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SOME BANACH SPACE EMBEDDINGS OF CLASSICAL FUNCTION SPACES

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Banach space embeddings of the Orlicz space $L_p + L_q$ and the Lorentz space $L_{p,q}$ into the Lebesgue–Bochner space $L_r(l_s)$ are demonstrated for appropriate ranges of the parameters.

1. INTRODUCTION

The purpose of this paper is to explain how the existence of isomorphic embeddings of particular classical Banach function spaces into the Lebesgue–Bochner spaces $L_r(l_s)$ may be deduced quite easily from some known stochastic and analytic facts. Our approach has at its foundation a moment inequality of Rosenthal [**9**] for sums of independent random variables. Indeed, our Theorem 1 below, which describes an embedding of the Orlicz space $L_r + L_s$ into $L_r(l_s)$ in the range $0 < r < s < \infty$, is in effect a disguised form of a particular version (Theorem A below) of Rosenthal's inequality. Theorem A is a special case of a recent, very general result of Johnson and Schechtman [**7**], although the idea of using Rosenthal's inequality to define Banach space embeddings dates back at least to [**6**] (see also [**8**]). From Theorem 1 just a few short steps are required to deduce the existence of an embedding of the Lorentz space $L_{p,q}$ into $L_q(l_s)$ in the range $q < p < s$. The latter implies the surprising theorem of Schütt, obtained by rather different finite-dimensional methods, on the existence of an embedding of $L_{p,q}$ into L_q for $1 \leqslant q < p < 2$. Schütt's theorem motivated our work to a great extent, and it is to be hoped that our approach helps to explain his theorem by exhibiting it in a different light.

2. NOTATION AND PRELIMINARY RESULTS

For $0 < p < \infty$, L_p denotes the usual Lebesgue space of real-valued functions on $[0, \infty)$ with its usual norm, denoted by $\|\cdot\|_p$. For $0 < r < s < \infty$ and $0 < t < \infty$,

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$L_r + t \cdot L_s$ denotes the Orlicz space $L_r + L_s$ equipped with the norm $\|f\|_{L_r+t\cdot L_s} = K(t, f; L_r, L_s)$, where

(1) $$K(t, f; L_r, L_s) = \inf\{\|g\|_r + t\,\|h\|_s : f = g + h\}$$

is the standard K-functional of interpolation theory. The Lorentz space $L_{p,q}$ is the space of real-valued functions on $[0, \infty)$ with the norm $\|f\|_{p,q} = \left[\int_0^\infty f^*(t)^q d(t^{q/p})\right]^{1/q}$, where f^* denotes the decreasing rearrangement of $|f|$. This form of the $L_{p,q}$ norm is not particularly suitable for our purposes, however, and we shall instead use the following formula which is derived from the Lions-Peetre K-method of interpolation (for example [1]):

(2) $$\|f\|_{p,q} \sim \left[\sum_{n=-\infty}^{+\infty} 2^{-n\theta q} K(2^n, f; L_r, L_s)^q\right]^{1/q}$$

provided $1/p = (1-\theta)/r + \theta/s$. (Throughout, $A \sim B$ means that there is a constant $c > 0$, depending only on the parameters p, q, et cetera, such that $(1/c)\,A \leqslant B \leqslant c\,A$.)

The Lebesgue–Bochner space $L_r(l_s)$, $0 < r,\ s < \infty$, is the space of sequences $(f_n)_{n=1}^\infty$ of functions on $[0, 1]$ equipped with the norm

$$\|(f_n)\|_{L_r(l_s)} = \left[\int_0^1 \left(\sum_{n=1}^\infty |f_n(t)|^s\right)^{r/s} dt\right]^{1/r}.$$

Given a family of Banach spaces $(X_n)_{n=-\infty}^\infty$ and $0 < p < \infty$, $\left(\sum_{-\infty}^\infty \oplus X_n\right)_p$ denotes the space of sequences $(x_n)_{n=-\infty}^\infty$, $x_n \in X_n$, with the norm $\left(\sum_{-\infty}^\infty \|x_n\|^p\right)^{1/p}$. When all the X_n's are the same space X, we write $l_p(X)$.

We turn now to state the moment inequality of Rosenthal [9] in a form which illuminates its linear character. First some notation: given a sequence of functions $(f_n)_{n=1}^\infty$ on $[0, 1]$, let $\sum_n \oplus f_n$ denote the function f on $[0, \infty)$ defined by $f(n-1+t) = f_n(t)$ for $n \geqslant 1$ and $0 < t \leqslant 1$. Identify $[0, 1]$ with the (measure-equivalent) product space $\Omega = [0, 1]^{\mathbf{N}}$, and denote a typical element of Ω by $(s_1, s_2, \ldots)$, so that $s_1, s_2, \ldots$ are independent coordinates. The version of Rosenthal's inequality which we shall use may be stated succinctly in the following form.

Theorem A. *Let $0 < p < 2$. Then*

(3) $$\|(f_n(s_n))_{n=1}^\infty\|_{L_p(l_2)} \sim \left\|\sum_{n=1}^\infty \oplus f_n\right\|_{L_p+L_2}.$$

This version of Rosenthal's inequality was first obtained by Johnson and Schechtman [7]. (See also [2] or [4] for an approach more in keeping with the present paper.)

3. The main results

Theorem 1. (See [4, Theorem 3.1]) *Let* $0 < r < s < \infty$. *Then* $L_r + L_s$ *is linearly isomorphic to a subspace of* $L_r(l_s)$.

Proof: Since $2r/s < 2$, we can apply (3) with f_n replaced by $|f_n|^{s/2}$ to obtain

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} \oplus f_n\right\|_{L_r+L_s} &\sim \left\|\sum_{n=1}^{\infty} \oplus |f_n|^{s/2}\right\|_{L_{2r/s}+L_2}^{2/s} \\ &\sim \left\|\left(\sum_{n=1}^{\infty} |f_n(s_n)|^s\right)^{1/2}\right\|_{2r/s}^{2/s} \\ &= \|(f_n(s_n))_{n=1}^{\infty}\|_{L_r(l_s)}.\end{aligned}$$

Hence the mapping $\sum_n \oplus f_n \to (f_n(s_n))_{n=1}^{\infty}$ defines a linear isomorphism from $L_r + L_s$ into $L_r(l_s)$. ∎

For our results on $L_{p,q}$ the following simple proposition is the key.

Proposition 1. *Let* $0 < r < s < \infty$ *and let* $0 < t < \infty$. *Then* $L_r + t \cdot L_s$ *is linearly isometric to* $L_r + L_s$.

Proof: By dilation of (1) by $\lambda > 0$, we have

$$K(t, f; L_r, L_s) = \inf\{\lambda^{1/r}\|g(\lambda x)\|_r + t\lambda^{1/s}\|h(\lambda x)\|_s : f = g + h\}.$$

Select $\lambda = t^{rs/(s-r)}$, so that $t = \lambda^{(s-r)/rs}$. Then

$$\begin{aligned}K(t, f; L_r, L_s) &= \lambda^{1/r}\inf\{\|g(\lambda x)\|_r + \|h(\lambda x)\|_s : f = g + h\} \\ &= \lambda^{1/r}K(1, f(\lambda x); L_r, L_s) \\ &= t^{s/(s-r)}K\left(1, f\left(t^{rs/(s-r)}x\right); L_r, L_s\right).\end{aligned}$$

Thus the mapping $f(x) \to t^{s/(s-r)}f\left(t^{rs/(s-r)}x\right)$ defines a linear isometry from $L_r + t \cdot L_s$ onto $L_r + L_s$. ∎

Remark. The above dilation argument rests on the fact that the function spaces are defined not on $[0, 1]$ but on $[0, \infty)$.

Corollary 1. *Let* $0 < r < p < s < \infty$ *and let* $0 < q < \infty$. *Then* $L_{p,q}$ *is linearly isomorphic to a subspace of* $l_q(L_r + L_s)$.

Proof: (2) says that $L_{p,q}$ is isomorphic to a subspace of $\left[\sum_{-\infty}^{\infty} \oplus(L_r + 2^n \cdot L_s)\right]_q$.
By Proposition 1 each $L_r + t \cdot L_s$ is linearly isometric to $L_r + L_s$, and so the desired result follows. ∎

Theorem 2. *Let $0 < q < p < s$. Then $L_{p,q}$ is linearly isomorphic to a subspace of $L_q(l_s)$.*

Proof: Applying Corollary 1 with $r = q$ shows that $L_{p,q}$ embeds into $l_q(L_q + L_s)$. Hence, by Theorem 1, $L_{p,q}$ embeds into $l_q(L_q(l_s))$ which is of course isomorphic to $L_q(l_s)$. ∎

Corollary 2. [10] *Let $0 < q < p < 2$. Then $L_{p,q}$ is linearly isomorphic to a subspace of L_q.*

Proof: Applying Theorem 2 with $s = 2$ shows that $L_{p,q}$ embeds into $L_q(l_2)$. It is a well-known consequence of Khintchine's inequality that the mapping $(f_n(s))_{n=1}^{\infty} \to \sum_n f_n(s) r_n(t)$, where $(r_n(t))_{n=1}^{\infty}$ are the Rademacher functions, defines an isomorphic embedding of $L_q(l_2)$ into $L_q([0,1]^2)$ (which is isomorphic to L_q), and the desired conclusion follows. ∎

Remark. It is also proved in [10] that if $1 < p < q < 2$ then $L_{p,q}$ is not isomorphic to a subspace of L_1. We indicate here a short proof of this fact. Since $L_{p,q}$ has "type" p in this range (see for example [8] for the definition of this notion), it follows from the main result of [5] that if $L_{p,q}$ embeds into L_1, then l_p embeds into $L_{p,q}$. But by [3, Theorem 2.8], l_p does not embed into $L_{p,q}$.

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