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ON POSITIVELY COMPLEMENTED SUBSPACES OF c_0

by PANAYOTIS C. TSEKREKOS

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1. Introduction

It has been proved, see [**1**], that a closed infinite dimensional subspace of c_0 is isomorphic to c_0 if and only if it is the range of a bounded linear projection. In [**6**] we proved half of the order-theoretic analogue of this result. In fact we showed that an infinite dimensional subspace of c_0 which is the range of a positive projection is order-isomorphic to c_0. We left open the question whether the converse holds also true. In this paper we answer this question negatively by providing an example in Section 4. In Section 3 we give necessary and sufficient conditions in order that an ordered-subspace of c_0 be the range of a positive projection.

2. Terminology

By c_0 we denote the linear space of all real sequences $x=(x(1), x(2), \dots)$ which converge to zero, ordered by the natural coordinatewise ordering which makes c_0 a Banach lattice ($|x| \leqq |y|$ implies $\|x\| \leqq \|y\|$, where $|x| = x V(-x)$ denotes the supremum of x and $-x$). The positive cone of c_0 defined by this ordering is denoted by c_0^+ and the unit ball by U. By an *ordered-subspace* of c_0 we mean a closed, infinite dimensional subspace X of c_0 such that $X = X_+ - X_+$, where $X_+ = X \cap c_0^+$. Such a subspace is considered everywhere in this paper to be ordered by the cone X_+. The closed unit ball of X is denoted by U_X. A bipositive topological isomorphism between two ordered topological linear spaces is called an *order-isomorphism* and then the spaces are said to be *order-isomorphic*. By a *projection* we mean a continuous linear idempotent operator. We write *U-basis* for an unconditional basis. The ordering associated with a basis (x_n) is given by the cone $X_+ = \{x \in X: x = \sum \lambda_n x_n, \lambda \geqq 0 \text{ for all } n \in \mathbb{N}\}$. For terminology and notation used here and not defined here we refer to [**3**], [**4**] and [**6**].

3. The main results

Our first theorem concerns those ordered-subspaces of c_0 which are order-isometric to c_0. Unlike the situation for l_p spaces (see [**6**]) these are not necessarily sublattices. For example, the subspace

$$X = \{x \in c_0: x(1) = \tfrac{1}{2}(x(2) + x(3))\}$$

is order-isometric to c_0 without being a sublattice.

Theorem 1. *Let X be an ordered-subspace of c_0 Then, X is order-isometric to c_0 if and only if X is the range of a positive contractive projection.*

Proof. Suppose that X is order-isometric to c_0 and let (x_n) be the sequence in X which corresponds to (e_n) under the order-isometry T. Evidently $\|x_n\|=1$ for every $n\in\mathbb{N}$, and for each $n\in\mathbb{N}$, there exists a minimum natural number k_n such that $x_n(k_n)=1$. Since T is an isometry

$$\|x_n \pm x_m\|=1 \quad \text{for all} \quad n\neq m,\ n,\ m=1,2,\ldots$$

which implies that

$$x_n(k_m)=\begin{cases}1 & n=m\\ 0 & n\neq m.\end{cases}$$

Now, for every $x\in c_0$, $x(k_n)\to 0$, and since (e_n) and (x_n) are equivalent, $\sum x(k_n)x_n\in X$.

The mapping P from c_0 into X defined by

$$Px=\sum x(k_n)x_n$$

is linear, $Px_n=x_n$ for every $n\in\mathbb{N}$, and $Px=x$ for every $x\in X$. Also

$$\|P\|=\sup_{\|x\|\leqq 1}\|Px\|\leqq \sup_{n\in\mathbb{N}}|x(k_n)|\leqq\|x\|.$$

This implies that $\|P\|\leqq 1$ and, since P is a projection, finally that $\|P\|=1$.

To prove the converse, suppose that X is an ordered-subspace of c_0 so that it is the range of a positive contractive projection. By virtue of Theorem 6 of [**6**], X is order-isomorphic to c_0, and more precisely the order of X is induced by a U-basis. So, let (x_n) be a normalised U-basis defining the ordering of X. If $z_1\nabla z_2$ denotes the supremum of two elements of X in the ordering defined by X_+, then, since $\|P\|=1$ and $P(z_1Vz_2)=z_1\nabla z_2$, we have that $\|z_1\nabla z_2\nabla\ldots\nabla z_n\|\leqq 1$ for every $n\in\mathbb{N}$ and every $z_1, z_2,\ldots,z_n\in U_X$. Now, it can be easily proved (see also Theorem 2 of [**5**]) that for the order-isomorphism T from X to c_0 defined by

$$T(\lambda_1x_1+\lambda_2x_2+\ldots)=(\lambda_1,\lambda_2,\ldots)$$

we have

$$\tfrac{1}{2}\|x\|\leqq\|Tx\|\leqq\|x\|.$$

This implies that $\|T\|\leqq 1$. For each $n\in\mathbb{N}$ and $\lambda_1,\lambda_2,\ldots,\lambda_n$ real numbers, we have

$$T(\lambda_1x_1\nabla\ldots\nabla\lambda_nx_n)=\lambda_1e_1V\ldots V\lambda_ne_n,$$

which implies

$$|\lambda_1|V\ldots V|\lambda_n|=\|\lambda_1e_1V\ldots V\lambda_ne_n\|\leqq\|T\|\,\|\lambda_1x_1\nabla\ldots\nabla\lambda_nx_n\|\leqq\|\lambda_1x_1\nabla\ldots\nabla\lambda_nx_n\|. \quad (1)$$

On the other hand, the relation

$$\frac{1}{|\lambda_1|V\ldots V|\lambda_n|}|\lambda_1x_1\nabla\ldots\nabla\lambda_nx_n|\leqq|x_1\nabla\ldots\nabla x_n|$$

implies, since c_0 is a Banach lattice, that

$$\frac{1}{|\lambda_1| V \ldots V |\lambda_n|} \|\lambda_1 x_1 \nabla \ldots \nabla \lambda_n x_n\| \leqq \|x_1 \nabla \ldots \nabla x_n\| \leqq 1$$

or

$$\|\lambda_1 x_1 \nabla \ldots \nabla \lambda_n x_n\| \leqq |\lambda_1 V \ldots V |\lambda_n|. \tag{2}$$

Now, relations (1), (2) imply that

$$\|\lambda_1 x_1 \nabla \ldots \nabla \lambda_n x_n\| = |\lambda_1| V \ldots V |\lambda_n|.$$

Let $x = \sum \lambda_n x_n = \nabla \lambda_n x_n$ be an arbitrary element of X. Clearly, $\left\|\overset{n}{\underset{1}{\nabla}} \lambda_i x_i\right\| \overset{n}{\to} \|x\|$, so,

$$\|x\| = \overset{\infty}{\underset{1}{V}} |\lambda_i|. \tag{3}$$

On the other hand

$$\|Tx\| = \|\sum \lambda_i e_i\| = \overset{\infty}{\underset{1}{V}} |\lambda_i|. \tag{4}$$

Relations (3) and (4) imply $\|x\| = \|Tx\|$ for every $x \in X$, which completes the proof.

Before stating the next results we need first a definition.

Definition 1. Let X be an ordered-subspace of c_0 and let $1 \leqq \lambda < \infty$. We say that X has the λ-*positive extension property* (λ-P.E.P. in short) if λ is the least real number for which every positive linear functional x^* on X with $\|x^*\| = 1$ has a positive extension y^* on c_0 with $\|y^*\| \leqq \lambda$. An ordered-subspace X of c_0 is said to have the *bounded positive extension property* (B.P.E.P. in short) if it has the λ-P.E.P. for some λ.

Theorem 2. *Let X be an ordered-subspace of c_0 order-isomorphic to c_0. Then X is the range of a positive projection if and only if it has the B.P.E.P.*

Proof. If X is the range of a positive projection, then it clearly has the B.P.E.P. For the converse, suppose that X has the B.P.E.P. Now, let (x_n) be the basis of X which corresponds to the natural basis (e_n) of c_0 under the order-isomorphism, (x_n^*) the functionals associated with (x_n) with $m \leqq \|x_n^*\| \leqq M$, and y_n^* a positive extension of x_n^* on c_0 with $\|y_n^*\| \leqq \lambda M$ for all $n \in \mathbb{N}$ and some $\lambda \geqq 1$.

We can also suppose that $\operatorname{supp} y_n^* \subseteq \operatorname{supp} x_n$ for each $n \in \mathbb{N}$, where $\operatorname{supp} z = \{i \in \mathbb{N}\colon z(i) \neq 0\}$, for otherwise we can take another extension $y_n'^*$ of x_n^* with $y_n'^* \leqq y_n^*$ and satisfying the above condition. Indeed, suppose that $\operatorname{supp} y_n^* \not\subseteq \operatorname{supp} x_n$. Since $y_n^*(x_m) = x_n^*(x_m) = \delta_{nm}$, $\operatorname{supp} y_n^* \subseteq \mathbb{N} \setminus \bigcup_{m \in \mathbb{N}\setminus\{n\}} \operatorname{supp} x_m$. By nullifying those coordinates of y_n^* which do not belong to $\operatorname{supp} x_n$ we get another extension $y_n'^*$ of x_n^* with the required property. Notice that for each $n \in \mathbb{N}$, $m \leqq \|y_n'^*\| \leqq \lambda M$ and also that $\operatorname{supp} y_n'^* \cap \operatorname{supp} y_m'^* = \varnothing$ for all $n \neq m$, n, $m \neq 1, 2, \ldots$. It follows now easily that $y_n^* \to 0$ with respect to the

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weak-star topology $\sigma(c_0)$. The mapping P from c_0 onto X defined by

$$Px = \sum y_n^*(x)x_n$$

is clearly a positive projection.

The following theorem has been also proved in [**2**] in much greater generality, although I was not aware of this fact. However, I cite it here in the following form, for I think the explicit mention of the constants serves the purpose of this paper better.

Theorem 3. *Let X be an ordered-subspace of c_0. If X has the λ-P.E.P., then $X \cap (U - c_0^+) \subseteq \overline{\lambda U_X - X_+}$. Conversely such a relation implies that X has the μ-P.E.P. with $\mu \leqq \lambda$.*

Proof. Suppose that X has the λ-P.E.P. but the given inclusion does not hold. Then we can find an element $u - p \in X \cap (U - c_0^+)$, with $u \in U$ and $p \in c_0^+$, such that $u - p \notin \overline{\lambda U_X - X_+}$. By the separation theorem, there exists an $x^* \in X^*$ with $\|x^*\| = 1$ such that

$$x^*(u-p) > \sup x^*\overline{(\lambda U_X - X_+)} \geqq \lambda.$$

Take an extension $y^* \geqq 0$ of x^* with $\|y^*\| \leqq \lambda$. Then,

$$\lambda < x^*(u-p) = y^*(u-p) \leqq y^*(u) \leqq \|y^*\| \leqq \lambda$$

which of course cannot be true.

To prove the converse, take a positive linear functional x^* on X with $\|x^*\| = 1$. If q denotes the Minkowski functional of $U - c_0^+$, then, since $X \cap (U - c_0^+) \subseteq \overline{\lambda U_X - X_+}$, we have that $x^*(x) \leqq \lambda q(x)$ for every $x \in X$. By the Hahn–Banach theorem x^* can be extended to a linear functional y^* such that $y^*(x) \leqq \lambda q(x)$ for every $x \in c_0$. If follows that y^* is positive and, since $q(x) \leqq \|x\|$ for all $x \in c_0$, $\|y^*\| \leqq \lambda$.

A simple calculation shows that an ordered-subspace of c_0 which is order-isomorphic to c_0 and has the λ-P.E.P. is the range of a positive projection P with $\|P\| \leqq \lambda$. So, recalling that a closed, infinite-dimensional sublattice X of c_0 is lattice-isometric to c_0 and has the 1-P.E.P., [**4**, prop. 33.15], we immediately conclude that X is the range of a positive contractive projection. It is also tempting to see how the "only if" part of Theorem 1 follows from Theorems 2 and 3. To this end it is enough to show that $X \cap (U - c_0^+) \subseteq \overline{U_X - X_+}$. Notice that the unit ball U_X of X is an upward directed subset of c_0. Suppose then that there is $u - p \in X \cap (U - c_0^+)$ such that $u - p \notin \overline{U_x - X_+}$. Then, according to Theorem 3.1.12 [**3**], there exists an $y^* \in c_0^*$ with $\|y^*\| = 1$ and $y^*(u-p) > \sup y^*(U_X - X_+) \geqq 1$. Then, $1 < y^*(u-p) \leqq y^*(u) \leqq 1$ which cannot be true. Whence, X has the 1-P.E.P. and consequently it admits a positive projection of norm one.

Before stating the next lemma, we explain some of the terminology and notation used in it.

Given an ordered-subspace X of c_0, a lattice in its own ordering, and a subset A of X, we denote by ∇A the set $\{x \in X$: there exist $\alpha_1, \ldots, \alpha_n \in A$ such that $\alpha_1 \nabla \ldots \nabla \alpha_n = x\}$.

We say that A *admits finitely many suprema*, if for every $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in A$, $\alpha_1 \nabla \ldots \nabla \alpha_n \in A$.

Lemma 1. *Let X be an ordered-subspace of c_0, order-isomorphic to c_0. Then, the following relations hold true:*

(α) $X\cap(U-c_0^+)\subseteq\{\nabla[\overline{X\cap(U-c_0^+)}]\}\cap X_+-X_+=\nabla K-X_+$ *where* $K=X\cap(U-c_0^+)$;

(β) *the sets* ∇U_X-X_+, $\overline{\nabla U_X-X_+}$ *admit finitely many suprema;*

(γ) *the sets* $(\nabla U_X-X_+)\cap X_+$, $\overline{\nabla U_X-X_+}\cap X_+$ *are bounded.*

Proof. (α) Take $u-p\in X\cap(U-c_0^+)$. Since $0\in X\cap(U-c_0^+)$,

$$(u-p)^+=(u-p)\nabla 0\in K$$
$$(u-p)^-=(p-u)\nabla 0\in X_+$$

Hence, $u-p=(u-p)^+-(u-p)^-\in K-X_+$

(β) Let u_1-x_1, $u_2-x_2\in\nabla U_X-X_+$. Then, $u_1\nabla u_2-x_1\Delta x_2\in\nabla U_X-X_+$. On the other hand $(u_1-x_1)\nabla(u_2-x_2)\leqq u_1\nabla u_2-x_1\Delta x_2$. So, $[u_1\nabla u_2-x_1\Delta x_2]-(u_1-x_1)\nabla(u_2-x_2)=p\in X_+$, and finally

$$(u_1-x_1)\nabla(u_2-x_2)=u_1\nabla u_2-(x_1\Delta x_2+p)\in\nabla U_X-X_+.$$

To prove that the second set has the required property, take x_1, x_2 from ∇U_X-X_+. There exist sequences $(u_n^1-x_n^1)$, $(u_n^2-x_n^2)$ from the set ∇U_X-X_+ such that

$$u_n^1-x_n^1\to x_1,\quad u_n^2-x_n^2\to x_2.$$

It follows that $(u_n^1-x_n^1)\nabla(u_n^2-x_n^2)\to x_1\nabla x_2$, and consequently, $x_1\nabla x_2\in\overline{\nabla U_X-X_+}$.

(γ) It is clear.

Theorem 4. *Let X be an ordered-subspace of c_0, order-isomorphic to c_0 and $K=X\cap(U-c_0^+)$. Then, X has the B.P.E.P. if and only if $M(K)<+\infty$. where $M(K)=$* $\sup\{\|x\|:x\in\nabla K\cap X_+\}$.

Proof. Suppose $M(K)<+\infty$. Then, there exists $\lambda>0$ such that

$$\nabla K\cap X_+\subseteq\lambda U_X.$$

Hence, $\nabla K\cap X_+\subseteq\lambda U_X-X_+$, and by Lemma 1

$$X\cap(U-c_0^+)\subseteq\nabla K\cap X_+-X_+\subseteq\lambda U_X-X_+.$$

This implies, by Theorem 3, that X has the B.P.E.P.

Suppose now that X has the B.P.E.P. By Theorem 3, there exists $\lambda>0$ such that

$$X\cap(U-c_0^+)\subseteq\overline{\lambda U_X-X_+}\subseteq\overline{\lambda(\nabla U_X)-X_+}$$

Hence, $\{\nabla[X\cap(U-c_0^+)]\}\cap X_+\subseteq\overline{\lambda(\nabla U_X)-X_+}\cap X_+$.

By Lemma 1, the set at the right side of the above inclusion is bounded, so $M(K)<+\infty$.

4. The example

We are going to construct a sequence (x_n) of positive elements of c_0 such that
(i) (x_n) is an unconditional basic sequence;
(ii) $X=[x_n]$, the closed linear span of (x_n), is an ordered-subspace of c_0 with

$$X_+=\left\{\sum_1^\infty \lambda_n x_n : \lambda_n \geqq 0 \quad \text{for all} \quad n\in N\right\};$$

(iii) $\|x_1+\ldots+x_n\|<M$ for all $n\in N$ and some positive real M. These conditions and Theorem 2 of [**5**] will imply that $[x_n]$ is order-isomorphic to c_0. However, X, as we shall see, cannot be the range of a positive projection. Consider the element

$$x_1=\left(1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots\right).$$

Now, to each prime number p_n, $p_n<p_{n+1}$ for all $n\in N\backslash\{1\}$, we correspond the element

$$x_n=\left(0,0,\ldots,0,1,0,0,\ldots,0,\frac{1}{\sqrt{p_n^2}},0,\ldots\right),$$

where the only non-zero coordinates are those corresponding to the positions $p_n, p_n^2, p_n^3,\ldots,p_n^k,\ldots, k\in\mathbb{N}$. More specifically, the p_n-coordinate is equal to 1 and the p_n^kth to $1/\sqrt{p_n^k}$. Clearly $\|x_n\|=1$ for all the $n\in\mathbb{N}$ and (iii) holds true for $M=2$. To show that (x_n) is a basic sequence it is sufficient to show that

$$\|\lambda_1x_1+\ldots+\lambda_nx_n\|<\|\lambda_1x_1+\ldots+\lambda_nx_n+\ldots+\lambda_mx_m\| \tag{A}$$

for all $n, m\in\mathbb{N}$ with $n<m$ and $\lambda_1,\lambda_2,\ldots,\lambda_m$ arbitrary real numbers. Indeed, put

$$x=\lambda_1x_1+\ldots+\lambda_nx_n \quad \text{and} \quad y=\lambda_1x_1+\ldots+\lambda_nx_n+\ldots+\lambda_mx_m.$$

and let i_0 be the coordinate at which $\|x\|=|x(i_0)|$. We distinguish the following two cases:

(a) $i_0\notin\bigcup_{j=2}^n \operatorname{supp} x_j$. Then, $i_0=1$ and since $x(1)=y(1)$, $\|x\|\leqq\|y\|$.

(b) $i_0\in\bigcup_{j=2}^n \operatorname{supp} x_j$. Then, since $\operatorname{supp} x_k\cap\operatorname{supp} x_1=\varnothing$, $k\neq 1$, we have that $x(i_0)=y(i_0)$ and consequently $\|x\|\leqq\|y\|$.

Hence the proof of the inequality (A) has been completed. To prove that (x_n) is an unconditional basic sequence, it is sufficient to show that the convergence of each series $x=\sum_1^\infty \lambda_nx_n$ is unconditional. By virtue of [**4**, Cor. 31.2], it is sufficient to show that the series $\sum_1^\infty \lambda_nx_n$ is $\bigcup$-Cauchy. Indeed, since $\operatorname{supp} x_n\cap\operatorname{supp} x_m=\varnothing$, $n\neq m\neq 1$, $[x_n]_2^\infty$ is a closed sublattice which, as is well known, is isomorphic to c_0; hence $(x_n)_2^\infty$ is a $\bigcup$-basic sequence equivalent to the usual basis of c_0. So, for the sequence $(x_n)_2^\infty$ we have that, given $\varepsilon>0$, there exists a finite subset Φ_0' of $\mathbb{N}$ such that all finite subsets

$\Phi' \supseteqq \Phi'_0$ of $\mathbb{N}$, $\|\sum_{\Phi} \lambda_i x_i - \sum_{\Phi'_0} \lambda_i x_i\| < \varepsilon$. But then

$$\left\|\sum_{\Phi'} \lambda_i x_i - \sum_{\Phi'_0} \lambda_i x_i\right\| = \left\|\sum_{\Phi} \lambda_i x_i + \lambda_1 x_1 - \sum_{\Phi'_0} \lambda_i x_i - \lambda_1 x_1\right\| = \left\|\sum_{\Phi} \lambda_i x_i - \sum_{\Phi_0} \lambda_i x_i\right\| < \varepsilon$$

where $\Phi = \Phi' \cup \{1\}$ and $\Phi_0 = \Phi'_0 \cup \{1\}$, which proves the required result. We are going now to prove that

$$X_+ = \left\{\sum_1^\infty \lambda_n x_n : \lambda_n \geqq 0 \quad \text{for all} \quad n \in \mathbb{N}\right\}.$$

Take $x \in X_+$, Since (x_n) is a $\cup$-basis for X, $x = \sum_1^\infty \lambda_n x_n$. The fact that $x \geqq 0$ implies that each coordinate is a non-negative real number. Since $x(1) = \lambda_1$, we have $\lambda_1 \geqq 0$. Moreover, for each $n \in \mathbb{N}$, the coordinates of x at the positions $p_n^2, p_n^3, \ldots, p_n^k, \ldots$ must also be non-negative numbers i.e.

$$\lambda_1/p_n^k + \lambda_n/\sqrt{p_n^k} \geqq 0 \quad \text{for all} \quad k \in \mathbb{N},$$

or

$$\lambda_1/\sqrt{p_n^k} + \lambda_n \geqq 0.$$

As $k \to \infty$, the above inequality gives $\lambda_n \geqq 0$, as required. Finally, relation (iii) implies that the M-constants of X are bounded, so, by Theorem 2 of [**5**], $[x_n]$ is order-isomorphic to c_0.

However, X cannot be the range of a positive projection. For if P is such a projection consider $P[kx_1 \wedge x_2]$, $k \in \mathbb{N}$, where the infimum is calculated in c_0.

As we have

$$0 \leqq P[kx_1 \wedge x_2] \leqq kPx_1, Px_2$$

and $Px_1 = x_1$, $Px_2 = x_2$ are disjoint in $[x_n]$, $P[kx_1 \wedge x_2] = 0$. But now observe that P is norm continuous and that $kx_1 \wedge x_2 \to x_2$ in norm, so $Px_2 = 0$, a contradiction.

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National Technical University of Athens
42 Patission Street
Athens 147
Greece