}(x,y)\,dy \right)^{\frac{q_1}{q_2}} dx \right)^{\frac{1}{q_1}},$$

where E, F are multidimensional spherical cones that, along with other symbols, are defined in Section 3. Note that in (1.3) x and y are multidimensional vectors, whereas in (1.1) they are real numbers, but there should be no confusion since it is clear from the context. The other aim of this paper is to study (1.3). We show that (1.3) holds if and only if (1.1) holds under the suitable range of parameters. In fact, this equivalence is also new for the inequalities considered in the usual direction. These results are collected in Section 3.

Such reductions from higher dimensional problems to the corresponding one dimensional situations were first considered in [5, 8–10, 17, 18]. In [17, 18], Sinnamon studied the boundedness of the Hardy operator $(\mathcal{H}f)(x) = \int_{S_x} f(y)dy$ and the Hardy–Steklov operator $(\mathcal{T}f)(x) = \int_{b(|x|)S \setminus a(|x|)S} f(y)dy$ in terms of the boundedness of the corresponding one dimensional operators

$$(Hf)(x) = \int_0^x f(y)dy \quad \text{and} \quad (Tf)(x) = \int_{a(x)}^{b(x)} f(y)dy,$$

respectively, while the compactness property has been studied in [5, 9]. Let us mention that the boundedness of T has been characterized in [6], while its compactness was studied in [7]. We also point out in this paper (see Section 4) that the equivalence of inequalities (1.1) and (1.3) is still valid if the Hardy operators in the two inequalities are replaced by the corresponding Hardy–Steklov operators.

Throughout, primes over constants denote conjugate indices, *e.g.*, $p' = \frac{p}{p-1}$ *etc.*. Primes are also used over variables, *e.g.*, x', y' *etc.*, but these are just real numbers (not conjugate to x, y). In fact, the usage of primes will be clear from the context.

2 Mixed Norm Type Inequalities in Two Dimensions

Let us fix some notation and terminology that will be used throughout the paper.

The symbols $w, w_1, w_2, U_1, U_2, V_1, V_2$ will denote weight functions on $(0,\infty)$ (or, simply, weights), *i.e.,* Lebesgue measurable, locally integrable, not identically zero, a.e. finite, and positive functions on $(0,\infty)$.

Let $0 < p < \infty$, $p \neq 0$. We denote by L^p_w, the weighted Lebesgue space that consists of all Lebesgue measurable, real functions f on $(0,\infty)$ such that

$$\|f\|_{L^p_w} := \left(\int_0^\infty w(y)|f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

It is known that for $p \geq 1$, L^p_w is a Banach space, and for $0 < p < 1$, it is only a normed linear space. We will omit the symbol w in L^p_w in the case $w \equiv 1$. Moreover, we have not used the interval $(0,\infty)$ in the notation L^p_w. We will be writing $L^p_w(a,b)$ only when (a,b) is an interval other than $(0,\infty)$.

Let $0 < p_1, p_2 < \infty$. The Lebesgue space with mixed norm $[L^{p_1}_{w_1}, L^{p_2}_{w_2}]$ consists of all Lebesgue measurable, real functions $f = f(s,t)$ on $(0,\infty)^2$ such that

$$f(x,\,\cdot\,) \in L^{p_2}_{w_2} \qquad \forall\, x \in (0,\infty) \quad a.e. \tag{2.1}$$

and the function

$$g(x) = \|f(x,\,\cdot\,)\|_{L^{p_2}_{w_2}} \in L^{p_1}_{w_1} \tag{2.2}$$

and for any such function f we set

$$\begin{aligned} \|f(x,y)\|_{[L^{p_1}_{w_1}, L^{p_2}_{w_2}]} &= \| \|f(x,\,\cdot\,)\|_{L^{p_2}_{w_2}} \|_{L^{p_1}_{w_1}} \\ &= \left(\int_0^\infty w_1(x) \left(\int_0^\infty w_2(y)|f(x,y)|^{p_2} dy \right)^{\frac{p_1}{p_2}} dx \right)^{\frac{1}{p_1}}. \end{aligned}$$

The unweighted Lebesgue space with mixed norm was introduced by Benedek and Panzone [3]. The general construction of mixed norm spaces built using two normed spaces is now considered classical, we refer the reader to Maligranda [14, 15].

The next theorem deals with inequality (1.1). Before the statement, let us make a few considerations, which will suggest some natural assumptions to be made, to avoid trivial cases.

Remark 2.1 Let us emphasize that inequality (1.1) is interesting for those functions f for which its right hand side is finite, *i.e.,* when the following conditions are both true (compare with (2.1), (2.2)):

$$H_2 f(x,\,\cdot\,) \in L^{q_2}_{U_2} \qquad \forall\, x \in (0,\infty), \tag{2.3}$$

$$\|H_2 f(x,\,\cdot\,)\|_{L^{q_2}_{U_2}} \in L^{q_1}_{U_1}. \tag{2.4}$$

We stress that in (2.3) we should have considered $x \in (0,\infty)$ *almost everywhere*, but we wrote "for all x" because inequality (1.1) has been considered for non-negative f, and in such case $H_2 f(x,y)$ is a non-decreasing function in x.

The nontriviality assumptions to be made when considering inequality (1.1) will be clear after the proof of the following two propositions.

Proposition 2.2 *Assume that there exists a function $f \geq 0$, not identically zero, such that* (2.3) *holds. Then $U_2 \in L^1(k, \infty)$ $\forall k > 0$.*

Proof Let $\lambda > 0$ and let $E \subset (0,\infty)^2$ be a bounded set of positive measure such that

$$f(s,t) \geq \lambda \chi_E(s,t) \qquad \forall\,(s,t) \in (0,\infty)^2.$$

Of course from (2.3) we get

$$H_2\chi_E(x,y) \in L^{q_2}_{U_2} \qquad \forall\, x \in (0,\infty). \tag{2.5}$$

Let $k_1 > 0$ be such that $E \subset (0,k_1)^2$. We have

$$x > k_1, \quad y > k_1 \Rightarrow H_2\chi_E(x,y) = H_2\chi_E(k_1,k_1) = |E| > 0,$$

and therefore

$$\begin{aligned}\Big(\int_{k_1}^{\infty} U_2(y)dy\Big)^{\frac{1}{q_2}} &= \Big(\frac{1}{|E|^{q_2}}\int_{k_1}^{\infty} U_2(y)|E|^{q_2}dy\Big)^{\frac{1}{q_2}}\\ &= \frac{1}{|E|}\Big(\int_{k_1}^{\infty} U_2(y)H_2\chi_E(x,y)^{q_2}dy\Big)^{\frac{1}{q_2}}\\ &\leq \frac{1}{|E|}\Big(\int_{0}^{\infty} U_2(y)H_2\chi_E(x,y)^{q_2}dy\Big)^{\frac{1}{q_2}}\\ &= \frac{1}{|E|}\|H_2\chi_E(x,y)\|_{L^{q_2}_{U_2}} \quad \forall\, x > k_1.\end{aligned}$$

From (2.5) we get that the last term of the previous chain must be finite, and this implies that $U_2 \in L^1(k_1,\infty)$ and obviously also that $U_2 \in L^1(k,\infty)$ $\forall k > k_1$.

On the other hand, if $0 < k < k_1$, taking into account that U_2 is a weight and therefore $U_2 \in L^1(k,k_1)$, we have again $U_2 \in L^1(k,\infty)$, and the proposition is proved. ∎

Mutatis mutandis it is possible to prove the following.

Proposition 2.3 *Assume that there exists a function $f \geq 0$, not identically zero, such that* (2.4) *holds. Then $U_1 \in L^1(k,\infty)$ for all $k > 0$.*

Propositions 2.2 and 2.3 make clear a basic assumption when studying inequality (1.1): the existence of a single function f for which the right-hand side is finite implies (2.3) and (2.4), which in turn imply $U_1, U_2 \in L^1(k,\infty)$ $\forall k > 0$.

We are now ready to prove the following result.

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Theorem 2.4 *Let $0 < p_i, q_i < \infty$, $i = 1, 2$, and let U_1, U_2, V_1, V_2 be weight functions on $(0, \infty)$. Assume furthermore that $U_1, U_2 \in L^1(k, \infty)$ for all $k > 0$. Then a necessary condition for the validity of the inequality* (1.1) *is that the following inequalities hold for all measurable functions g and h on $(0, \infty)$:*

$$\left(\int_0^\infty V_1(x) g^{p_1}(x)dx\right)^{\frac{1}{p_1}} \le C\left(\int_0^\infty U_1(x)\left(\int_0^x g(t)dt\right)^{q_1} dx\right)^{\frac{1}{q_1}}, \tag{2.6}$$

$$\left(\int_0^\infty V_2(x) h^{p_2}(x)dx\right)^{\frac{1}{p_2}} \le C\left(\int_0^\infty U_2(x)\left(\int_0^x h(t)dt\right)^{q_2} dx\right)^{\frac{1}{q_2}}. \tag{2.7}$$

Proof Suppose (1.1) holds for all non-negative measurable functions f on $(0, \infty)^2$. Then it also holds, in particular, for $f(s, t) = g(s)h(t)$, where g and h are non-negative measurable functions on $(0, \infty)$. Inequality (1.1) then reduces to

$$\begin{aligned}(2.8)\quad &\left(\int_0^\infty V_1(x) g^{p_1}(x)dx\right)^{\frac{1}{p_1}} \left(\int_0^\infty V_2(x) h^{p_2}(x)dx\right)^{\frac{1}{p_2}} \le \\ &C\left(\int_0^\infty U_1(x)\left(\int_0^x g(t)dt\right)^{q_1} dx\right)^{\frac{1}{q_1}} \left(\int_0^\infty U_2(x)\left(\int_0^x h(t)dt\right)^{q_2} dx\right)^{\frac{1}{q_2}}.\end{aligned}$$

Since U_2 is a weight, let $\lambda_u > 0$ and let $F_u \subset (0, \infty)$, $\inf F_u > 0$, be a bounded set of positive measure such that $U_2(x) \ge \lambda_u \chi_{F_u}(x)$, so that $\chi_{F_u} \in L^{q_2}_{U_2}$ and $\int_{F_u} U_2(x)dx > 0$. Since V_2 is a weight, let $\lambda_v > 0$ and let $F_v \subset (0, \infty)$, $\inf F_v > 0$, be a bounded set of positive measure such that $V_2(x) \ge \lambda_v \chi_{F_v}(x)$, so that $\chi_{F_v} \in L^{p_2}_{V_2}$ and $\int_{F_v} V_2(x)dx > 0$.

Now set $k = \inf F_u \cup F_v$ and observe that $0 < k < \infty$, and let $h = h_0 = \chi_{F_u \cup F_v}$ in (2.8). Of course we have

$$\begin{aligned} 0 &< \int_{F_v} V_2(x)dx \le \int_k^\infty V_2(x) h_0^{p_2}(x)dx \\ &< \int_k^{\max(\sup F_u, \sup F_v)} V_2(x)dx \le \|V_2\|_{L^1(k,\infty)} < \infty, \end{aligned}$$

$$\begin{aligned} 0 &< \int_{F_u} U_2(x)|(0, x) \cap F_u| dx \\ &\le \int_{F_u} U_2(x)\left(\int_k^x h_0\right)^{q_2} dx < \int_k^\infty U_2(x)\left(\int_k^x h_0\right)^{q_2} dx \\ &< \int_k^\infty U_2(x)|F_u \cup F_v|^{q_2} dx \\ &= |F_u \cup F_v|^{q_2} \|U_2\|_{L^1(k,\infty)} < \infty. \end{aligned}$$

The previous chains of inequalities show that h_0 is such that

$$0 < \int_0^\infty V_2(x) h_0^{p_2}(x)dx = \int_k^\infty V_2(x) h_0^{p_2}(x)dx < \infty$$

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and

$$0 < \int_0^\infty U_2(x)\Big(\int_0^x h_0\Big)^{q_2} dx = \int_k^\infty U_2(x)\Big(\int_0^x h_0\Big)^{q_2} dx < \infty.$$

Now, dividing both sides of (2.8) by $(\int_0^\infty V_2(x)h_0^{p_2}(x)dx)^{1/p_2}$, we get (2.6) (with a different constant C). Similarly, choosing the corresponding $g = g_0$ we get that (2.7) holds as well, and the assertion follows. ∎

Following the proof of Appell and Kufner [1] and taking into account the fact that Minkowskii's inequality holds in the reverse direction for index less than 1, we at once obtain the following result giving a sufficient condition for inequality (1.1) to hold.

Theorem 2.5 *Let $0 < p_i, q_i < 1$, $i = 1, 2$, and let U_1, U_2, V_1, V_2 be weight functions on $(0,\infty)$. Assume in addition that either $q_1 < p_2 < p_1$ or $q_1 < q_2 < p_1$. Then a sufficient condition for the validity of* (1.1) *is that both* (2.6) *and* (2.7) *hold.*

In view of Theorems 2.4 and 2.5, we immediately obtain the following characterization for the inequality (1.1) to hold.

Theorem 2.6 *Let $0 < p_i, q_i < 1$, $i = 1, 2$, and let U_1, U_2, V_1, V_2 be weight functions on $(0,\infty)$ with $U_1, U_2 \in L^1(k,\infty)$ for all $k > 0$. Assume in addition that either $q_1 < p_2 < p_1$ or $q_1 < q_2 < p_1$. Then inequality* (1.1) *holds for all measurable functions f defined on $(0,\infty)^2$ if and only if inequalities* (2.6) *and* (2.7) *hold for all measurable functions g and h defined on $(0,\infty)$.*

Remark 2.7 For the case $1 < p_i < \infty$, $0 < q_i < \infty$, $i = 1, 2$, Appell and Kufner [1] proved that a necessary condition for the validity of inequality (1.2) is that at least one of the inequalities (2.6) and (2.7) in opposite direction holds; *i.e.,* one of the following inequalities hold:

$$\Big(\int_0^\infty U_1(x)\Big(\int_0^x g(t)dt\Big)^{q_1} dx\Big)^{\frac{1}{q_1}} \le C\Big(\int_0^\infty V_1(x)g^{p_1}(x)dx\Big)^{\frac{1}{p_1}}, \tag{2.9}$$

$$\Big(\int_0^\infty U_2(x)\Big(\int_0^x h(t)dt\Big)^{q_2} dx\Big)^{\frac{1}{q_2}} \le C\Big(\int_0^\infty V_2(x)h^{p_2}(x)dx\Big)^{\frac{1}{p_2}}. \tag{2.10}$$

However, depicting the proof of Theorem 2.4 with obvious modifications, it can be shown that the validity of both (2.9) and (2.10) is necessary for inequality (1.2) to hold. Already, for certain choices of indices, both (2.9) and (2.10) are sufficient for (1.2) (see [1]). Summarizing, we have the following characterization for inequality (1.2).

Theorem 2.8 *Let $1 < p_i < \infty$, $0 < q_i < \infty$, $q_i \neq 1$, $i = 1, 2$, and let U_1, U_2, V_1, V_2 be weight functions on $(0,\infty)$ with $V_1, V_2 \in L^1(k,\infty)$ for all $k > 0$. Assume in addition that either $p_1 \le p_2 \le q_1$ or $p_1 \le q_2 \le q_1$. Then inequality* (1.2) *holds for all measurable functions f defined on $(0,\infty)^2$ if and only if inequalities* (2.9) *and* (2.10) *hold for all measurable functions g and h defined on $(0,\infty)$.*

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In the literature, the one dimensional Hardy inequality

$$\left(\int_0^\infty U(x)\left(\int_0^x f(t)dt\right)^q dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty V(x)f^p(x)dx\right)^{\frac{1}{p}} \tag{2.11}$$

and the corresponding reverse inequality

$$\left(\int_0^\infty V(x)f^p(x)dx\right)^{\frac{1}{p}} \le C\left(\int_0^\infty U(x)\left(\int_0^x f(t)dt\right)^q dx\right)^{\frac{1}{q}} \tag{2.12}$$

have been studied extensively for their weight conditions. Using those conditions and Theorem 2.6 (and also Theorem 2.8), precise weight conditions for inequality (1.1) (and also (1.2)) can be obtained. With regard to inequality (1.2), we use the weight characterization of (2.11) from any of the standard monographs [11–13] and apply Theorem 2.8 to obtain the following.

Theorem 2.9 *Let $1 < p_i < \infty$, $0 < q_i < \infty$, $q_i \neq 1$, $i = 1,2$, and let U_1, U_2, V_1, V_2 be weight functions on $(0,\infty)$ with $V_1, V_2 \in L^1(k,\infty)$ for all $k > 0$. Assume in addition that either $p_1 \le p_2 \le q_1$ or $p_1 \le q_2 \le q_1$. Then a necessary and sufficient condition for the validity of inequality* (1.2) *is the following:*

(a) *In case $p_2 \le q_2$*

$$B_i := \sup_{0<x<\infty}\left(\int_x^\infty U_i(t)dt\right)^{1/q_i}\left(\int_0^x V_i^{1-p_i'}(t)dt\right)^{1/p_i'} < \infty;$$

(b) *In case $q_2 \le p_2$*

$$\left(\int_0^\infty\left(\int_x^\infty U_i(t)dt\right)^{r_i/q_i}\left(\int_0^x V_i^{1-p_i'}(t)dt\right)^{r_i/p_i'} V_i^{1-p_i'}(t)dt\right)^{1/r_i},$$

where $\frac{1}{r_i} = \frac{1}{q_i} - \frac{1}{p_i}$.

In order to obtain weight conditions for inequality (1.1), we mention that Beesack and Heinig [2] studied inequality (2.12) for the case $0 < p, q < 1$. Applying their results to Theorem 2.6, we can state the weight conditions for (1.1).

For $i = 1, 2$, denote

$$B_i = \inf_{x>0} J_i(x),$$

where

$$J_i(x) = \left(\int_x^\infty U_i(t)dt\right)^{1/q_i}\left(\int_x^\infty V_i^{1-p_i'}(t)dt\right)^{1/p_i'}.$$

Theorem 2.10 *Let $0 < p_i, q_i < 1$, and let U_i, V_i be weight functions on $(0,\infty)$ with $U_1, U_2 \in L^1(0,\infty)$ and for all $x > 0$, $0 < \int_x^\infty U_i(t)dt < \infty$, $\int_x^\infty V_i^{1-p_i'}(t)dt < \infty$, $i = 1,2$. Then a necessary condition for* (1.1) *to hold is that $min(B_1, B_2) > 0$.*

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Theorem 2.11 *Let $0 < p_i, q_i < 1$, and let U_i, V_i be weight functions on $(0,\infty)$ and for all $x > 0$, $0 < \int_x^\infty U_i(t)dt < \infty$, $\int_x^\infty V_i^{1-p_i'}(t)dt < \infty$, $i = 1, 2$. Assume in addition that either $q_1 < p_2 < p_1$ or $q_1 < q_2 < p_1$. Then a sufficient condition for the validity of* (1.1) *is the following:*

(a) *In case $q_2 \le p_2$, we have that $J_i(x)$ is non-increasing and $B_i > 0$, $i = 1, 2$.*
(b) *In case $p_2 < q_2$, we have that $J_1(x)$ is non-increasing, $B_1 > 0$, and any of the following equivalent conditions is satisfied:*

(i)

$$\int_0^\infty U_2(x)\Big(\int_x^\infty U_2(t)dt\Big)^{-r_2/p_2}\Big(\int_x^\infty V_2^{1-p_2'}(t)dt\Big)^{-r_2/p_2'}dx < \infty$$

and, in addition, $\int_0^\infty V_2^{1-p_2'}(t)dt < \infty$ if $\int_0^\infty U_2 < \infty$.

(ii)

$$\int_0^\infty V_2^{1-p_2'}(x)\Big(\int_x^\infty V_2^{1-p_2'}(t)dt\Big)^{-r_2/q_2'}\Big(\int_x^\infty U_2(t)dt\Big)^{-r_2/q_2}dx < \infty,$$

(iii)

$$\int_0^\infty U_2(x)\Big(\int_0^x V_2^{1-p_2'}(t)\Big(\int_t^\infty U_2(s)ds\Big)^{p_2'-1}dt\Big)^{-r_2/p_2'}dx < \infty,$$

where $\frac{1}{r_2} = \frac{1}{p_2} - \frac{1}{q_2}$.

Remark 2.12 In [4], Bennett studied inequality (2.12) for the case $0 < p < 1$, $p < q < \infty$. One can use his results in Theorem 2.6 to obtain a different set of weight conditions for inequality (1.1).

Remark 2.13 It would also be of interest to study Theorems 2.4, 2.5, and 2.6 for negative indices. Also one could consider these theorems as well as Theorem 2.8 for general measures. The one dimensional inequalities for negative indices have been studied by Prokhorov [16] and for all range of indices with general measure by Sinnamon [19]. Using their results, the precise weight conditions for (1.1) and (1.2) can be obtained.

3 Higher Dimensional Inequalities

In this section, we will study inequality (1.3) and its variants in terms of inequality (1.1) and its corresponding variants. We first give some notation and terminology.

Let $\sum_M$ be the unit ball in $\mathbb{R}^M$, *i.e.*, $\sum_M = \{x \in \mathbb{R}^M : |x| = 1\}$. Let B_M be a measurable subset of $\sum_M$ and let $E \subset \mathbb{R}^M$ be the corresponding spherical cone, *i.e.*,

$$E = \{x \in \mathbb{R}^M : x = s\sigma,\ 0 \le s < \infty,\ \sigma \in B_M\}.$$

Let S_{M_x}, $x \in E$ denote the part of E with 'radius' $\le |x|$, *i.e.*,

$$S_{M_x} = \{y \in \mathbb{R}^M : y = s\sigma,\ 0 \le s \le |x|,\ \sigma \in B_M\}.$$

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Further, we denote by αS_M, $\alpha > 0$, the part of E with radius $\leq \alpha$. Note that $E = \bigcup_{\alpha>0} \alpha S_M$. For $x \in E \setminus \{0\}$, we denote by $|S_{M_x}|$ the volume of S_{M_x}. The symbols B_N, F, S_{N_y}, and $|S_{N_y}|$ are defined similarly for an N-dimensional setting.

Now, we give a characterization of inequality (1.3) in terms of (1.1).

Theorem 3.1 *Let $0 < p_i, q_i < \infty$, $i = 1, 2$, let u_1, v_1 be weight functions on E and let u_2, v_2 be weight functions on F. Then inequality (1.3) holds for all $f \geq 0$ if and only if the inequality*

$$(3.1)\quad \left(\int_0^\infty V_1(x_0)\left(\int_0^\infty V_2(y_0)g^{p_2}(x_0, y_0)dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}} \leq C\left(\int_0^\infty U_1(x_0)\left(\int_0^\infty U_2(y_0)(H_2g)^{q_2}(x_0, y_0)dy_0\right)^{\frac{q_1}{q_2}} dx_0\right)^{\frac{1}{q_1}}$$

holds for all $g \geq 0$ with

$$(3.2)\qquad U_1(x_0) = \int_{B_M} u_1(x_0x')x_0^{M-1}dx', \quad x_0 > 0$$

$$(3.3)\qquad U_2(y_0) = \int_{B_N} u_2(y_0y')y_0^{N-1}dy', \quad y_0 > 0$$

$$(3.4)\qquad V_1(x_0) = \left(\int_{B_M} v_1^{1-p_1'}(x_0x')x_0^{M-1}dx'\right)^{1-p_1}, \quad x_0 > 0$$

$$(3.5)\qquad V_2(y_0) = \left(\int_{B_N} v_2^{1-p_2'}(y_0y')y_0^{N-1}dy'\right)^{1-p_2}, \quad y_0 > 0.$$

Moreover, the constants in (1.3) and (3.1) are the same.

Proof Suppose (3.1) holds. Let $x' \in B_M$ and $y' \in B_N$. Fix a locally integrable function $f : E \times F \to \mathbb{R}$. Define

$$(3.6)\qquad g(x_0, y_0) = \int_{B_M}\int_{B_N} f(x_0x', y_0y')x_0^{M-1}y_0^{N-1}dy'dx', \quad x_0, y_0 > 0.$$

For $x \in E$, we use polar coordinates $x = x_0x'$, $x_0 \in (0, \infty)$, $x' \in B_M$ so that $x_0 = |x|$. Similarly, we use $y = y_0y'$, $s = s_0s'$, $t = t_0t'$. Thus, we have

$$(3.7)\qquad (H_{E,F}f)(x, y) = \int_0^{x_0}\int_0^{y_0}\int_{B_M}\int_{B_N} f(s_0s', t_0t')s_0^{M-1}t_0^{N-1}dt'ds'dt_0ds_0 = \int_0^{x_0}\int_0^{y_0} g(s_0, t_0)dt_0ds_0 = (H_2g)(x_0, y_0).$$

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Therefore, using (3.4), Hölder's inequality, Minkowskii's integral inequality, (3.5), again applying Hölder's inequality to the inner integral and using (3.6), we get

$$
\begin{aligned}
&\left(\int_E v_1(x)\left(\int_F v_2(y) f^{p_2}(x,y)\,dy\right)^{\frac{p_1}{p_2}} dx\right)^{\frac{1}{p_1}}\\
&=\left(\int_0^\infty \int_{B_M}\left(\int_F f^{p_2}(x_0x',y)v_2(y)\,dy\right)^{\frac{p_1}{p_2}} v_1(x_0x')x_0^{M-1}\,dx'\,dx_0\right)^{\frac{1}{p_1}}\\
&=\left(\int_0^\infty V_1(x_0)\int_{B_M}\left(\int_F f^{p_2}(x_0x',y)v_2(y)\,dy\right)^{\frac{p_1}{p_2}} v_1(x_0x')x_0^{M-1}\,dx'\right.\\
&\qquad\left.\times\left(\int_{B_M} v_1^{1-p_1'}(x_0x')x_0^{M-1}\,dx'\right)^{p_1-1} dx_0\right)^{\frac{1}{p_1}}\\
&\le\left(\int_0^\infty V_1(x_0)\left(\int_{B_M}\left(\int_F f^{p_2}(x_0x',y)v_2(y)\,dy\right)^{\frac{1}{p_2}} x_0^{M-1}\,dx'\right)^{p_1} dx_0\right)^{\frac{1}{p_1}}\\
&=\left(\int_0^\infty V_1(x_0)\left(\int_{B_M}\left(\int_0^\infty\int_{B_N} f^{p_2}(x_0x',y_0y')v_2(y_0y')\right.\right.\right.\\
&\qquad\left.\left.\left.\times\, y_0^{N-1}\,dy'(x_0^{M-1})^{p_2}\,dy_0\right)^{\frac{1}{p_2}} dx'\right)^{p_1} dx_0\right)^{\frac{1}{p_1}}\\
&\le\left(\int_0^\infty V_1(x_0)\left(\int_0^\infty\left(\int_{B_M}\left(\int_{B_N} f^{p_2}(x_0x',y_0y')v_2(y_0y')y_0^{N-1}\,dy'\right)^{\frac{1}{p_2}}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\, x_0^{M-1}\,dx'\right)^{p_2} dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}}\\
&=\left(\int_0^\infty V_1(x_0)\left(\int_0^\infty V_2(y_0)\left(\int_{B_M}\left(\int_{B_N} f^{p_2}(x_0x',y_0y')v_2(y_0y')y_0^{N-1}\,dy'\right)^{\frac{1}{p_2}}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\left(\int_{B_N} v_2^{1-p_2'}(y_0y')y_0^{N-1}\,dy'\right)^{\frac{1}{p_2'}} x_0^{M-1}\,dx'\right)^{p_2} dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}}\\
&\le\left(\int_0^\infty V_1(x_0)\left(\int_0^\infty V_2(y_0)\left(\int_{B_M}\int_{B_N} f(x_0x',y_0y')\right.\right.\right.\\
&\qquad\left.\left.\left.\times\, y_0^{N-1}\,dy'\,x_0^{M-1}\,dx'\right)^{p_2} dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}}\\
&=\left(\int_0^\infty V_1(x_0)\left(\int_0^\infty V_2(y_0)g^{p_2}(x_0,y_0)\,dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}}.
\end{aligned}
$$

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Next, we use (3.1), (3.2), (3.3) and (3.7) to get

$$
\begin{aligned}
&\Big(\int_E v_1(x)\Big(\int_F v_2(y) f^{p_2}(x,y)\,dy\Big)^{\frac{p_1}{p_2}} dx\Big)^{\frac{1}{p_1}}\\
&\le C\Big(\int_0^\infty U_1(x_0)\Big(\int_0^\infty U_2(y_0)(H_2 g)^{q_2}(x_0,y_0)\,dy_0\Big)^{\frac{q_1}{q_2}} dx_0\Big)^{\frac{1}{q_1}}\\
&= C\Big(\int_0^\infty\int_{B_M} u_1(x_0x')\Big(\int_0^\infty\int_{B_N} u_2(y_0y')(H_2 g)^{q_2}(x_0,y_0)\\
&\quad\times y_0^{N-1}\,dy'\,dy_0\Big)^{\frac{q_1}{q_2}} x_0^{M-1}\,dx'\,dx_0\Big)^{\frac{1}{q_1}}\\
&= C\Big(\int_E u_1(x)\Big(\int_F u_2(y)(H_{E,F} f)^{q_2}(x,y)\,dy\Big)^{\frac{q_1}{q_2}} dx\Big)^{\frac{1}{q_1}}.
\end{aligned}
$$

Conversely, now assume that (1.3) holds. Fix a locally integrable function $g\colon (0,\infty)\times(0,\infty)\to\mathbb{R}$ and define $f\colon E\times F\to\mathbb{R}$ by

$$
f(x_0x', y_0y') = g(x_0,y_0)V_2^{p_2'-1}(y_0)v_2^{1-p_2'}(y_0y')V_1^{p_1'-1}(x_0)v_1^{1-p_1'}(x_0x'),
$$

where $x_0, y_0 > 0,\ x'\in B_M,\ y'\in B_N$.

Then (3.4) and (3.5) give

$$
\int_{B_M}\int_{B_N} f(x_0x', y_0y')x_0^{M-1}y_0^{N-1}\,dy'\,dx' = g(x_0,y_0)
$$

and consequently, we get

$$
\begin{aligned}
&\Big(\int_0^\infty V_1(x_0)\Big(\int_0^\infty V_2(y_0) g^{p_2}(x_0,y_0)\,dy_0\Big)^{\frac{p_1}{p_2}} dx_0\Big)^{\frac{1}{p_1}}\\
&= \Big(\int_0^\infty V_1^{p_1'}(x_0)\Big(\int_{B_M} v_1^{1-p_1'}(x_0x')x_0^{M-1}\,dx'\Big)\\
&\quad\times\Big(\int_0^\infty g^{p_2}(x_0,y_0)V_2(y_0)\,dy_0\Big)^{\frac{p_1}{p_2}} dx_0\Big)^{\frac{1}{p_1}}\\
&= \Big(\int_0^\infty\int_{B_M} v_1(x_0x')\Big(\int_0^\infty g^{p_2}(x_0,y_0)V_2^{p_2'}(y_0)\\
&\quad\times\int_{B_N} v_2^{1-p_2'}(y_0y')y_0^{N-1}\,dy'\,dy_0\Big)^{\frac{p_1}{p_2}} V_1^{p_1'}(x_0)v_1^{-p_1'}(x_0x')x_0^{M-1}\,dx'\,dx_0\Big)^{\frac{1}{p_1}}\\
&= \Big(\int_0^\infty\int_{B_M} v_1(x_0x')\Big(\int_0^\infty\int_{B_N} v_2(y_0y')f^{p_2}(x_0x', y_0y')y_0^{N-1}\,dy'\,dy_0\Big)^{\frac{p_1}{p_2}}\\
&\quad\times x_0^{M-1}\,dx'\,dx_0\Big)^{\frac{1}{p_1}}
\end{aligned}
$$

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$$= \Big(\int_E v_1(x) \Big(\int_F v_2(y) f^{p_2}(x,y)dy \Big)^{\frac{p_1}{p_2}} dx \Big)^{\frac{1}{p_1}}$$

$$\leq C \Big(\int_E u_1(x) \Big(\int_F u_2(y) (H_{E,F} f)^{q_2}(x,y)dy \Big)^{\frac{q_1}{q_2}} dx \Big)^{\frac{1}{q_1}}$$

$$= C \Big(\int_0^\infty U_1(x_0) \Big(\int_0^\infty U_2(y_0)(H_2 g)^{q_2}(x_0,y_0)dy_0 \Big)^{\frac{q_1}{q_2}} dx_0 \Big)^{\frac{1}{q_1}},$$

and we are done. ∎

Using Theorems 2.10, 2.11, and 3.1, precise weight conditions can be given for inequality (1.3) to hold. We state the results.

Denote

$$\mathcal{J}_1(x) = \Big(\int_{E \setminus S_{M_x}} u_1(y)dy \Big)^{1/q_1} \Big(\int_{E \setminus S_{M_x}} v_1^{1-p_1'}(y)dy \Big)^{1/p_1'}, \quad x \in E$$

$$\mathcal{J}_2(x) = \Big(\int_{F \setminus S_{N_y}} u_2(y)dy \Big)^{1/q_2} \Big(\int_{F \setminus S_{N_y}} v_2^{1-p_2'}(y)dy \Big)^{1/p_2'}, \quad y \in F$$

$$\mathcal{B}_1 = \inf_{x \in E} \mathcal{J}_1(x), \quad \mathcal{B}_2 = \inf_{y \in F} \mathcal{J}_2(y).$$

Theorem 3.2 *Let $0 < p_i, q_i < 1$, $p_i, q_i \neq 0$, $i = 1, 2$, let u_1, v_1 be weight functions on E, and let u_2, v_2 be weight functions on F, $u_1 \in L^1(E)$, $u_2 \in L^1(F)$*

$$(3.8) \quad 0 < \int_{E \setminus S_{M_x}} u_1(z)dz < \infty, \qquad 0 < \int_{E \setminus S_{M_x}} v_1^{1-p_1'}(z)dz < \infty \qquad x \in E,$$

and

$$(3.9) \quad 0 < \int_{F \setminus S_{N_y}} u_2(z)dz < \infty, \qquad 0 < \int_{F \setminus S_{N_y}} v_2^{1-p_2'}(z)dz < \infty \qquad y \in F.$$

Then a necessary condition for inequality (1.3) *to hold is that* $\min(\mathcal{B}_1, \mathcal{B}_2) > 0$.

Theorem 3.3 *Let $0 < p_i, q_i < 1$, $i = 1, 2$, let u_1, v_1 be weight functions on E, let u_2, v_2 be weight functions on F, and let* (3.8), (3.9) *be satisfied . Assume, in addition, that either $q_1 < p_2 < p_1$ or $q_1 < q_2 < p_1$. Then a sufficient condition for the validity of* (1.3) *is:*

(a) *In case $q_2 \leq p_2$, we have that $\mathcal{J}_i(x)$ is non-increasing in $|x|$ and $\mathcal{B}_i > 0$, $i = 1, 2$.*
(b) *In case $p_2 < q_2$, we have that $\mathcal{J}_1(x)$ is non-increasing in $|x|$, $\mathcal{B}_1 > 0$, and any of the following equivalent conditions is satisfied:*

(i)

$$\int_F u_2(y) \Big(\int_{F \setminus S_{N_y}} u_2(z)dz \Big)^{-r_2/p_2} \Big(\int_{F \setminus S_{N_y}} v_2^{1-p_2'}(z)dz \Big)^{-r_2/p_2'} dy < \infty,$$

and, in addition, $\int_F v_2^{1-p_2'}(z)dz < \infty$ if $\int_F u_2(z)dz < \infty$;

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(ii)

$$\int_F v_2^{1-p_2'}(y)\Big(\int_{F\setminus S_{N_y}} v_2^{1-p_2'}(z)dz\Big)^{-r_2/q_2'}\Big(\int_{F\setminus S_{N_y}} u_2(z)dz\Big)^{-r_2/q_2} dy < \infty;$$

(iii)

$$\int_F u_2(y)\Big(\int_{S_{N_y}} v_2^{1-p_2'}(z)\Big(\int_{F\setminus S_{N_z}} u_2\Big)^{p_2'-1} dz\Big)^{-r_2/p_2'} dy < \infty,$$

where $\frac{1}{r_2} = \frac{1}{p_2} - \frac{1}{q_2}$.

Remark 3.4 A result similar to Theorem 3.1 can be obtained for the range $1 < p_i < \infty$, $0 < q_i < \infty$, $i = 1, 2$. The proof is identical to that of Theorem 3.1 with the obvious change in the direction of the inequalities where Hölder and Minkowskii inequalities are used. We state the result below.

Theorem 3.5 *Let* $0 < q_1, q_2 < \infty$, $1 < p_1, p_2 < \infty$, *let* u_1, v_1 *be weight functions on E, and let* u_2, v_2 *be weight functions on F. Then the inequality*

$$(3.10)\quad \Big(\int_E u_1(x)\Big(\int_F u_2(y)(H_{E,F}f)^{q_2}(x,y)dy\Big)^{\frac{q_1}{q_2}} dx\Big)^{\frac{1}{q_1}} \le C\Big(\int_E v_1(x)\Big(\int_F v_2(y)f^{p_2}(x,y)dy\Big)^{\frac{p_1}{p_2}} dx\Big)^{\frac{1}{p_1}}$$

holds for all $f \ge 0$ *if and only if the inequality*

$$(3.11)\quad \Big(\int_0^\infty U_1(x_0)\Big(\int_0^\infty U_2(y_0)(H_2 g)^{q_2}(x_0,y_0)dy_0\Big)^{\frac{q_1}{q_2}} dx_0\Big)^{\frac{1}{q_1}} \le C\Big(\int_0^\infty V_1(x_0)\Big(\int_0^\infty V_2(y_0)g^{p_2}(x_0,y_0)dy_0\Big)^{\frac{p_1}{p_2}} dx_0\Big)^{\frac{1}{p_1}}$$

holds for all $g \ge 0$ *with* U_i, V_i *as given in Theorem* 3.1*. Moreover, the constants in the two inequalities are the same.*

Remark 3.6 Inequality (3.11) is the one characterized in Theorem 2.9 in terms of two one dimensional standard Hardy inequalities. Consequently, the precise weight conditions for (3.10) can be written accordingly.

4 Final Result and Remarks

If we consider $1 < q_1, q_2 < \infty$ in Theorem 3.5, then both the sides of inequalities (3.10) and (3.11) can be regarded as norms in certain mixed normed spaces. Consequently, we immediately obtain the following deduction from Theorem 3.5.

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Corollary 4.1 *Let $1 < p_1, p_2, q_1, q_2 < \infty$, u_1, v_1 be weight functions on E, u_2, v_2 be weight functions on F. Then the operator $H_{E,F}$ is bounded between the mixed normed spaces $[L^{p_1}(E, v_1), L^{p_2}(F, v_2)]$ and $[L^{q_1}(E, u_1), L^{q_2}(F, u_2)]$ if and only if the operator H_2 is bounded between the mixed normed spaces $[L^{p_1}((0,\infty), V_1), L^{p_2}((0,\infty), V_2)]$ and $[L^{q_1}((0,\infty), U_1), L^{q_2}((0,\infty), U_2)]$ with the weights U_1, U_2, V_1, V_2 as given in Theorem* 3.1.

Throughout the paper, we have considered the operators $H_{E,F}$ and H_2. As a matter of fact, we can consider more general operators

$$(T_{E,F}f)(x,y) = \int_{b(|x|)S_M \setminus a(|x|)S_M} \int_{d(|y|)S_N \setminus c(|y|)S_N} f(s,t)dt\, ds, \ x \in E, \ y \in F$$

and

$$(T_2 f)(x,y) = \int_{a(x)}^{b(x)} \int_{c(y)}^{d(y)} f(s,t)dtds, \quad x, y \in (0,\infty),$$

where a, b, c, d are strictly increasing differentiable functions on $[0,\infty]$ satisfying

$$a(0) = b(0) = 0, \quad a(x) < b(x) \quad \text{for} \quad 0 < x < \infty, \quad a(\infty) = b(\infty) = \infty$$

and

$$c(0) = d(0) = 0, \quad c(x) < d(x) \quad \text{for} \quad 0 < x < \infty, \quad c(\infty) = d(\infty) = \infty.$$

Remarks 4.2 (i) Theorem 2.6 is still valid if the operator H_2 is replaced by T_2. In that case inequalities (2.6) and (2.7) will be replaced by the ones involving the Hardy–Steklov operator $(Tf)(x) = \int_{a(x)}^{b(x)} f(t)dt$. But the corresponding inequalities have not been studied in the literature to our knowledge (Heinig and Sinnamon [5] studied these inequalities for $p_i > 1$, $q_i > 0$). Once these inequalities are studied, one can write the results corresponding to Theorems 2.9, 2.10, and 2.11.

(ii) In Theorem 3.1, $H_{E,F}$ and H_2 can be replaced respectively by $T_{E,F}$ and T_2. However, in view of the above remark, the precise weight conditions for the operator $T_{E,F}$ cannot be written unless we know the corresponding conditions for T_2.

(iii) Finally, in Theorem 3.5 as well, the operators $H_{E,F}$ and H_2 can be replaced respectively by $T_{E,F}$ and T_2. The corresponding inequality has already been studied, see [8, Corollary 3], [9, Proposition 2.5], and thus the precise weight conditions for the two inequalities are already known. This result shows that the two inequalities can be studied in terms of each other.

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