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RESEARCH ARTICLE

On the combined imperfect repair process

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Abstract
In this paper, a new point process is introduced. It combines the nonhomogeneous Poisson process with the generalized Polya process (GPP) studied in recent literature. In reliability interpretation, each event (failure) from this process is minimally repaired with a given probability and GPP-repaired with the complementary probability. Characterization of the new process via the corresponding bivariate point process is presented. The mean numbers of events for marginal processes are obtained via the corresponding rates, which are used for considering an optimal replacement problem as an application.

1. Introduction

Stochastic point processes have been widely used in the reliability literature for modeling the processes of failures and repairs of repairable items. The most popular and the simplest repair in practical implementation is the perfect repair when an item is "good as new" after the repair (replacement). In what follows, for convenience, we will consider instantaneous repair; therefore, the processes of failure and repair coincide forming the renewal process in case of the perfect repair.

Let X denote the lifetime of an item described by its absolutely continuous cumulative distribution function (c.d.f.) $F(t)$, the probability density function (p.d.f.) $f(t)$, the failure rate $\lambda(t)$ and the survival function $\overline{F}(t) = 1 - F(t)$. The other type of repair that is widely used in practice is the minimal repair, introduced in Barlow and Hunter [6]. After the minimal repair, an item is in the "bad as old" state meaning that the distribution of its remaining lifetime is the same as just before the failure, that is,

$$F\,(t|x) = \frac{\bar{F}\,(x) - \bar{F}(x+t)}{\bar{F}(x)} = 1 - \exp\left\{-\int_x^{x+t} \lambda\,(u)\;\mathrm{d}u\right\}, \tag{1}$$

where x is the time of failure/repair. It is well-known that the process of minimal repairs is described by the corresponding nonhomogeneous Poisson process (NHPP) with intensity function/rate $\lambda(t)$. Minimal repair is, obviously, already an imperfect repair.

Numerous models of imperfect repair has been reported in the literature (see, e.g., Badia and Berrade [3–5], Navarro et al. [16], Cha and Finkelstein [11] and references therein). Note that, most of the papers deal with the intermediate case when the repair is better than minimal but worse than perfect. On the other hand, in practice, there are situations when the repair is worse than minimal. For example, often the failure of a component in a multicomponent system can increase stress, temperature, humidity, etc., which results in the increase in the overall failure rate of a system. Also, similar to debugging of software, the new "bugs" can be inserted during repair. See the relevant discussion and examples in

Lee and Cha [14]. The generalized Polya process (GPP) introduced and characterized in Cha [9] is an efficient mathematical tool to model the worse than minimal repair (see the formal definition in the next section). This process generalizes the NHPP and its failure intensity/intensity process increases with each event.

In practice, however, at many instances, the type of repair is not predetermined and depends on the “properties” of each failure. The famous Brown-Proschan model [8] was developed to deal with this randomness for a combination of minimal and perfect repair. Specifically, according to this model, each time an item fails, it is perfectly repaired with probability p and is minimally repaired with probability $(1-p)$. The Brown–Proschan model was extended by Block et al. [7] to a model with an age-dependent probability $p(t)$, where t is the time since the last perfect repair. This process can be simply characterized. For instance, the c.d.f. of the time between consecutive perfect repairs that form the corresponding renewal process, $F_P(t)$, is defined as

$$F_P(t) = 1 - \exp\left\{-\int_0^t p(u)\,\lambda(u)\,\mathrm{d}u\right\}. \tag{2}$$

The goal of our paper is to consider a model similar to the Brown–Proschan model but with two imperfect repair options, that is, to combine in the described way the minimal and the GPP (worse than minimal) repairs. This model can be applied at many instances in practice, as some failures are easily minimally repaired, whereas others result in a worse than minimal repair. Therefore, it will define a new stochastic point process with each event (failure) minimally repaired with probability $p(t)$ and GPP-repaired with the complementary probability $1 - p(t)$.

This appears to be a much more difficult task, as we do not have renewal points now, and the stochastic intensity of this process is changing with each GPP event (it stays the same for minimal repairs). To do so, we will characterize the new combined repair process via introducing the relevant bivariate process and the pooled process, which is the sum of two marginal processes. It will be also shown that some relevant expected values for the events in the processes can be obtained in a simpler way considering not the stochastic intensities of processes but the corresponding intensity functions (rates). The latter is sufficient for considering, as an application, the optimal replacement problem: to find the optimal time of replacement (perfect repair) that minimizes the long-run cost rate.

The rest of the paper is organized as follows. In Section 2, we provide necessary definitions and descriptions. Section 3 is devoted to the characterization of the new bivariate counting process. In Section 4, we derive intensity functions of the processes of interest. Section 5 considers the optimal replacement problem for items with the described types of repair. Finally, brief conclusions are given in Section 6.

2. Preliminaries

This supplementary section provides general concepts and main definitions for description of univariate and multivariate processes via the stochastic intensity. There can be other ways of characterization; however, for our study, this is the most appropriate one. The discussion in this section basically follows that of Cha [9] and Cha and Giorgio [12] and contains descriptions that are necessary for the presentation of further results.

Description of a univariate counting process

Denote by $\{N(t),\ t \geq 0\}$ an orderly point process, and let $\mathcal{H}_{t-} \equiv \{N(u),\ 0 \leq u < t\}$ be its history in $[0, t)$, that is, the set of all previous point events in $[0, t)$. Observe that $\mathcal{H}_{t-}$ can equivalently be defined in terms of $N(t-)$ and the sequential arrival points of the events $0 \leq T_1 \leq T_2 \leq \cdots \leq T_{N(t-)} < t$ in $[0, t)$, where T_i is the time from 0 until the arrival of the ith event in $[0, t)$. At many instances and especially in reliability studies, it is useful to define point processes using the concept of the stochastic intensity λ_t, $t \geq 0$ (the intensity process) (Aven and Jensen [1, 2]). As discussed in Cha and Finkelstein [10], the stochastic intensity λ_t of an orderly point process $\{N(t),\ t \geq 0\}$ is defined as the following limit:

$$\begin{aligned}\lambda_t &\equiv \lim_{\Delta t \to 0} \frac{P\left(N(t, t+\Delta t) = 1 | \mathcal{H}_{t-}\right)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{E\left[N(t, t+\Delta t) | \mathcal{H}_{t-}\right]}{\Delta t},\end{aligned} \tag{3}$$

where $N(t_1, t_2), t_1 < t_2$, is the number of events in $[t_1, t_2)$. It has the following infinitesimal interpretation (heuristic)

$$\lambda_t \, \mathrm{d}t = E\left[\mathrm{d}N(t) | \mathcal{H}_{t-}\right], \tag{4}$$

which is similar to the ordinary failure (hazard) rate of a random variable (Aven and Jensen [1]).

It is well known that the intensity function (rate) of a point process $\{N(t),\ t \geq 0\}$, that distinct from the stochastic intensity does not fully characterize it, is defined as

$$\phi(t) \equiv \frac{\mathrm{d}E[N(t)]}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{E[N(t+\Delta t)] - E[N(t)]}{\Delta t} = \lim_{\Delta t \to 0} \frac{E[N(t, t+\Delta t)]}{\Delta t}.$$

For orderly processes,

$$\phi(t) = \lim_{\Delta t \to 0} \frac{P(N(t, t+\Delta t) = 1)}{\Delta t}.$$

Therefore, the intensity function $\phi(t)$ can be considered as an unconditional version of the stochastic intensity λ_t.

Using the concept of stochastic intensity, the GPP can be defined as follows.

Definition 2.1. (Generalized Polya process). *A counting process* $\{N(t),\ t \geq 0\}$ *is called the GPP with the set of parameters* $(\lambda(t), \alpha, \beta)$, $\alpha \geq 0$, $\beta > 0$, *if*

(i) $N(0) = 0$;
(ii) $\lambda_t = (\alpha N(t-) + \beta)\lambda(t)$.

It can be easily seen that the GPP with $(\lambda(t), \alpha = 0, \beta = 1)$ reduces to the NHPP with the rate function $\lambda(t)$.

Let now $\{N(t),\ t \geq 0\}$ be interpreted as the failure/instantaneous repair process for a repairable item with a lifetime characterized by the failure rate $\lambda(t)$, where $N(t)$ is the total number of failures/repairs in $(0, t]$.

Definition 2.2. (GPP repair). *The failure/repair process for an item with the failure rate* $\lambda(t)$ *is called the "GPP repair" with parameter* α *if* $\{N(t), t \geq 0\}$ *is the GPP with the parameter set* $(\lambda(t), \alpha, 1)$.

Description of a bivariate counting process

We will use the bivariate counting process for description of the new process that will be defined in the next section. Let $\{\mathbf{N}(t), t \geq 0\}$, where $\mathbf{N}(t) = (N_1(t), N_2(t))$, be a bivariate process. The corresponding "pooled" point process $\{M(t),\ t \geq 0\}$ is defined as the sum $M(t) = N_1(t) + N_2(t)$, whereas the marginal point processes $\{N_i(t),\ t \geq 0\}$, for convenience, will be called type i point process, $i = 1, 2$, respectively. Furthermore, the events from type i point process $\{N_i(t),\ t \geq 0\}$ will also be called type i events.

There are two types of regularity that occur in multivariate point processes: (i)marginal regularity and (ii)regularity (see Cha and Giorgio [12] for corresponding definitions). In this paper, we will consider *regular* (also known as *orderly*) multivariate point processes.

Let $\mathcal{H}_{Pt-} \equiv \{M(u),\ 0 \le u < t\}$ be the history (internal filtration) of the pooled process in $[0,t)$, that is, the set of all point events in $[0,t)$. Observe that $\mathcal{H}_{Pt-}$ can equivalently be defined in terms of $M(t-)$ and the sequential arrival points of the events $0 \le T_1 \le T_2 \le \cdots \le T_{M(t-)} < t$ in $[0,t)$, where $M(t-)$ is the total number of events in $[0,t)$ and T_i is the time from 0 until the arrival of the ith event in $[0,t)$ of the pooled process $\{M(t), t \ge 0\}$. Similarly, define the marginal histories of the marginal processes $\mathcal{H}_{it-} \equiv \{N_i(u), 0 \le u < t\}$, $i = 1,2$. Then, $\mathcal{H}_{it-} \equiv \{N_i(u), 0 \le u < t\}$ can also be completely defined in terms of $N_i(t-)$ and the sequential arrival points of the events $0 \le T_{i1} \le T_{i2} \le \cdots \le T_{iN_i(t-)} < t$ in $[0,t)$, $i = 1,2$, where $N_i(t-)$ is the total number of events of type i point process in $[0,t)$, $i = 1,2$.

Although multivariate point processes can be defined in different ways, the most convenient general characterization (especially for applications) can also be done through the stochastic intensity approach. A "marginally regular bivariate process" can be specified by

$$\begin{aligned}
\lambda_{1t} &\equiv \lim_{\Delta t\to 0}\frac{P\left(N_1(t,t+\Delta t)\ge 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t} = \lim_{\Delta t\to 0}\frac{P\left(N_1(t,t+\Delta t)= 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t},\\
\lambda_{2t} &\equiv \lim_{\Delta t\to 0}\frac{P\left(N_2(t,t+\Delta t)\ge 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t} = \lim_{\Delta t\to 0}\frac{P\left(N_2(t,t+\Delta t)= 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t},\\
\lambda_{12t} &\equiv \lim_{\Delta t\to 0}\frac{P\left(N_1(t,t+\Delta t)N_2(t,t+\Delta t)\ge 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t},
\end{aligned} \tag{5}$$

where $N_i(t_1,t_2)$, $t_1 < t_2$, represent the number of events in $[t_1,t_2)$, $i = 1,2$, respectively (see Cox and Lewis [13]). The functions in Eq. (5) are called the *complete intensity functions*. For a regular bivariate process, $\lambda_{12t} = 0$, and it is sufficient to specify just λ_{1t} and λ_{2t} in Eq. (5).

3. Characterization of a new bivariate counting process

Let $\{M(t),\ t \ge 0\}$ be an orderly univariate counting process with sequential arrival points of the events $0 \le T_1 \le T_2 \le \cdots$. Consider a sequence of random variables $\{I_1, I_2, \ldots\}$ with $I_j = 1$ with probability $p(t_j)$ and $I_j = 0$ with probability $1 - p(t_j)$, $j = 1,2,\ldots$, where t_j is the realization of T_j. We assume that, given $\{T_1, T_2, \ldots, T_n\}$, $\{I_1, I_2, \ldots, I_n\}$ are independent, $n = 1,2,\ldots$. Furthermore, given $T_{M(t-)+1}$, the random variable $I_{M(t-)+1}$ does not depend on $\{T_1, T_2, \ldots, T_{M(t-)}, M(t-); I_1, I_2, \ldots, I_{M(t-)}\}$, $t > 0$. We assume that the conditional stochastic intensity of $\{M(t), t \ge 0\}$ with additionally given $\{I_1, I_2, \cdots\}$ is specified by

$$\begin{aligned}
\lambda_{t|I_{t-}} &\equiv \lim_{\Delta t\to 0}\frac{P\left(M(t,t+\Delta t)=1|T_1,T_2,\ldots,T_{M(t-)},M(t-);I_1,I_2,\ldots,I_{M(t-)}\right)}{\Delta t}\\
&= \left(\alpha\sum_{j=1}^{M(t-)} I_j + 1\right)\lambda(t).
\end{aligned} \tag{6}$$

The interpretation of the conditional stochastic intensity in Eq. (6) is as follows. A counting process starts at time 0 with baseline intensity $\lambda(t)$. On each event occurrence, with probability $p(t)$, the stochastic intensity increases according to the GPP pattern (GPP repair, type 1 event) and, with probability $1 - p(t)$, the stochastic intensity is the same as that just before the occurrence of the event (minimal repair, type 2 event), where t is the occurrence time of the event. Clearly, when $p(t) = 1$ for all $t > 0$, $I_j = 1, j = 1,2,\ldots$ (thus the conditional part $\{I_1, I_2, \ldots, I_{M(t-)}\}$ is not necessary anymore), the conditional stochastic intensity (6) reduces to the usual stochastic intensity in Definition 2.1 and the counting process model becomes the GPP. On the other hand, when $p(t) = 0$ for all $t > 0$, $I_j = 0, j = 1,2,\cdots$, and the counting process model becomes the NHPP with rate $\lambda(t)$. Thus, the counting process model in Eq. (6) can be used as the model for a generalized minimal and GPP repairs.

From the counting process model in Eq. (6), it can be seen that, on each event, the type of event can be classified into two types: type 1 and type 2. Thus, we can now define a bivariate counting process $\{(N_1(t), N_2(t)), t \geq 0\}$. For convenience, we use the notations defined in the previous section to describe this bivariate process and the pooled process. To formally characterize the bivariate process $\{(N_1(t), N_2(t)), t \geq 0\}$ generated from the pooled process $\{M(t), t \geq 0\}$, as mentioned in the previous section, we need to specify the stochastic intensities λ_{1t} and λ_{2t}.

Proposition 3.1. *The stochastic intensity* λ_{1t} *and* λ_{2t} *for* $\{(N_1(t), N_2(t)), t \geq 0\}$ *are given by*

$$\lambda_{1t} = (\alpha N_1(t-) + 1)p(t)\lambda(t);$$

$$\lambda_{2t} = (\alpha N_1(t-) + 1)(1 - p(t))\lambda(t).$$

Proof. Note that each event in $\{M(t), t \geq 0\}$ is classified into two types of events independently of everything else. Note also that the joint history $\{\mathcal{H}_{1t-}; \mathcal{H}_{2t-}\}$ can be equivalently defined by $\{T_1, T_2, \ldots, T_{M(t-)}, M(t-); I_1, I_2, \ldots, I_{M(t-)}\}$. Thus,

$$\begin{aligned}
\lambda_{1t} &= \lim_{\Delta t \to 0} \frac{P\left(N_1(t, t+\Delta t) = 1 | \mathcal{H}_{1t-}; \mathcal{H}_{2t-}\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{P\left(\{M(t, t+\Delta t) = 1\} \cap \{\text{The event is type 1}\} | \mathcal{H}_{1t-}; \mathcal{H}_{2t-}\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} P\left(\text{The event is type } 1 | M(t, t+\Delta t) = 1; \mathcal{H}_{1t-}; \mathcal{H}_{2t-}\right) \\
&\quad \times \frac{P\left(M(t, t+\Delta t) = 1 | \mathcal{H}_{1t-}; \mathcal{H}_{2t-}\right)}{\Delta t} \\
&= p(t) \lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | \mathcal{H}_{1t-}; \mathcal{H}_{2t-}\right)}{\Delta t} \\
&= p(t) \cdot \lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | T_1, T_2, \ldots, T_{M(t-)}, M(t-); I_1, I_2, \ldots, I_{M(t-)}\right)}{\Delta t} \\
&= \left(\alpha \sum_{j=1}^{M(t-)} I_j + 1\right) p(t)\lambda(t) = (\alpha N_1(t-) + 1)p(t)\lambda(t).
\end{aligned}$$

By similar arguments, we have $\lambda_{2t} = (\alpha N_1(t-) + 1)(1 - p(t))\lambda(t)$. □

The following proposition further characterizes the pooled process $\{M(t), t \geq 0\}$ and the marginal processes $\{N_i(t), t \geq 0\}$, $i = 1, 2$. Denote by λ_{Mt}, $\lambda_{N_i t}$, $i = 1, 2$, the stochastic intensities of the pooled process $\{M(t),\ t \geq 0\}$ and the marginal processes $\{N_i(t),\ t \geq 0\}$, $i = 1, 2$, respectively.

Proposition 3.2. *For the pooled process* $\{M(t), t \geq 0\}$ *and the marginal processes* $\{N_i(t), t \geq 0\}$, $i = 1, 2$, *the following properties hold.*

(i) The stochastic intensity λ_{Mt} *of the pooled process* $\{M(t),\ t \geq 0\}$ *is given by*

$$\lambda_{Mt} = \lambda(t), \quad \text{if } M(t-) = 0;$$

otherwise,

$$\lambda_{Mt} = \sum_{i_1=0,1} \sum_{i_2=0,1} \cdots \sum_{i_{M(t-)}=0,1} \left(\alpha \sum_{j=1}^{M(t-)} i_j + 1\right) \lambda(t)$$

$$\times \frac{g(i_1, i_2, \ldots, i_{M(t-)}, T_1, T_2, \ldots, T_{M(t-)}, M(t-))}{\sum_{i_1=0,1} \sum_{i_2=0,1} \cdots \sum_{i_{M(t-)}=0,1} g(i_1, i_2, \ldots, i_{M(t-)}, T_1, T_2, \ldots, T_{M(t-)}, M(t-))},$$

where

$$g(i_1, i_2, \ldots, i_m, t_1, t_2, \ldots, t_m, m) \equiv \left\{ \prod_{j=1}^{m} p(t_j)^{i_j} (1 - p(t_j))^{1-i_j} \right\}$$
$$\times \left\{ \prod_{k=1}^{m} \exp\left\{ -\int_{t_{k-1}}^{t_k} \left(\alpha \sum_{l=1}^{k-1} i_l + 1 \right) \lambda(u)\, \mathrm{d}u \right\} \left(\alpha \sum_{l=1}^{k-1} i_l + 1 \right) \lambda(t_k) \right\}$$
$$\times \exp\left\{ -\int_{t_{t_m}}^{t} \left(\alpha \sum_{l=1}^{m} i_l + 1 \right) \lambda(u)\, \mathrm{d}u \right\},$$

with $t_0 = 0$.

(ii) The stochastic intensity $\lambda_{N_1 t}$ *of the marginal processes* $\{N_1(t), t \geq 0\}$ *is given by*

$$\lambda_{N_1 t} = (\alpha N_1(t-) + 1) p(t) \lambda(t).$$

Thus, the marginal process $\{N_1(t), t \geq 0\}$ *is the GPP with the set of parameters* $(p(t)\lambda(t), \alpha, 1)$.

(iii) Given the history of type 1 process, type 2 process is conditionally NHPP with the rate function

$$\lambda(t|\mathcal{H}_{1t-}) = (\alpha N_1(t-) + 1)\lambda(t), \quad t > 0. \tag{7}$$

Thus, type 2 process is mixing of the NHPP with respect to the mixing rate function in Eq. (7).

Proof. It is clear that $\lambda_{Mt} = \lambda(t)$ if $M(t-) = 0$. Now suppose that $M(t-) \geq 1$. The pooled process $\{M(t), t \geq 0\}$ can be characterized by its stochastic intensity

$$\lambda_{Mt} = \lim_{\Delta t \to 0} \frac{P\left(M(t, t + \Delta t) = 1 | T_1, T_2, \ldots, T_{M(t-)}, M(t-)\right)}{\Delta t}.$$

From the conditional stochastic intensity in Eq. (6), λ_{Mt} can be obtained by

$$\lambda_{Mt} = E[\lambda_{t|I_{t-}} | T_1, T_2, \ldots, T_{M(t-)}, M(t-)].$$

Thus, we need the conditional joint distribution of $(I_1, I_2, \ldots, I_{M(t-)} | T_1, T_2, \ldots, T_{M(t-)}, M(t-))$. The joint distribution of $(I_1, I_2, \ldots, I_{M(t-)}; T_1, T_2, \ldots, T_{M(t-)}, M(t-))$ is given by

$$g(i_1, i_2, \ldots, i_m, t_1, t_2, \ldots, t_m, m) \equiv \left\{ \prod_{j=1}^{m} p(t_j)^{i_j} (1 - p(t_j))^{1-i_j} \right\}$$
$$\times \left\{ \prod_{k=1}^{m} \exp\left\{ -\int_{t_{k-1}}^{t_k} \left(\alpha \sum_{l=1}^{k-1} i_l + 1 \right) \lambda(u)\, \mathrm{d}u \right\} \left(\alpha \sum_{l=1}^{k-1} i_l + 1 \right) \lambda(t_k) \right\}$$
$$\times \exp\left\{ -\int_{t_{t_m}}^{t} \left(\alpha \sum_{l=1}^{m} i_l + 1 \right) \lambda(u)\, \mathrm{d}u \right\},$$

where $t_0 = 0$. Then, the joint distribution of $(T_1, T_2, \ldots, T_{M(t-)}, M(t-))$ is given by

$$\sum_{i_1=0,1} \sum_{i_2=0,1} \cdots \sum_{i_m=0,1} g(i_1, i_2, \ldots, i_m, t_1, t_2, \ldots, t_m, m).$$

Thus, the conditional joint distribution of $(I_1, I_2, \ldots, I_{M(t-)} | T_1, T_2, \ldots, T_{M(t-)}, M(t-))$ is given by

$$\frac{g(i_1, i_2, \ldots, i_m, t_1, t_2, \ldots, t_m, m)}{\sum_{i_1=0,1} \sum_{i_2=0,1} \cdots \sum_{i_m=0,1} g(i_1, i_2, \ldots, i_m, t_1, t_2, \ldots, t_m, m)}.$$

Then, λ_{Mt} can be obtained by

$$\begin{aligned}\lambda_{Mt} &= \sum_{i_1=0,1} \sum_{i_2=0,1} \cdots \sum_{i_m=0,1} \left(\alpha \sum_{j=1}^{M(t-)} i_j + 1 \right) \lambda(t) \\ &\quad \times \frac{g(i_1, i_2, \ldots, i_{M(t-)}, T_1, T_2, \ldots, T_{M(t-)}, M(t-))}{\sum_{i_1=0,1} \sum_{i_2=0,1} \cdots \sum_{i_m=0,1} g(i_1, i_2, \ldots, i_{M(t-)}, T_1, T_2, \ldots, T_{M(t-)}, M(t-))}.\end{aligned}$$

From Proposition 3.1,

$$\lambda_{1t} = \lim_{\Delta t \to 0} \frac{P\left(N_1(t, t+\Delta t) = 1 | \mathcal{H}_{1t-}; \mathcal{H}_{2t-}\right)}{\Delta t} = (\alpha N_1(t-) + 1) p(t) \lambda(t),$$

which does not depend on $\mathcal{H}_{2t-}$. Thus,

$$\lambda_{1t} = \lim_{\Delta t \to 0} \frac{P\left(N_1(t, t+\Delta t) = 1 | \mathcal{H}_{1t-}\right)}{\Delta t} = \lambda_{N_1 t}$$

From Proposition 3.1, given a fixed history $\mathcal{H}_{1t-}$, the stochastic intensity λ_{2t} is given by a deterministic function with respect to the history $\mathcal{H}_{2t-}$

$$(\alpha N_1(t-) + 1)(1 - p(t)) \lambda(t).$$

This implies that type 2 process is defined via mixing of the NHPP with respect to the mixing rate function (7). In other words, given the history of type 1 process, type 2 process is conditionally NHPP with the rate function in Eq. (7). □

Remark 3.3.

(i) Although the stochastic intensity of the pooled process $\{M(t), t \geq 0\}$ can be expressed in explicit form, the pooled process $\{M(t), t \geq 0\}$ does not belong to a known class of processes.
(ii) The marginal process $\{N_2(t), t \geq 0\}$ belongs to a class of mixed NHPP, but its stochastic intensity cannot be expressed in a closed form.

In Propositions 3.1 and 3.2, the stochastic intensity functions for the bivariate counting process $\{(N_1(t), N_2(t)), t \geq 0\}$ and the corresponding marginal processes have been derived. In the following result, we derive the expressions for the joint distribution for $(N_1(t), N_2(t))$ and the corresponding marginal distributions. For the description of the following proposition, we define $\Lambda(t) \equiv \int_0^t \lambda(x)\,\mathrm{d}x$, $\Lambda_p(t) \equiv \int_0^t p(x)\lambda(x)\,\mathrm{d}x$, $t_{10} \equiv 0$, $t_{1,n_1+1} \equiv t$,

$$\mathrm{Poi}(n;\phi) \equiv \frac{\phi^n}{n!}\exp\{-\phi\},$$

$$\begin{aligned} &g(t_{11}, t_{12}, \ldots, t_{1n_1}, n_1) \\ &\equiv n_1! \prod_{i=1}^{n_1} \frac{\alpha p(t_{1i})\lambda(t_{1i})\,\exp\{\alpha\Lambda_p(t_{1i})\}}{\exp\{\alpha\Lambda_p(t_{1i})\} - 1}, n_1 \geq 1, 0 < t_{11} < t_{12} < \cdots < t_{1n_1} < t. \end{aligned}$$

Proposition 3.4. *The joint distribution for* $(N_1(t), N_2(t))$ *and the corresponding marginal distributions are given as follows.*

$$\begin{aligned} (i)\ P(N_1(t) = n_1, N_2(t) = n_2) = &\int_0^t \int_0^{t_{1n_1}} \cdots \int_0^{t_{12}} Poi\left(n_2; \sum_{i=0}^{n_1} (i\alpha + 1)[\Lambda(t_{1,i+1}) - \Lambda(t_{1i})]\right) \\ &\times g(t_{11}, t_{12}, \ldots, t_{1n_1}, n_1)\mathrm{d}t_{11} \cdots \mathrm{d}t_{1n_1} \\ &\times \frac{\Gamma(1/\alpha + n_1)}{\Gamma(1/\alpha)n_1!}(1 - \exp\{-\alpha\Lambda_p(t)\})^{n_1}(\exp\{-\alpha\Lambda_p(t)\})^{1/\alpha}, \\ &n_1 = 1, 2, \ldots, \end{aligned}$$

and

$$P(N_1(t) = 0, N_2(t) = n_2) = Poi(n_2; \Lambda(t)) \cdot \exp\{-\Lambda_p(t)\}.$$

$$(ii)\ P(N_1(t) = n_1) = \frac{\Gamma(1/\alpha + n_1)}{\Gamma(1/\alpha)n_1!}(1 - \exp\{-\alpha\Lambda_p(t)\})^{n_1}(\exp\{-\alpha\Lambda_p(t)\})^{1/\alpha}.$$

$$\begin{aligned} (iii)\ P(N_2(t) = n_2) = &Poi(n_2; \Lambda(t)) \cdot \exp\{-\Lambda_p(t)\} \\ &+ \sum_{n_1=1}^{\infty}\left(\int_0^t \int_0^{t_{1n_1}} \cdots \int_0^{t_{12}} Poi\left(n_2; \sum_{i=0}^{n_1}(i\alpha + 1)[\Lambda(t_{1,i+1}) - \Lambda(t_{1i})]\right)\right. \\ &\times g(t_{11}, t_{12}, \ldots, t_{1n_1}, n_1)\mathrm{d}t_{11} \cdots \mathrm{d}t_{1n_1} \\ &\left.\times \frac{\Gamma(1/\alpha + n_1)}{\Gamma(1/\alpha)n_1!}(1 - \exp\{-\alpha\Lambda_p(t)\})^{n_1}(\exp\{-\alpha\Lambda_p(t)\})^{1/\alpha}\right). \end{aligned}$$

Proof. Denote by $0 < T_{11} < T_{12} < \cdots$ the sequential arrival times of the marginal process $\{N_1(t),\ t \geq 0\}$. Clearly, if $n_1 = 0$, $P(N_2(t) = n_2 | N_1(t-) = n_1) = \mathrm{Poi}(n_2; \Lambda(t))$. If $n_1 \geq 1$, from Proposition 3.2 (iii), given $(T_{11} = t_{11}, T_{12} = t_{12}, \ldots, T_{1N_1(t-)} = t_{1n_1}, N_1(t-) = n_1)$,

$$\begin{aligned} &\int_0^t (\alpha N_1(x-) + 1)\lambda(x)\mathrm{d}t \\ &= \int_0^{t_{11}} \lambda(x)\,\mathrm{d}x + \int_{t_{11}}^{t_{12}} (\alpha + 1)\lambda(x)\,\mathrm{d}x + \cdots + \int_{t_{1n_1}}^t (n_1\alpha + 1)\lambda(x)\,\mathrm{d}x \\ &= \sum_{i=0}^{n_1} (i\alpha + 1)[\Lambda(t_{1,i+1}) - \Lambda(t_i)], \end{aligned}$$

and thus the conditional distribution of $N_2(t)$ is given by

$$P(N_2(t) = n_2 | T_{11} = t_{11}, T_{12} = t_{12}, \ldots, T_{1N_1(t-)} = t_{1n_1}, N_1(t-) = n_1)$$
$$= \frac{\left(\sum_{i=0}^{n_1} (i\alpha + 1)[\Lambda(t_{1,i+1}) - \Lambda(t_{1i})]\right)^{n_2}}{n_2!} \exp\left\{-\sum_{i=0}^{n_1} (i\alpha + 1)[\Lambda(t_{1,i+1}) - \Lambda(t_{1i})]\right\}.$$

As the marginal process $\{N_1(t), t \geq 0\}$ is the GPP with the set of parameters $(p(t)\lambda(t), \alpha, 1)$, from Cha [9], the joint conditional arrival time distribution of $(T_{11}, T_{12}, \ldots, T_{1N_1(t-)} | N_1(t-))$ is given by

$$n_1! \prod_{i=1}^{n_1} \frac{\alpha p(t_{1i})\lambda(t_{1i}) \exp\{\alpha \Lambda_p(t_{1i})\}}{\exp\{\alpha \Lambda_p(t_{1i})\} - 1}.$$

Thus, for $n_1 \geq 1$, the conditional distribution of $(N_2(t)|N_1(t))$ is given by

$$P(N_2(t) = n_2 | N_1(t) = n_1) = \int_0^t \int_0^{t_{1n_1}} \cdots \int_0^{t_{12}} \frac{\left(\sum_{i=0}^{n_1} (i\alpha + 1)[\Lambda(t_{1,i+1}) - \Lambda(t_{1i})]\right)^{n_2}}{n_2!}$$
$$\times \exp\left\{-\sum_{i=0}^{n_1} (i\alpha + 1)[\Lambda(t_{1,i+1}) - \Lambda(t_{1i})]\right\}$$
$$\times n_1! \prod_{i=1}^{n_1} \frac{\alpha p(t_{1i})\lambda(t_{1i}) \exp\{\alpha \Lambda_p(t_{1i})\}}{\exp\{\alpha \Lambda_p(t_{1i})\} - 1} \mathrm{d}t_{11} \cdots \mathrm{d}t_{1n_1}. \tag{8}$$

On the other hand, from Cha [9], the marginal distribution of $N_1(t)$ is given by

$$P(N_1(t) = n_1) = \frac{\Gamma(1/\alpha + n_1)}{\Gamma(1/\alpha) n_1!} (1 - \exp\{-\alpha \Lambda_p(t)\})^{n_1} (\exp\{-\alpha \Lambda_p(t)\})^{1/\alpha}. \tag{9}$$

Combining Eqs. (8) and (9), we can obtain the desired results. □

4. Intensity function approach

In the previous section, the bivariate process $\{(N_1(t), N_2(t)), t \geq 0\}$, the pooled process $\{M(t), t \geq 0\}$ and the marginal processes $\{N_i(t), t \geq 0\}$, $i = 1, 2$, have been characterized. As mentioned before, $\{M(t), t \geq 0\}$ does not belong to a known class of processes and it is difficult to obtain further properties of $\{N_2(t), t \geq 0\}$ in explicit forms. On the other hand, in various applications (e.g., as in the optimal replacement problem considered as a reliability application in the next section), we need only the expected values of $N_i(t)$, $i = 1, 2$. Thus, in this section, we derive $E[M(t)]$, $E[N_i(t)]$, $i = 1, 2$, relying on the intensity function approach.

Recall that the intensity function (rate) of an orderly counting process $\{N(t),\ t \geq 0\}$ is defined by

$$\phi(t) \equiv \frac{\mathrm{d}E[N(t)]}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{E[N(t, t + \Delta t)]}{\Delta t} = \lim_{\Delta t \to 0} \frac{P(N(t, t + \Delta t) = 1)}{\Delta t}.$$

Thus, if $\phi(t)$ is given, $E[N(t)]$ can be obtained as

$$E[N(t)] = \int_0^t \phi(u)\, \mathrm{d}u.$$

In the following, we denote by $\phi_M(t)$, $\phi_{N_1}(t)$ and $\phi_{N_2}(t)$ the intensity functions of $\{M(t), t \geq 0\}$ and $\{N_i(t), t \geq 0\}$, $i = 1, 2$, respectively.

Proposition 4.1. *The intensity functions $\phi_M(t)$, $\phi_{N_1}(t)$ and $\phi_{N_2}(t)$ are given by*

$$\phi_M(t) = \lambda(t)\ \exp\{\alpha\Lambda_p(t)\},$$

$$\phi_{N_1}(t) = p(t)\lambda(t)\ \exp\{\alpha\Lambda_p(t)\},$$

$$\phi_{N_2}(t) = (1-p(t))\lambda(t)\ \exp\{\alpha\Lambda_p(t)\},$$

and the expectations are given by

$$E[M(t)] = \int_0^t \lambda(u)\ \exp\{\alpha\Lambda_p(u)\}\,\mathrm{d}u,$$

$$E[N_1(t)] = \frac{1}{\alpha}\left(\exp\{\alpha\Lambda_p(t)\} - 1\right),$$

$$E[N_2(t)] = \int_0^t (1-p(u))\lambda(u)\ \exp\{\alpha\Lambda_p(u)\}\mathrm{d}u.$$

Proof. As $\{N_1(t), t \geq 0\}$ is the GPP with the set of parameters $(p(t)\lambda(t), \alpha, 1)$,

$$P(N_1(t) = n) = \frac{\Gamma(1/\alpha + n)}{\Gamma(1/\alpha)n!}\left(1 - \exp\{-\alpha\Lambda_p(t)\}\right)^n\left(\exp\{-\alpha\Lambda_p(t)\}\right)^{\frac{1}{\alpha}},$$

where $\Lambda_p(t) = \int_0^t p(u)\lambda(u)\,\mathrm{d}u$ (see Cha [9]). Thus,

$$E[N_1(t)] = \frac{1}{\alpha}\left(\exp\{\alpha\Lambda_p(t)\} - 1\right)$$

and

$$\phi_{N_1}(t) = p(t)\lambda(t)\ \exp\{\alpha\Lambda_p(t)\}.$$

For the pooled process $\{M(t), t \geq 0\}$,

$$\lim_{\Delta t\to 0}\frac{P(M(t+\Delta t, t) = 1|N_1(t) = n)}{\Delta t} = (\alpha n + 1)\lambda(t).$$

Thus,

$$\begin{aligned}
\phi_M(t) &= \lim_{\Delta t\to 0}\frac{P(M(t+\Delta t, t) = 1)}{\Delta t}\\
&= \lim_{\Delta t\to 0}\frac{E[P(M(t+\Delta t, t) = 1|N_1(t))]}{\Delta t}\\
&= E\left[\lim_{\Delta t\to 0}\frac{P(M(t+\Delta t, t) = 1|N_1(t))}{\Delta t}\right]\\
&= E\left[(\alpha N_1(t) + 1)\lambda(t)\right]\\
&= \lambda(t)\exp\{\alpha\Lambda_p(t)\}.
\end{aligned}$$

The expectation of $M(t)$ then can be obtained by

$$E[M(t)] = \int_0^t \lambda(u)\ \exp\{\alpha\Lambda_p(u)\}\mathrm{d}u.$$

As the involved processes are all orderly processes and $M(t) = N_1(t) + N_2(t)$,

$$\begin{aligned}\phi_M(t) &= \lim_{\Delta t\to 0}\frac{P(M(t+\Delta t,t)=1)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{P(N_1(t+\Delta t,t)+N_2(t+\Delta t,t)=1)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{P(N_1(t+\Delta t,t)=1 \text{ or } N_2(t+\Delta t,t)=1)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{P(N_1(t+\Delta t,t)=1)+P(N_2(t+\Delta t,t)=1)}{\Delta t}\\ &= \phi_{N_1}(t)+\phi_{N_2}(t).\end{aligned}$$

Thus,

$$\phi_{N_2}(t) = \phi_M(t) - \phi_{N_1}(t) = (1-p(t))\lambda(t)\ \exp\{\alpha\Lambda_p(t)\},$$

and

$$E[N_2(t)] = \int_0^t (1-p(u))\lambda(u)\ \exp\{\alpha\Lambda_p(u)\}\,\mathrm{d}u.$$

□

It is interesting to compare $E[N_2(t)]$ with the expected number of events in the NHPP with the intensity function $\int_0^t (1-p(u))\lambda(u)\,\mathrm{d}u$. Thus, we see the "influence" of the process $N_1(t)$ on this characteristic, which is increasing exponentially in the integrand.

In the next section, we will illustrate these results by considering the corresponding optimal replacement problem. Note that it is possible because we need only the expected values for the numbers of GPP and minimal repairs.

5. Optimal replacement problem

A system with the baseline failure rate $\lambda(t)$ starts operation at $t=0$. On each failure, with probability $p(t)$, the system is GPP-repaired and with probability $1-p(t)$, it is minimally repaired. The corresponding costs are c_{GPP} and c_m, with $c_{\mathrm{GPP}} > c_m$. The system is replaced by a new one at its age T. The cost for the replacement is $c_{\mathrm{r}} > c_{\mathrm{GPP}}$. The latter is a natural assumption in preventive maintenance. Otherwise, all GPP repairs should be replacements in a cost-wise approach. Then, in accordance with periodic (with period T) replacement policy (see, e.g., Nakagawa [15]), the long-run expected cost rate, which is also a cost rate for the replacement cycle, is given by the following expression

$$c(T) = \frac{\frac{1}{\alpha}\left(\exp\{\alpha\Lambda_p(T)\}-1\right)\cdot c_{\mathrm{GPP}} + \int_0^T (1-p(u))\lambda(u)\ \exp\{\alpha\Lambda_p(u)\}\mathrm{d}u\cdot c_m + c_{\mathrm{r}}}{T}.$$

It is easy to see that $\lim_{T\to 0} C(T) = \infty$. Assume also that $\lim_{T\to\infty} C(T) = \infty$. This is absolutely nonrestrictive and even decreasing failure rates $\lambda(t)$, which are usually not considered in the PM modeling, can comply with this condition. It means that the optimal problem

$$C(T_m) = \min_{T\geq 0} C(T)$$

has a finite solution in $[0,\infty)$.

Let us rewrite the objective function in the original form when the first term in the numerator is not integrated. This will help us to see the needed pattern

$$C(T) = \frac{\int_0^T p(u)\lambda(u)\exp\left\{\alpha\Lambda_p(u)\right\}\,\mathrm{d}u\cdot c_{\mathrm{GPP}} + \int_0^T (1-p(u))\lambda(u)\exp\left\{\alpha\Lambda_p(u)\right\}\,\mathrm{d}u\cdot c_m + c_r}{T}$$

Consider now two supplementary objective functions

$$C^{**}(T) = \frac{\int_0^T \lambda(u)\,\exp\left\{\alpha\Lambda_p(u)\right\}\,\mathrm{d}u\cdot c_{\mathrm{GPP}} + c_r}{T} \tag{10}$$

$$C^{*}(T) = \frac{\int_0^T \lambda(u)\,\exp\left\{\alpha\Lambda_p(u)\right\}\mathrm{d}u\cdot c_m + c_{\mathrm{r}}}{T} \tag{11}$$

The first one is obtained from $C(T)$ by setting $c_{\mathrm{GPP}} = c_m$ in the second integral in the numerator, whereas the second one, in the first integral. Thus, in one case, all failures are GPP repaired, in the other-minimally repaired. Obviously, in accordance with our assumptions on the costs

$$C^{**}(T) > C^{*}(T), T > 0. \tag{12}$$

As we are considering only analytical expressions for expected values for the numbers of events, it is not important from what real stochastic processes they have originated, what really matters is these expressions. Therefore, the integral in Eqs. (10) and (11) can be also considered as the expected value of the number of events in the process of minimal repairs (NHPP) with the intensity function $\lambda(u)\exp\left\{\alpha\Lambda_p(u)\right\}$, where as previously, $\Lambda_p(t) = \int_0^t p(u)\lambda(u)\,\mathrm{d}u$. However, this periodic optimal replacement problem for the process of minimal repairs and replacements at $t = T, 2T, 3T\ldots$ is well known and has a finite, unique optimal solution under mild conditions, which in our case can be formulated for both cases as (Nakagawa [15]):

$$\int_0^\infty u\,\mathrm{d}(\lambda(u)\,\exp\left\{\alpha\Lambda_p(u)\right\}) > \frac{c_{\mathrm{r}}}{c_{\mathrm{GPP}}},$$
$$\int_0^\infty u\,\mathrm{d}(\lambda(u)\,\exp\left\{\alpha\Lambda_p(u)\right\}) > \frac{c_{\mathrm{r}}}{c_m}.$$

Those are also absolutely nonrestrictive assumptions (see also Remark 5.1). Therefore, denote by T^{**} the optimal solution minimizing the objective function (10) and by T^* that for (11). Our case is intermediate between two boundary cases when all events are either minimally or GPP repaired. From this and general considerations (the smaller costs of repair always result in the larger replacement time) and also from the fact that under our assumptions, we already know that the optimal replacement time exists and finite, the following bounds can be obtained for T_m:

$$T^{**} < T_m < T^{*}.$$

These bounds can be very useful in practice especially when there is no sufficient information with respect to probability $p(t)$.

Remark 5.1. Note that the additional assumption for the Brown–Proschan model discussed in the Introduction to hold was

$$\lim_{t\to\infty}\int_0^t p(u)\lambda(u)\,\mathrm{d}u = \infty$$

Table 1. *The upper and lower bounds T^* and T^{**} when $c_m = 1$, $c_r = 10$, $p(t) = 0.5$, $t \geq 0$*

		α		
$\lambda(t)$	c_{GPP}	0.1	0.2	0.3
t	2	$T^* = 3.6, T^{**} = 2.8$	$T^* = 3.2, T^{**} = 2.5$	$T^* = 2.9, T^{**} = 2.4$
	3	$T^* = 3.6, T^{**} = 2.3$	$T^* = 3.2, T^{**} = 2.2$	$T^* = 2.9, T^{**} = 2.1$
	5	$T^* = 3.6, T^{**} = 1.9$	$T^* = 3.2, T^{**} = 1.8$	$T^* = 2.9, T^{**} = 1.7$
2t	2	$T^* = 2.5, T^{**} = 1.9$	$T^* = 2.2, T^{**} = 1.8$	$T^* = 2.1, T^{**} = 1.7$
	3	$T^* = 2.5, T^{**} = 1.7$	$T^* = 2.2, T^{**} = 1.5$	$T^* = 2.1, T^{**} = 1.5$
	5	$T^* = 2.5, T^{**} = 1.3$	$T^* = 2.2, T^{**} = 1.3$	$T^* = 2.1, T^{**} = 1.2$
3t	2	$T^* = 2.1, T^{**} = 1.6$	$T^* = 1.8, T^{**} = 1.5$	$T^* = 1.7, T^{**} = 1.4$
	3	$T^* = 2.1, T^{**} = 1.3$	$T^* = 1.8, T^{**} = 1.3$	$T^* = 1.7, T^{**} = 1.2$
	5	$T^* = 2.1, T^{**} = 1.1$	$T^* = 1.8, T^{**} = 1.0$	$T^* = 1.7, T^{**} = 1.0$

that guarantees that the distribution described by the failure rate $p(t)\lambda(t)\,dt$ is proper. This condition is also relevant for our assumption $\lim_{T\to\infty} C(T) = \infty$.

To investigate the range between the lower and upper bounds T^{**} and T^* in the optimization problem, numerical analysis has been performed. The results are summarized in Table 1.

As can be seen from Table 1, (i) as c_{GPP} decreases; (ii) as α increases; and (iii) as $\lambda(t)$ increases for each t, the range between the upper and lower bounds T^* and T^{**} becomes smaller.

6. Concluding remarks

In this paper, we propose a new stochastic point process. It combines the NHPP with the GPP and models the process of failures and instantaneous repairs of a repairable item. Each failure from this process is minimally repaired with a given probability and GPP-repaired with the complementary probability.

Characterization of the new process via the corresponding bivariate point process is presented. The marginal and the pooled processes are defined and described. The latter is the sum of the marginal processes.

Although the stochastic intensity of the pooled process $\{M(t), t \geq 0\}$ can be expressed in explicit form, the pooled process $\{M(t), t \geq 0\}$ does not belong to a known class of processes. On the other hand, the marginal process $\{N_2(t), t \geq 0\}$ belongs to a class of mixed NHPP, but its stochastic intensity cannot be expressed in a closed form.

It is also shown that expected numbers of failures/repairs for marginal processes (and, therefore, for the pooled one) can be obtained in a simpler way using only the rates of these processes. This enables to consider the minimization of the expected long-run cost rate for the corresponding optimal replacement problem. Simple, effective bounds for this characteristic are obtained.

A possible generalization of the developed approach could be in adding the renewal points to the process while defining the probabilities of each type of repair on each failure (i.e., minimal, GPP and perfect). However, mathematical feasibility of this setting is still not clear.

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