Notes

86.60 Immediate successors and predecessors of Fibonacci and Lucas numbers

Fibonacci numbers $F_n$ are often defined by the recursive formula

$$F_n = F_{n-1} + F_{n-2} \quad (1)$$

where $F_1 = F_2 = 1$, and $n \geq 3$. Lucas numbers $L_n$ are also often defined recursively: $L_n = L_{n-1} + L_{n-2}$, where $L_1 = 1, L_2 = 3$, and $n \geq 3$. Both formulas, although easy to work with, require two consecutive elements in either sequence to compute the next one in the sequence.

Interestingly, the immediate successor $F_{n+1}$ of $F_n$ can be computed without knowing $F_{n-1}$. To see this, let $\alpha = (1 + \sqrt{5})/2$, the golden ratio, and $\beta = (1 - \sqrt{5})/2 = 1 - \alpha = -1/\alpha$. Using the recurrence relation (1), we have:

$$F_{n+1} - \alpha F_n = (1 - \alpha)F_n + F_{n-1} = \beta F_n + F_{n-1} = \beta(F_n - \alpha F_{n-1})$$

$$= \beta^2(F_{n-1} - \alpha F_{n-2})$$

$$= \beta^3(F_{n-2} - \alpha F_{n-3})$$

$$\vdots$$

$$= \beta^{n-1}(F_2 - \alpha F_1)$$

$$= \beta^{n-1}(1 - \alpha)$$

$$= \beta^n.$$ 

Since $|\beta| < 0.62$, $|\beta|^2 < \frac{1}{2}$; so $|\beta|^n < \frac{1}{2}$ for $n \geq 2$. This implies that

$$-\frac{1}{2} < F_{n+1} - \alpha F_n < \frac{1}{2}$$

and hence

$$\alpha F_n - \frac{1}{2} < F_{n+1} < \alpha F_n + \frac{1}{2} \quad (2)$$

for $n \geq 2$. Thus

$$F_{n+1} = \left\lfloor \alpha F_n + \frac{1}{2} \right\rfloor \quad (3)$$

for $n \geq 2$. See [1, 2] for an alternate proof of this.

For instance, the immediate successor of the Fibonacci number $F_{20} = 6765$ is given by $\left\lfloor 6765\alpha + \frac{1}{2} \right\rfloor = \left\lfloor 10946.4999... \right\rfloor = 10946 = F_{21}$.

The recursive formula (3) provides a bonus. It can be used to compute the ratios $F_{n+1}/F_n$ as $n \to \infty$, without resorting to the method of continued fractions.

It follows from (3) that

$$F_{n+1} = \alpha F_n + \frac{1}{2} - \theta_n$$

for some real number $\theta_n$, where $0 \leq \theta_n < 1$. Then

$$\frac{F_{n+1}}{F_n} = \alpha + \frac{1}{2F_n} - \frac{\theta_n}{F_n}.$$
Therefore, \( \lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \alpha + 0 + 0 = \alpha \)
a fact that is well-known [1, 2].

Interestingly, formula (3) can be used in the reverse direction also. Using the double inequality (2), it can be shown that

\[
F_n = \left\lfloor \frac{F_{n+1} + \frac{1}{2}}{\alpha} \right\rfloor
\]

(4)

This formula computes the immediate predecessor of a given Fibonacci number.

For example, the immediate predecessor of \( F_{21} = 10946 \) is given by \( \left\lfloor 10946.5/\alpha \right\rfloor = \left\lfloor 6765.3090\ldots \right\rfloor = 6765 = F_{20} \).

Formula (4) shows that \( \lim_{n \to \infty} \frac{F_n}{F_{n+1}} = \frac{1}{\alpha} \). We leave the details as an exercise.

Formulas (3) and (4) can be written in terms of the ceiling function \( \lceil x \rceil \), where \( \lceil x \rceil \) denotes the ceiling of \( x \), that is, the least integer \( \geq x \):

\[
F_{n+1} = \left\lfloor \alpha F_n - \frac{1}{2} \right\rfloor
\]

(5)

\[
F_n = \left\lceil \frac{F_{n+1} - \frac{1}{2}}{\alpha} \right\rceil, \quad n \geq 2
\]

(6)

For instance, using formula (5), the immediate successor of \( F_{17} = 1597 \) is given by \( \left\lceil 1597\alpha - \frac{1}{2} \right\rceil = \left\lceil 2583.5002\ldots \right\rceil = 2584 = F_{18} \).

Formulas (3) through (6) have analogous results to Lucas numbers. They can be obtained by simply changing \( F_m \) to \( L_m \) in each formula. For example, the immediate predecessor of the Lucas number \( L_{20} = 15127 \) is \( \left\lfloor 15127.5/\alpha \right\rfloor = \left\lfloor 9349.3091\ldots \right\rfloor = 9349 = L_{19} \).

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References

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