LMS J. Comput. Math. 17 (1) (2014) 360–378
doi:10.1112/S1461157014000096

Computing boundary extensions of conformal maps

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Abstract

We show that a computable and conformal map of the unit disk onto a bounded domain D has a computable boundary extension if D has a computable boundary connectivity function.

1. *Introduction*

We investigate what information can be used to compute the boundary extension of a conformal map. By the *boundary extension* of a conformal map we mean its continuous extension to the closure of its domain. The conditions under which a boundary extension (computable or otherwise) exists will be reviewed in § 2. Our main result is that if ϕ is a computable and conformal map of the unit disk onto a bounded domain D and if D has a computable boundary connectivity function, then the boundary extension of ϕ is computable as well. By a *boundary connectivity function* for D we mean a function $g : \mathbb{N} \to \mathbb{N}$ with the following property: whenever p and q are distinct points of the boundary of D such that $|p - q| \leqslant 2^{-g(k)}$, the boundary of D contains an arc from p to q whose diameter is smaller than 2^{-k}. (Here, $\mathbb{N}$ denotes the set of non-negative integers.) Roughly speaking, such a function predicts how close two boundary points must be in order to connect them with a small arc that is included in the boundary. We do not assume any amount of differentiability of the boundary of D. Thus, our results apply to domains bounded by fractal curves like the Koch snowflake.

Suppose that ϕ is a computable and conformal map of the unit disk onto a bounded domain D and that the boundary extension of ϕ exists. To understand why computing the boundary extension of ϕ may not be an entirely trivial matter, and might require some information beyond ϕ itself, let us begin by considering how we extend ϕ to the boundary of the unit disk. Namely, we set $\phi(\zeta) = \lim_{z\to\zeta} \phi(z)$ whenever ζ is unimodular. It is well known that limiting operations can churn incomputable behavior out of computable settings. For example, a theorem due to Specker states that it is possible to compute a sequence of rational numbers that is increasing and bounded but whose limit is incomputable [**28**]; that is, roughly speaking, it is not possible to write a computer program to compute the decimal expansion of the limit. In [**20**], it is shown that there is a computable and conformal map of the unit disk onto a Jordan domain whose boundary extension is incomputable. Thus, some information beyond ϕ itself must be utilized in order to compute the boundary extension of ϕ. We will make the case for considering boundary connectivity functions in § 2.

We now outline our strategy for proving the main theorem. Suppose that D has a computable boundary connectivity function. One natural approach to computing the boundary extension of ϕ is to first show that ϕ is computable on the unit circle and then merge an algorithm for computing ϕ on the unit circle with an algorithm for computing ϕ on the unit disk. The flaw in this approach is that an algorithm for computing the boundary extension of ϕ can only accept approximations of points (for example, approximations of the real and imaginary parts), and from an approximation of a point it is not always possible to determine if it lies on the unit circle. We work around this obstacle by first showing that ϕ is *strongly computable on the*

Received 24 April 2013; revised 27 February 2014.

2010 Mathematics Subject Classification 03D78 (primary), 30C30, 30E10, 03F60, 54D05 (secondary).

unit circle. Roughly speaking, this means that not only is ϕ computable on the unit circle, but also that our approximations of the values of ϕ on unimodular points hold for all nearby points as well. This term is precisely defined in § 3. We then produce an algorithm for computing the boundary extension of ϕ by merging an algorithm for computing ϕ on the unit disk and an algorithm for strongly computing ϕ on the unit circle.

The outline of the paper is as follows. In § 2 we summarize background information from complex analysis and the theory of computation. Our goal is to make our results accessible to readers in computer science and complex analysis. In § 3 we summarize the intermediate results of the paper and how they are combined to produce a proof of the main theorem. In § 4 we develop new estimates of the boundary values of ϕ in terms of a boundary connectivity function for D. In § 5 we make the case that these estimates can be used by an algorithm. In §6 we show that these results yield strong computability of ϕ on the unit circle and thereby complete the proof of the main theorem.

2. *Background and preliminaries*

We begin by summarizing background material from complex analysis.

A *domain* is a subset of the plane that is open and connected.

Let $D_r(z)$ denote the open disk whose center is z and whose radius is r. Let $\mathbb{D}$ denote the unit disk, that is, the open disk whose center is the origin and whose radius is 1. We refer to the boundary of $\mathbb{D}$ as the *unit circle* and to the closure of $\mathbb{D}$ as the *closed unit disk*.

The *Riemann mapping theorem* states that if D is a simply connected domain that omits at least one point, then there is an injective and analytic map of the unit disk onto D. Since this map is analytic and injective, it is also conformal. If w_0 is a point in D, then, among all such maps of the unit disk onto D, there is exactly one that maps the origin to w_0 and whose derivative at 0 is positive. We denote this map by ϕ_{D,w_0}. Such a map is called a *Riemann map* of D.

Suppose that ϕ is a conformal map of the unit disk onto a domain D. By a theorem of Pommerenke [**25**], ϕ has a boundary extension if and only if D is bounded and its boundary is locally connected. If ϕ has a boundary extension, then we will denote this extension by ϕ as well. The *Carathéodory theorem* states that if the boundary of D is a Jordan curve, then the boundary extension of ϕ is a homeomorphism. A very elegant proof of the Carathéodory theorem appears in [**11**, Chapter I].

By an *arc* we mean a homeomorphic image of $[0, 1]$. Such a homeomorphism is called a *parameterization* of the arc. It will simplify our discussion if we identify each arc with its parameterizations.

A metric space X is *uniformly locally arcwise connected* if, for every $\epsilon > 0$, there is a $\delta > 0$ so that whenever p and q are distinct points of X such that $d(p, q) < \delta$, X includes an arc from p to q whose diameter is smaller than ϵ. Thus, a domain D has a boundary connectivity function if and only if its boundary is uniformly locally arcwise connected. If X is compact and connected, then X is locally connected if and only if it is uniformly locally arcwise connected; see [**15**, Lemma 3-29, p. 129]. So, the requirement that D has a computable boundary connectivity function is a suitable substitute for local connectivity when pursuing a computable version of Pommerenke's theorem on boundary extensions.

We now summarize background material from computability theory. In general, the adjective 'computable' refers to the ability to solve some problem with an algorithm. By 'algorithm' we roughly mean a procedure that can be implemented on a computer. There are several ways to mathematically formalize this notion, such as Turing machines. All of these formalizations yield the same classes of computable objects. See [**8**] or [**24**] for a more expansive discussion. For our purposes, it will suffice to work with the informal notion of 'algorithm'.

We begin with the computability of various kinds of subsets of the plane. Let us call an interval *rational* if its end points are rational numbers, and let us call a rectangle rational if its vertices are rational points.

When U is an open subset of the plane, let $R(U)$ denote the set of all closed rational rectangles that are included in U. When C is a closed subset of the plane, let $R(C)$ denote the set of all open rational rectangles that contain at least one point of C. Whether X is open or closed, the set $R(X)$ completely identifies X. That is, $R(X) = R(X')$ if and only if $X = X'$.

Let us call an open subset of the plane U *computable* if $R(U)$ is computably enumerable, that is, if the elements of $R(U)$ can be arranged into a sequence $\{R_n\}_{n\in\mathbb{N}}$ in such a way that there is an algorithm that computes R_n from n for every $n \in \mathbb{N}$. Intuitively, as such an enumeration is run, it provides more and more information about what is in the set. We similarly define what it means for a closed subset of the plane to be computable. Again, by enumerating the rational rectangles that contain at least one point of a closed set C we obtain more and more information about what is in the set. As an example, the interior of the ellipse with equation $4x^2 + 9y^2 = 16$ is computable, as is its boundary. In fact, most naturally occurring open sets and closed sets are computable.

We now discuss computability of functions. A function $g : \mathbb{N} \to \mathbb{N}$ is *computable* if there is an algorithm that given any $k \in \mathbb{N}$ as input produces $g(k)$ as output.

Suppose that f is a function that maps complex numbers to complex numbers. We say that f is *computable* if there is an algorithm P that satisfies the following three criteria.

- *Approximation:* whenever P is given an open rational rectangle as input, it either does not halt or produces an open rational rectangle as output. (Here, the input rectangle is regarded as an approximation of a $z \in \mathrm{dom}(f)$ and the output rectangle is regarded as an approximation of $f(z)$.)
- *Correctness:* whenever P halts on an open rational rectangle R, the rectangle it outputs contains $f(z)$ for each $z \in R \cap \mathrm{dom}(f)$.
- *Convergence:* suppose that U is a neighborhood of a point $z \in \mathrm{dom}(f)$ and that V is a neighborhood of $f(z)$. Then there is an open rational rectangle R such that R contains z, R is included in U, and, when R is put into P, P produces a rational rectangle that is included in V.

For example, sin, cos, and exp are computable, as can be seen by considering their power series expansions and the bounds on the convergence of these series that can be obtained from Taylor's theorem. A consequence of this definition is that computable functions on the complex plane must be continuous. A comprehensive treatment of the computability of functions on continuous domains can be found in [**30**]. See also [**5**, **13**, **17**, **18**, **26**, **29**] and [**6**].

Suppose that f is a function of a complex variable and that X is included in the domain of f. We say that f is *computable on* X if its restriction to X is computable. If X is the unit circle, then, as remarked in the introduction, we will need a stronger version of this notion, which we now define.

Definition 2.1. Suppose that f is a function that maps complex numbers to complex numbers and is defined at every point on the unit circle. We say that f is *strongly computable on the unit circle* if there is an algorithm P with the following properties.

- *Approximation:* whenever an open rational rectangle is input to P, P either does not halt or outputs an open rational rectangle.
- *Strong correctness:* if P outputs a rational rectangle R_1 on input R, then $f(z) \in R_1$ whenever $z \in R \cap \mathrm{dom}(f)$.

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- *Convergence:* if U is a neighborhood of a unimodular point ζ, and if V is a neighborhood of $f(\zeta)$, then ζ belongs to an open rational rectangle $R \subseteq U$ so that P halts on input R and produces a rational rectangle that is contained in V.

Suppose that f is defined at every point of the closed unit disk. If we merely assert that f is computable on the unit circle, then the correctness criterion only requires our output rectangle to contain $f(z)$ for each *unimodular* point z in the input rectangle. But, if we assert that f is strongly computable on the unit circle, then our output rectangle must contain $f(z)$ whenever z is a point in the input rectangle that also belongs to the domain of f.

PROPOSITION 2.2. *Suppose $f : \overline{\mathbb{D}} \to \mathbb{C}$. Then f is computable if and only if f is both computable on the unit disk and strongly computable on the unit circle.*

Proof. If f is computable, then it trivially follows that f is both computable on the open unit disk and strongly computable on the unit circle; any algorithm which computes f on the closed unit disk works for each of these notions. So, suppose that f is both computable on the unit disk and strongly computable on the unit circle. Let P_1 be an algorithm that computes f on the unit disk and let P_2 be an algorithm that strongly computes f on the unit circle. We compute f on the closed unit disk by merging these algorithms as follows. Suppose that an open rational rectangle R is given as input. If R contains no point of the closed unit disk, then we choose not to halt. So, suppose that R contains at least one point of the closed unit disk. If R is contained in the unit disk, then we run algorithm P_1 on R. Suppose that R is not contained in the open unit disk; that is, that R contains at least one point of the unit circle. We then run algorithm P_2 on R.

It is clear that the approximation criterion is met. By considering the cases $z \in \mathbb{D}$ and $z \in \partial\mathbb{D}$, it is easily shown that the convergence criterion is met. It then follows from the strong correctness criterion of Definition 2.1 that the correctness criterion is met. □

We now review some related work. Suppose that D is a simply connected domain that omits at least one point. Extending the work of Koebe [**16**], Cheng [**7**], and Bishop and Bridges [**3**], Hertling proved that ϕ_{D,w_0} is computable if and only if w_0, D, and ∂D are computable [**14**]. The zipper algorithm of Marshall and Rohde provides a practical algorithm for computing Riemann maps of a Jordan domain with a sufficiently differentiable boundary [**19**]. The complexity of computing Riemann maps of a Jordan domain was determined by Binder *et al.* in [**2**]. In [**20**], it was shown that if the boundary of D is a Jordan curve, and if ϕ is a Riemann map of D, then ϕ has a computable boundary extension if and only if ϕ is computable and there is a computable homeomorphism of the unit circle with the boundary of D. Various versions of computable local connectivity properties are examined in [**4**], [**10**], and [**9**].

To facilitate exposition, let us make the following conventions. Throughout the rest of this paper, ϕ denotes a conformal map of the unit disk onto a bounded domain D whose boundary is locally connected. Let g denote a boundary connectivity function for D. We can assume that this map is increasing. Our main theorem states that if ϕ and g are computable, then the boundary extension of ϕ is computable.

3. *Outline of the proof of the main theorem*

3.1. *Analytical estimates*

We begin by developing approximations of the values of ϕ on unimodular points. We do so in terms of sides of crosscuts, which we now define.

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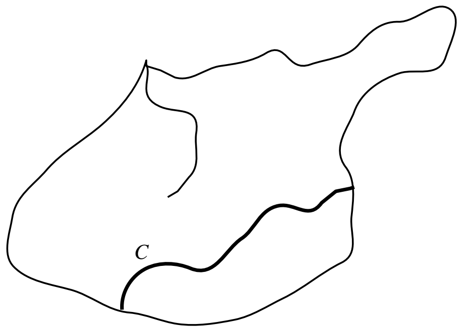

FIGURE 1. *A crosscut.*

Suppose that C is an arc in $\overline{D}$. If the only points of C that lie on the boundary of D are the end points of C, then C is called a *crosscut* of D. See Figure 1. If C is a crosscut of D, then $D-C$ has exactly two connected components. To see this, consider the map $z \mapsto (1-|\phi^{-1}(z)|)^{-1}$ under which the boundary of D is mapped to ∞ and C is mapped to a Jordan curve through ∞; apply the Jordan curve theorem. These components are called the *sides* of C. When C is a crosscut of D that does not contain $\phi(0)$, let C^- denote the side of C that contains $\phi(0)$, and let C^+ denote the other side.

Whenever $0 < s_0 < 1$ and $|\zeta| = 1$, let $A_{s_0,\zeta}$ denote the image of ϕ on $\partial D_{s_0}(\zeta)$. Thus, $A_{s_0,\zeta}$ is a crosscut of D. Note that $A^+_{s_0,\zeta}$ is the image of ϕ on $D_{s_0}(\zeta) \cap \mathbb{D}$. Also, $\phi(t\zeta) \in A^+_{s_0,\zeta}$ if $1 - s_0 < t < 1$.

Fix an integer N_0 that is larger than the area of D. When $0 < r_0 < s_0 < 1$, let

$$m(s_0, N_0, r_0) = \sqrt{\frac{\pi N_0}{\ln(s_0/r_0)}}.$$

Note that $m(s_0, N_0, r_0) \to 0^+$ when $r_0 \to 0^+$.

The central idea is to use appropriately constructed crosscuts to approximate $\phi(\zeta)$ when $|\zeta| = 1$; more precisely, to treat each point on such a crosscut as an approximation of $\phi(\zeta)$. Let C be such a crosscut. If $\phi(\zeta) \notin C$, then this leads to two considerations: determining which side of C the point $\phi(\zeta)$ abuts, and determining an upper bound on the diameter of this side. The crosscuts we introduce in Definition 3.1 contain enough information to resolve these issues.

DEFINITION 3.1. Suppose $|\zeta| = 1$. Let C be a crosscut of D. We say that C *recognizably bounds the value of* ϕ *on* ζ if there are rational numbers r_0, s_0 such that the following hold:

(i) $0 < r_0 < s_0 < 1/2$;
(ii) $\phi((1 - s_0)\zeta) \in C$;
(iii) $C \cap A_{s_0,\zeta}$ is connected and $C \cap A^+_{s_0,\zeta}$ has two connected components;
(iv) $|\phi(t\zeta) - z| > m(s_0, N_0, r_0)$ whenever $z \in \overline{A^+_{s_0,\zeta} \cap C}$ and $1 - s_0 \leqslant t \leqslant 1 - r_0$.

We say that (r_0, s_0) *witnesses* that C recognizably bounds the value of ϕ on ζ.

Note that it follows from condition (iii) that $C \subseteq A_{s_0,\zeta} \cup A^+_{s_0,\zeta}$. An illustration of Definition 3.1 appears in Figure 2.

In § 4 we prove the following two theorems.

THEOREM 3.2. *Suppose that* (r_0, s_0) *witnesses that* C *recognizably bounds the value of* ϕ *on* ζ. *Then* $A^+_{r_0,\zeta} \subseteq C^+$.

Thus, $\phi(\zeta)$ is a limit point of C^+.

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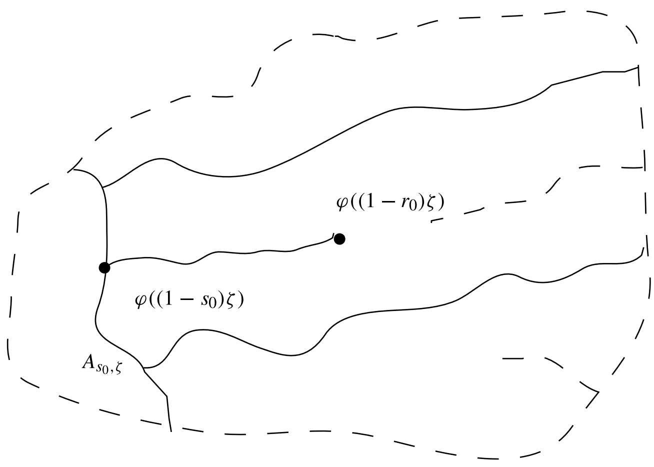

FIGURE 2. *Definition* 3.1.

THEOREM 3.3. *Suppose that C recognizably bounds the value of ϕ on ζ. If $2^{-k+1} < |\phi(0) - \phi(\zeta/2)|$, and if the diameter of C is smaller than $2^{-g(k)}$, then the diameter of C^+ is at most 2^{-k+1}.*

In § 4 we also prove the following theorem.

THEOREM 3.4. *Suppose $|\zeta| = 1$. Then there are crosscuts of arbitrarily small diameter that recognizably bound the value of ϕ on ζ. That is, for every $\epsilon > 0$, there is a crosscut that recognizably bounds the value of ϕ on ζ and whose diameter is smaller than ϵ.*

So, points on crosscuts that recognizably bound the value of ϕ on ζ can be used to approximate $\phi(\zeta)$ with arbitrarily small error.

3.2. *Computability issues*

To say that an algorithm computes with crosscuts is a chimera, since there are uncountably many crosscuts but algorithms proceed by manipulating strings from a fixed finite alphabet. So, we are led to consider the approximation of crosscuts. Since a crosscut is an arc, we first discuss how we approximate arcs. Our approach is drawn from the work on computable arcs in [**10**] and [**22**]. To begin, a finite sequence of sets $(S_1, \ldots, S_n)$ is a *chain* if $S_j \cap S_{j+1} \neq \emptyset$ whenever $1 \leqslant j < n$. In addition, $(S_1, \ldots, S_n)$ is a *simple chain* if $S_j \cap S_k \neq \emptyset$ only when $|j - k| = 1$. We then define a *wad* to be a union of a chain of open rational boxes and an *approximate arc* to be a simple chain of wads.

When $A_1, \ldots, A_n$ are subarcs of an arc A, we write $A = A_1 + \ldots + A_n$ if A_{j+1} contains exactly one point of A_j whenever $1 \leqslant j < n$. An approximate arc $(w_1, \ldots, w_n)$ *approximates* an arc A if A can be decomposed into a sum $A = A_1 + \ldots + A_n$ such that $A_j \subseteq w_j$ for all j; equivalently, if there are numbers $0 = t_0 < \ldots < t_n = 1$ such that A maps each number in $[t_{j-1}, t_j]$ into w_j. The largest diameter of a wad w_j will be referred to as the *error* in this approximation. In § 5 we show that every approximate arc actually approximates an arc, and that every arc can be approximated with arbitrarily small error.

We define an *approximate crosscut* of D to be an approximate arc $(w_1, \ldots, w_n)$ such that:

(i) $\overline{w_j} \subseteq D$ when $1 < j < n$; and

(ii) $w_j \cap \partial D \neq \emptyset$ if $j = 1, n$.

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It follows from the results in § 5 that every approximate crosscut indeed approximates a crosscut of D, and that every crosscut of D can be approximated with arbitrarily small error by an approximate crosscut.

So, when $|\zeta| = 1$, the computation of $\phi(\zeta)$ now reduces to producing approximate crosscuts that approximate, with arbitrarily small error, crosscuts of arbitrarily small diameter that recognizably bound the value of ϕ on ζ. This leads to the following two definitions and theorem.

DEFINITION 3.5. Suppose that $\mathcal{C}$ is a set of crosscuts of D and that $\mathcal{A}$ is a set of approximate crosscuts. We say that $\mathcal{A}$ *describes* $\mathcal{C}$ if the following two conditions are met.

(i) Every approximate crosscut in $\mathcal{A}$ approximates a crosscut in $\mathcal{C}$.

(ii) Every crosscut in $\mathcal{C}$ can be approximated with arbitrarily small error by an approximate crosscut in $\mathcal{A}$. That is, if C is a crosscut in $\mathcal{C}$, and if $\epsilon > 0$, then C is approximated by an approximate crosscut in $\mathcal{A}$ with error smaller than ϵ.

We say that an algorithm *enumerates* a set of approximate crosscuts $\mathcal{A}$ if it has the property that whenever an approximate arc is given as input the algorithm halts if and only if the approximate arc belongs to $\mathcal{A}$.

DEFINITION 3.6. Let $\mathcal{C}$ be a set of crosscuts of D. We say that an algorithm *recognizes* $\mathcal{C}$ if it enumerates a set of approximate crosscuts that describes $\mathcal{C}$. We say that $\mathcal{C}$ is *recognizable* if at least one algorithm recognizes it.

In § 4 we prove the following theorem.

THEOREM 3.7. *Suppose that s_0, r_0 are rational numbers and that ζ is a computable unimodular point. Let $\mathcal{C}$ be the set of all crosscuts C such that (s_0, r_0) witnesses that C recognizably bounds the value of ϕ on ζ. If ϕ is computable, then $\mathcal{C}$ is recognizable.*

The proof of Theorem 3.7 is uniform. That is, it produces an algorithm that from s_0, r_0, an algorithm that computes ζ, and an algorithm that computes ϕ, computes an algorithm that recognizes the set of all crosscuts C such that (s_0, r_0) witnesses that C recognizably bounds the value of ϕ on ζ. This uniformity allows us to prove the following by a covering argument in § 6.

THEOREM 3.8. *If ϕ and g are computable, then ϕ is strongly computable on the unit circle.*

In light of Proposition 2.2, this yields the proof of the main theorem.

THEOREM 3.9. *The boundary extension of a computable and conformal map of the unit disk onto a bounded domain with a computable boundary connectivity function is computable.*

4. *Recognizable bounding crosscuts*

Our first task is to prove Theorem 3.3. We use two principles of analysis: Schwarz's inequality and the Lusin area integral. For reference, we state these theorems here. The first is stated only for the case of Lebesgue measure on $\mathbb{R}$. Schwarz's inequality is a consequence of Hölder's inequality [**27**]. Greene and Krantz [**12**, Chapter 13, §1] contains a proof of Lusin's area integral. Recall that when $X \subseteq \mathbb{R}^2$, the *area* of X is defined to be

$$\int_X 1 \, dA,$$

where $\int_X f \, dA$ denotes the Riemann integral of f over X. We denote the area of X by $\text{Area}(X)$.

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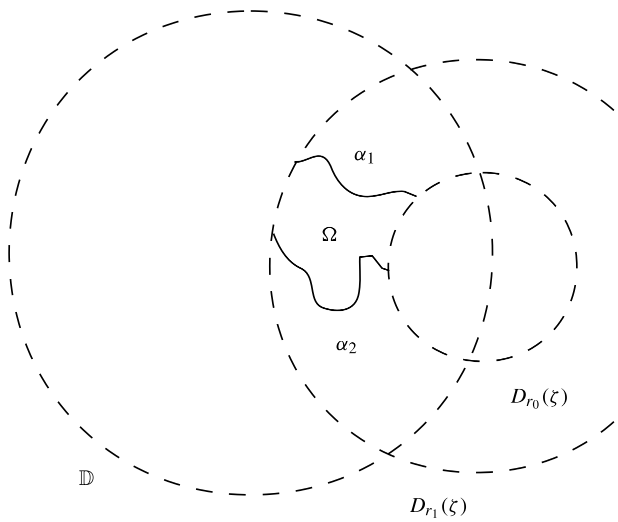

FIGURE 3. *Lemma 4.1.*

- *Schwarz's inequality:* let μ denote Lebesgue measure on the real line. Let $X \subseteq \mathbb{R}$ be measurable and suppose that f, g are non-negative measurable functions on X. Then

$$\left(\int_X fg\,d\mu\right)^2 \leqslant \int_X f^2\,d\mu \int_X g^2\,d\mu.$$

- *Lusin area integral:* suppose that U is a domain and that f is analytic and one-to-one on U. Then

$$\mathrm{Area}(f[U]) = \int_U |f'|^2\,dA.$$

We now set about proving Theorem 3.3. When $X_1, X_2 \subseteq \mathbb{C}$, let

$$d_{\inf}(X_1, X_2) = \inf\{|z_1 - z_2| : z_1 \in X_1 \wedge z_2 \in X_2\}.$$

LEMMA 4.1. *Suppose that ζ, r_0, r_1, α_1, α_2, and Ω are as in Figure 3. That is:*
(i) *$0 < r_0 < r_1 < 1$ and $|\zeta| = 1$;*
(ii) *α_1 and α_2 are disjoint crosscuts of*

$$\{z \in \mathbb{D} : r_0 < |z - \zeta| < r_1\}$$

that do not touch the boundary of $\mathbb{D}$;
(iii) *Ω consists of those points in the side of α_1 that includes α_2 that also belong to the side of α_2 that includes α_1.*

Then

$$\mathrm{Area}(\phi[\Omega]) \geqslant \frac{1}{\pi} d_{\inf}(\phi[\alpha_1], \phi[\alpha_2])^2 \ln\left(\frac{r_1}{r_0}\right).$$

Proof. By the Lusin area integral,

$$\mathrm{Area}(\phi[\Omega]) = \int_\Omega |\phi'|^2\,dA.$$

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We intend to write this integral in polar coordinates centered at ζ. To this end, let $\gamma_r(\theta) = \zeta + r\zeta e^{i\theta}$. Note that

$$\Omega \subseteq \left\{\gamma_r(\theta) : r_0 < r < r_1 \wedge \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}\right\}.$$

When $0 < r_0 < r < r_1$, let

$$S_r = \left\{\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] : \gamma_r(\theta) \in \Omega\right\}.$$

We now change to polar coordinates and obtain

$$\begin{aligned}\int_\Omega |\phi'|^2 \, dA &= \int_{r_0}^{r_1} \int_{S_r} |\phi'(\gamma_r(\theta))|^2 r \, d\theta \, dr \\ &= \int_{r_0}^{r_1} \left(r \int_{S_r} |\phi'(\gamma_r(\theta))|^2 d\theta \right) dr \\ &\geqslant \frac{1}{\pi} \int_{r_0}^{r_1} \frac{1}{r} \left(\int_{S_r} r^2 d\theta \int_{S_r} |\phi'(\gamma_r(\theta))|^2 d\theta \right) dr.\end{aligned}$$

By Schwarz's inequality,

$$\int_{S_r} r^2 \, d\theta \int_{S_r} |\phi'(\gamma_r(\theta))|^2 \, d\theta \geqslant \left(\int_{S_r} r|\phi'(\gamma_r(\theta))| \, d\theta \right)^2.$$

When $r_0 < r < r_1$, let

$$\begin{aligned}\theta_{r,1} &= \max \left\{\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] : \gamma_r(\theta) \in \alpha_1\right\}, \\ \theta_{r,2} &= \min \left\{\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] : \gamma_r(\theta) \in \alpha_2\right\}.\end{aligned}$$

Then

$$\begin{aligned}\int_{S_r} r|\phi'(\gamma_r(\theta))| \, d\theta &\geqslant \int_{\theta_{r,1}}^{\theta_{r,2}} r|\phi'(\gamma_r(\theta))| \, d\theta \\ &= \int_{\theta_{r,1}}^{\theta_{r,2}} \left| \frac{d}{d\theta} \phi(\gamma_r(\theta)) \right| d\theta.\end{aligned}$$

The latter integral is the length of the arc traced by $\phi(\gamma_r(\theta))$ as θ ranges from $\theta_{r,1}$ to $\theta_{r,2}$. This in turn is at least as large as the minimum distance between $\phi[\alpha_1]$ and $\phi[\alpha_2]$. Pulling all this together, we obtain

$$\begin{aligned}\text{Area}(\Phi[\Omega]) &\geqslant \frac{1}{\pi} \int_{r_0}^{r_1} \frac{1}{r} d_{\inf}(\phi[\alpha_1], \phi[\alpha_2])^2 \, dr \\ &= \frac{1}{\pi} d_{\inf}(\phi[\alpha_1], \phi[\alpha_2])^2 \ln \left(\frac{r_1}{r_0} \right).\end{aligned}$$ □

When $z_1, z_2 \in \mathbb{C}$ are distinct, let $[z_1, z_2]$ denote the line segment from z_1 to z_2.

LEMMA 4.2. *Suppose* $|\zeta| = 1$ *and* $0 < r_0 < s_0 < 1$. *Suppose that* C *is an arc from a point* $p \in A_{s_0,\zeta}$ *to a point* $q \in \partial D$ *such that* $C \cap \partial D = \{q\}$ *and such that* $|\phi(t\zeta) - z| \geqslant m(s_0, N_0, r_0)$ *whenever* $1 - s_0 \leqslant t \leqslant 1 - r_0$ *and* $z \in C$. *Then no point of* $A^+_{r_0,\zeta}$ *belongs to* C.

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Proof. By way of contradiction, suppose otherwise. Since $C \cap \partial D = \{q\}$, it follows that $\phi^{-1}[C - \{q\}]$ starts at a point ζ_0 on the boundary of $D_{s_0}(\zeta)$ and crosses the boundary of $D_{r_0}(\zeta)$; let ζ_1 be the first point at which it does so. Let α_1 be the subarc of $\phi^{-1}[C]$ from ζ_0 to ζ_1. Let $\alpha_2 = [(1 - s_0)\zeta, (1 - r_0)\zeta]$. It follows from Lemma 4.1 that

$$\text{Area}(D) \geqslant \frac{1}{\pi} m(s_0, N_0, r_0)^2 \ln\left(\frac{s_0}{r_0}\right) \geqslant N_0 > Area(D).$$

This is a contradiction and the proof is complete. □

Proof of Theorem 3.2. We first note that if U is a connected subset of D that contains no point of C, then U must be included in a side of C. Since $r_0 < s_0$, $\phi((1 - r_0)\zeta) \in C^+$. In addition, $\phi((1 - r_0)\zeta)$ is a boundary point of $A^+_{r_0,\zeta}$. Thus, C^+ contains at least one point of $A^+_{r_0,\zeta}$. Since $A^+_{r_0,\zeta}$ is connected, if $A^+_{r_0,\zeta}$ is not included in C^+, then it must contain a point of C. Let $C_2 = C \cap A_{s_0,\zeta}$, and let C_1 and C_3 be the connected components of $A^+_{s_0,\zeta} \cap C$. Since $s_0 < r_0$, $A^+_{r_0,\zeta}$ contains no point of C_2. It follows from Lemma 4.2 and Definition 3.6 that $C_1 \cup C_2$ contains no point of $A^+_{r_0,\zeta}$. Since $A^+_{r_0,\zeta} \cap C = \emptyset$, it follows that $A^+_{r_0,\zeta} \subseteq C^+$. □

Proof of Theorem 3.3. Let $\text{diam}(X)$ denote the diameter of X. Let τ be an arc in the boundary of D that joins the end points of C. Since the diameter of C is not larger than $2^{-g(k)}$, we can assume that the diameter of τ is smaller than 2^{-k}. Let $J = C \cup \tau$. Thus, J is a Jordan curve. Since g is increasing, $g(x) \geqslant k$. Thus, the diameter of J is at most 2^{-k+1}. Note that the diameter of the interior of J is identical to the diameter of J. Since $s_0 < 1/2$ (by Definition 3.6), $\phi(\zeta/2) \in A^-_{s_0,\zeta}$. However, $A^-_{s_0,\zeta} \subseteq C^-$ and so $\phi(\zeta/2) \in C^-$. On the other hand, since $2^{-k+1} < |\phi(0) - \phi(\zeta/2)|$ (by assumption), the interior of J does not include C^-.

We now claim that the interior of J contains a point of $D - C$. For, let $p \in C \cap D$. Thus, $p \in J$. So, p is a boundary point of the interior of J. Since $p \in D$, D includes an open disk centered at p. Thus, this disk contains a point in the interior of J; let q denote such a point. Therefore, $q \notin C$ (since $C \subseteq J$) and $q \in D$.

Since $q \in D - C$, q belongs to one and only one side of C; let S denote this side. We claim that the interior of J includes S. For, suppose that q_1 is a point in S besides q. Since S is open and connected, it includes an arc σ from q to q_1. Since S includes σ, σ contains no point of C. Since D includes σ, and since the boundary of D includes τ, σ contains no point of τ. Thus, σ never crosses J and so q_1 belongs to the interior of J. Thus, the interior of J includes S.

It now follows that $S = C^+$. Since the diameter of J is at most 2^{-k+1}, the diameter of C^+ is at most 2^{-k+1}. □

We now show that there are arbitrarily small crosscuts that recognizably bound the value of ϕ on a unimodular ζ. We use the following proposition.

PROPOSITION 4.3. *The pre-image of ϕ on a finite subset of the boundary of D has empty interior* (*in the relative topology on $\partial\mathbb{D}$*).

Proof. By way of contradiction, suppose otherwise. It follows that there is a point ζ that belongs to the boundary of D and whose pre-image under ϕ includes an arc G. Let C be a crosscut of the unit disk whose end points are the end points of G. Then $\phi[C] \cup \{\zeta\}$ is a Jordan curve and ϕ conformally maps the interior of $G \cup C$ onto the interior of $\phi[C] \cup \{\zeta\}$. It follows from the Carathéodory theorem that the boundary extension of ϕ is injective. This is a contradiction, since ϕ maps all of G onto ζ. □

Actually, much more than Proposition 4.3 is true: if $\zeta \in \partial D$, then $\phi^{-1}[\{\zeta\}]$ has measure zero. However, the pre-image of ϕ on a boundary point may be uncountable. See Beurling [**1**].

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Proof of Theorem 3.4. Without loss of generality, we assume $\zeta = 1$. The general claim then follows by applying the following argument to the map ψ such that $\psi(z) = \phi(\zeta z)$ for all $z \in \mathbb{D}$. Fix a positive number s_0 that is smaller than $\frac{1}{2}$. Suppose $\delta > 0$. It follows from Proposition 4.3 that there is a positive number θ_0 that is smaller than δ and $\pi/2$ and such that $\phi(e^{i\theta_0}) \neq \phi(1)$. It also follows that there is a negative number θ_1 that is larger than $-\delta$ and $-\pi/2$ and such that $\phi(e^{i\theta_1}) \neq \phi(1), \phi(e^{i\theta_0})$.

Choose δ small enough so that the lines with equations $y = \mathrm{Im}(e^{i\theta_0})$ and $y = \mathrm{Im}(e^{i\theta_1})$ cross $\partial D_{s_0}(1)$. Let σ_j denote the intersection of the line with equation $y = \mathrm{Im}(e^{i\theta_j})$ with the closure of $\mathbb{D} \cap D_{s_0}(1)$. Let p_j denote the end point of σ_j on $\partial D_{s_0}(1)$. Let τ denote the subarc of $\partial D_{s_0}(1) \cap \mathbb{D}$ from p_1 to p_2. Thus, since $\phi(e^{i\theta_0}) \neq \phi(e^{i\theta_1})$, the image of ϕ on $\sigma_0 \cup \tau \cup \sigma_1$ is a crosscut of D. Denote this crosscut by C.

By allowing s_0 to approach zero from the right while allowing δ to approach zero from the right, we can make the diameter C as small as we like. We can also choose s_0 to be rational.

Let $C_j = \phi[\sigma_j]$. Thus, C_0 and C_1 are the components of $\overline{C \cap A^+_{s_0,1}}$. The key point now is that $\phi(t) \notin C_0 \cup C_1$ whenever $1 - s_0 \leqslant t \leqslant 1$. The task now is to choose r_0. We begin by letting δ_1 denote the minimum of $|\phi(t) - z|$ as t ranges from $1 - s_0$ to 1 and z ranges over $\sigma_0 \cup \sigma_1$. We can then choose r_0 so that $m(s_0, N_0, r_0) < \delta_1$. It follows that there is a rational number r_0 between 0 and s_0 such that $d(\phi(t), \phi[\sigma_1 \cup \sigma_2]) > m(s_0, N_0, r_0)$ whenever $1 - s_0 \leqslant t \leqslant 1 - r_0$. It follows that C, s_0, and r_0 meet all conditions of Definition 3.1. □

5. *Approximating crosscuts*

Our next task is to prove Theorem 3.7. We begin with the following results on arc approximation.

THEOREM 5.1. *Suppose that $(w_1, \ldots, w_n)$ is an approximate arc and that p, q are points in w_1, w_n, respectively. Then $(w_1, \ldots, w_n)$ approximates an arc from p to q.*

Proof. Set $p_0 = p$ and $p_n = q$. Choose a point p_j in $w_j \cap w_{j+1}$ for each $j \in \{1, \ldots, n-1\}$. We can assume $p_0 \neq p_1$ and $p_{n-1} \neq p_n$. Since $(w_1, \ldots, w_n)$ is a simple chain, it follows that $p_0, \ldots, p_n$ are pairwise distinct.

Since a wad is a union of a chain of open rational rectangles, every wad is an open and connected set. So, each w_j includes an arc from p_{j-1} to p_j; call this arc B_j.

If we join the arcs $B_1, \ldots, B_n$ together we do not necessarily get an arc, since, for example, B_2 may intersect B_1 at one or more points besides p_1. So, let p'_j be the first point on B_j that belongs to B_{j+1} for each $j \in \{1, \ldots, n-1\}$. Let $p'_0 = p_0$ and let $p'_n = p_n$. Let A_j be the subarc of B_j from p'_{j-1} to p'_j. It then follows that $A_1 \cup \ldots \cup A_n$ is an arc that is approximated by $(w_1, \ldots, w_n)$. □

In the proof of our next theorem, we use the following, which is [**15**, Theorem 3-4].

THEOREM 5.2. *If a, b are two points of a connected space S, and if $\{U_\alpha\}_{\alpha \in I}$ is a family of open sets that covers S, then there exist $\alpha_1, \ldots, \alpha_n \in I$ so that $(U_{\alpha_1}, \ldots, U_{\alpha_n})$ is a simple chain such that $a \in U_{\alpha_1} - U_{\alpha_2}$ and such that $b \in U_{\alpha_n} - U_{\alpha_{n-1}}$.*

In the following proof, we will also use the fact that the connected components of an open subset of a locally connected space are open. For example, see [**15**, Theorem 3-2].

THEOREM 5.3. *If A is an arc from p to q, then, for every positive number ϵ, there is an approximation of A, $(w_1, \ldots, w_n)$, with error smaller than ϵ so that $p \in w_1 - \overline{w_2}$ and $q \in w_n - \overline{w_{n-1}}$.*

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Proof. As a function, A is uniformly continuous. It follows that there are numbers $0 = t_0 < \ldots < t_n = 1$ so that $|A(s) - A(t)| < \epsilon/3$ whenever $s, t \in [t_{j-1}, t_j]$. Let A_j denote the image of A on $[t_{j-1}, t_j]$. Then $A_j \cap A_k = \emptyset$ if $|j - k| > 1$. So, when $|j - k| > 1$, let $\delta_{j,k}$ denote

$$\min\{|z_1 - z_2| : z_1 \in A_j, z_2 \in A_k\}.$$

Let δ denote the minimum of all $\delta_{j,k}$.

Fix j for the moment. Let $\mathcal{R}$ be the set of all open rational rectangles that contain at least one point of A_j and whose diameter is smaller than $\epsilon/3$ and $\delta/2$. If $j \in \{2, n-1\}$, then we also require $p, q \notin \overline{R}$. We claim that there is a chain of rectangles in $\mathcal{R}$ that covers A_j. For, let $\mathcal{S}$ be the set of all U for which there is an $R \in \mathcal{R}$ such that U is a connected component of $R \cap A_j$. Then each set in $\mathcal{S}$ is open (in the relative topology on A_j). Let a, b be the end points of A_j. Let $U_{\alpha_1}, \ldots, U_{\alpha_m}$ be as given by Theorem 5.2. Since $(U_{\alpha_1}, \ldots, U_{\alpha_m})$ is a simple chain, its union is connected. Since a, b are the end points of A_j, it follows that $\bigcup_k U_{\alpha_k} = A_j$. For each k, there is a rectangle $R_k \in \mathcal{R}$ such that U_{α_k} is a connected component of $R_k \cap A_j$. It follows that $(R_1, \ldots, R_m)$ is a chain that covers A_j. Set $w_j = \bigcup_k R_k$.

By the choice of δ and the diameters of the R, $(w_1, \ldots, w_n)$ is a simple chain. It follows that $(w_1, \ldots, w_n)$ approximates A. It follows from the choice of ϵ that the diameter of each w_j is smaller than ϵ. □

We define an arc to be *computable* if it is the image of a map on the unit interval that is computable and injective. We then have the following lemma.

LEMMA 5.4. *If A is a computable arc, then there is an algorithm that enumerates the set of all approximations of A.*

Proof. Let f be a computable homeomorphism of $[0, 1]$ with A. Fix an algorithm that computes f.

Let $(w_1, \ldots, w_n)$ be an approximate arc that is given as input. We first note that $(w_1, \ldots, w_n)$ approximates A if and only if there are rational numbers $t_0 = 0 < t_1 < \ldots < t_k = 1$ so that, for each j, f maps each point in $[t_{j-1}, t_j]$ into w_j. We then note that f maps an interval $[a, b]$ into an open set U just in the case where there are open rational rectangles $R_1, \ldots, R_m, S_1, \ldots, S_m$ so that $[a, b]$ is covered by $\{R_1, \ldots, R_m\}$, $\overline{S_j} \subseteq U$ for each j, and for each j the algorithm that computes f produces S_j on input R_j. By putting these two observations together, we arrive at a search procedure that terminates if and only if $(w_1, \ldots, w_n)$ approximates A. □

We note that the proof of Lemma 5.4 is uniform. That is, it provides an algorithm that, given any algorithm that computes an arc A as input, produces an algorithm that enumerates all approximations of A.

Throughout the rest of this section, let $\mathcal{C}_{(s_0,r_0,\zeta)}$ denote the set of all crosscuts C such that (s_0, r_0) witnesses that C recognizably bounds the value of ϕ on ζ. In order to prove Theorem 3.7, we need to define a set of approximate arcs that describes $\mathcal{C}_{(s_0,r_0,\zeta)}$ (see Definition 3.5). To this end, we make the following definition.

DEFINITION 5.5. Let $\mathcal{A}_{(s_0,r_0,\zeta)}$ denote the set of all approximate crosscuts of D $(w_1, \ldots, w_n)$ for which there exist integers j_1, j_2 so that the following conditions are met:

(i) $1 < j_1 < j_2 < n$ and $0 < r_0 < s_0 < 1/2$;

(ii) $\{w_j\}_{j=j_1}^{j_2}$ approximates a subarc of $A_{s_0,\zeta}$ that contains $\phi((1 - s_0)\zeta)$; let L denote the connected component of $\phi((1 - s_0)\zeta)$ in $A_{s_0,\zeta} \cap \bigcup_{j_1 \leqslant j \leqslant j_2} w_j$;

(iii) $\overline{w_j} \subseteq A^+_{s_0,\zeta}$ whenever $1 < j < j_1$ and whenever $j_2 < j < n$;

(iv) there is a component E_1 of $w_{j_1} \cap A^+_{s_0,\zeta}$ such that $L \cap w_{j_1} \cap \partial E_1 \neq \emptyset$ and $E_1 \cap w_{j_1 - 1} \neq \emptyset$;

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(v) there is a component E_2 of $w_{j_2} \cap A^+_{s_0,\zeta}$ such that $L \cap w_{j_2} \cap \partial E_2 \neq \emptyset$ and $E_2 \cap w_{j_2+1} \neq \emptyset$;
(vi) $|\phi(t\zeta) - z| > m(r_0, N_0, s_0)$ whenever $1 - s_0 \leqslant t \leqslant 1 - r_0$ and

$$z \in \bigcup_{j=1}^{j_1} \overline{w_j} \cup \bigcup_{j=j_2}^{n} \overline{w_j}.$$

We now show that $\mathcal{A}_{(s_0,r_0,\zeta)}$ describes $\mathcal{C}_{(s_0,r_0,\zeta)}$. We begin with the following two lemmas.

LEMMA 5.6. *Suppose that $(w_1, \ldots, w_n)$ is an arc approximation and $1 \leqslant k \leqslant n-1$. Suppose $p_1 \in w_1$, $p_2 \in w_k \cap w_{k+1}$, and $p_3 \in w_n$. Suppose that $(w_1, \ldots, w_k)$ approximates an arc A from p_1 to p_2 and that $(w_{k+1}, \ldots, w_n)$ approximates an arc B from p_2 to p_3. Then $(w_1, \ldots, w_n)$ approximates an arc $C \subseteq A \cup B$ from p_1 to p_2.*

Proof. Let $A = A_1 + \ldots + A_k$ be a decomposition of A with the property that $A_j \subseteq w_j$ whenever $1 \leqslant j \leqslant k$. Let $B = B_{k+1} + \ldots + B_n$ be a decomposition of B so that $B_j \subseteq w_j$ whenever $k+1 \leqslant j \leqslant n$. Then let p_2' be the first point on A that belongs to B. Since $(w_1, \ldots, w_n)$ is a simple chain, $p_2' \in w_k \cap w_{k+1}$. So, $p_2' \notin A_1 \cup \ldots \cup A_{k-1}$. So, let A_k^* be the subarc of A_k from A_{k-1} to p_2'. Since $p_2' \in w_k$, $p_2' \notin B_{k+2} \cup \ldots \cup B_n$. Let B_{k+1}^* be the subarc of B_{k+1} from p_2' to B_{k+2}. Let $C = A_1 \cup \ldots \cup A_{k-1} \cup A_k^* \cup B_{k+1}^* \cup B_{k+2} \cup \ldots \cup B_n$. Then C is an arc and is approximated by $(w_1, \ldots, w_n)$. ☐

In the following proof we use the fact that an open and connected subset of the plane is arcwise connected.

LEMMA 5.7. *Suppose $(w_1, \ldots, w_n) \in \mathcal{A}_{(s_0,r_0,\zeta)}$. Let j_1, j_2, L, E_1, and E_2 be as in Definition 5.5. Then:*
(i) *there is an arc G_1 from a point in $E_1 \cap w_{j_1-1} \cap w_{j_1}$ to a point q_1 in $w_{j_1} \cap L \cap \partial E_1$ so that $G_1 \subseteq E_1 \cup \{q_1\}$;*
(ii) *there is an arc G_2 from a point in $E_2 \cap w_{j_2} \cap w_{j_2+1}$ to a point q_2 in $w_{j_2} \cap L \cap \partial E_2$ so that $G_2 \subseteq E_2 \cup \{q_2\}$.*

Proof. We first note that each boundary point of E_1 belongs either to $A_{s_0,\zeta}$ or to the boundary of w_{j_1}. For, let p be a boundary point of E_1. Suppose $p \notin A_{s_0,\zeta}$. Since $E_1 \subseteq A^+_{s_0,\zeta}$, $p \notin A^-_{s_0,\zeta}$. Since $(w_1, \ldots, w_n)$ is an approximate crosscut of D, $\overline{w_{j_1}} \subseteq D$. So, $\partial E_1 \subseteq D$. Thus, $p \in D$ and so $p \in A^+_{s_0,\zeta}$. But, $p \notin E_1$ since E_1 is open. It follows that $p \notin w_{j_1}$. For, if $p \in w_{j_1}$, then its component in $A^+_{s_0,\zeta} \cap w_{j_1}$ is an open set that contains p but no point of E_1. It now follows that $p \in \partial w_{j_1}$.

By condition (iv) of Definition 5.5, there is a point $q_1' \in w_{j_1} \cap L \cap \partial E_1$. Let ϵ be a positive number such that $D_\epsilon(q_1') \subseteq w_{j_1}$ and $D_\epsilon(q_1') \cap A_{s_0,\zeta} \subseteq L$. By [**15**, Theorem 3-18], there are a point $q_1 \in \partial E_1$ and a point $p \in E_1$ so that $|q_1 - q_1'| < \epsilon$ and $[p, q_1] \subseteq E_1 \cup \{q_1\}$. Thus, $q_1 \in L$. Let $p' \in E_1 \cap w_{j_1-1} \cap w_{j_1}$. Then E_1 includes an arc G from p' to p. Let p'' be the first point on G that belongs to $[p, q]$. Let G^* be the subarc of G from p' to p''. Then take $G_1 = G^* \cup [p'', q]$.

Part (ii) is proved similarly. ☐

THEOREM 5.8. *$\mathcal{A}_{(s_0,r_0,\zeta)}$ describes $\mathcal{C}_{(s_0,r_0,\zeta)}$.*

Proof. To begin, suppose that $(w_1, \ldots, w_n)$ is an approximate crosscut in $\mathcal{A}_{(s_0,r_0,\zeta)}$. We construct a crosscut in $\mathcal{C}_{(s_0,r_0,\zeta)}$ that is approximated by $(w_1, \ldots, w_n)$. Let j_1, j_2, and L be as in the definition of $\mathcal{A}_{(s_0,r_0,\zeta)}$.

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We first show that $(w_1, \dots, w_{j_1})$ approximates an arc C_1 such that $C_1 \cap (A_{s_0,\zeta} \cup \partial D) \subseteq \{C_1(0), C_1(1)\}$ and $C_1(1) \in L$. By Lemma 5.7, there is an arc $G \subseteq w_{j_1}$ from a point $p \in E_1 \cap w_{j_1} \cap w_{j_1-1}$ to a point $q \in w_{j_1} \cap L$ so that $G \cap (A_{s_0,\zeta} \cup \partial D) = \{q\}$. By Theorem 5.1, $(w_1, \dots, w_{j_1-1})$ approximates an arc H from a point $p_1' \in \partial D$ to p. Let $H = H_1 + \dots + H_{j_1-1}$ be a decomposition of H so that $H_j \subseteq w_j$ for all j. Let p_1 be the last point on H that belongs to ∂D. Then $p_1 \in w_1$. Since $(w_1, \dots, w_n)$ is an approximate crosscut of D, it follows that $p_1 \notin H_2 \cup \dots \cup H_{j_1-1}$. Let H_1^* be the subarc of H_1 from p_1 to H_2. Then $(w_1, \dots, w_{j_1-1})$ approximates $H_1^* \cup H_2 \cup \dots \cup H_{j_1-1}$. The existence of C_1 now follows from Lemma 5.6.

We can similarly show that $\{w_j\}_{j=j_2}^n$ approximates an arc C_3 such that $C_3 \cap (\partial D \cup A_{s_0,\zeta}) = \{C_3(0), C_3(1)\}$ and such that $C_3(0) \in L$. Let C_2 be the subarc of $A_{s_0,\zeta}$ from $C_1(1)$ to $C_3(0)$. Then $C := C_1 \cup C_2 \cup C_3$ is a crosscut that is approximated by $(w_1, \dots, w_n)$. Furthermore, it follows from the conditions of Definition 5.5 that (s_0, r_0) witnesses that C recognizably bounds the value of ϕ on ζ.

Now suppose $C \in \mathcal{C}_{(s_0,r_0,\zeta)}$. Let $\epsilon > 0$. We construct an approximate crosscut in $\mathcal{A}_{(s_0,r_0,\zeta)}$ that approximates C with error less than ϵ. Let C_1, C_3 denote the components of $\overline{C \cap A^+_{s_0,\zeta}}$. Let C_2 denote $C \cap A_{s_0,\zeta}$. Let C_2' be a subarc of C from an intermediate point of C_1 to an intermediate point of C_3. Let A_j be a subarc of C_j that omits C_2' and that contains a boundary point of D. Let $C_j' = \overline{C_j - (C_2' \cup A_j)}$.

Let a_1 be the end point of A_1 that lies on the boundary of D. Let a_2 be the other end point of A_1. Let a_3 be the other end point (besides a_2) of C_1'. Let a_4 be the other end point of C_2'. Let a_5 be the other end point of C_3'. Let a_6 be the other end point of A_3, and let a_7 be the end point of A_3 that lies on the boundary of D.

We now apply Theorem 5.3. Let $(w_2, \dots, w_{k_1})$ be an approximation of C_1' with error smaller than ϵ so that $a_2 \in w_2 - \overline{w_3}$ and $a_3 \in w_{k_1} - \overline{w_{k_1-1}}$. Note that $a_2 \notin \overline{w_4} \cup \dots \cup \overline{w_{k_1}}$ and $a_3 \notin \overline{w_2} \cup \dots \cup \overline{w_{k_1-2}}$. Let $(w_1', \dots, w_m')$ be an approximation of C_3' with error smaller than ϵ so that $a_4 \in w_1' - \overline{w_2'}$ and $a_5 \in w_m' - \overline{w_{m-1}'}$. We can suppose that ϵ is small enough so that $\overline{w_j} \subseteq A^+_{s_0,\zeta}$ for all j and $\overline{w_j'} \subseteq A^+_{s_0,\zeta}$ for all j. Fix a positive number $\delta > 0$. Let $\mathcal{R}_j$ be a finite set of open rational rectangles so that $A_j \subseteq \bigcup \mathcal{R}_j$, $R \cap A_j \neq \emptyset$ for each $R \in \mathcal{R}_j$, and the diameter of each rectangle in $\mathcal{R}_j$ is smaller than δ. We choose δ so that

$$\overline{\bigcup \mathcal{R}_1 \cup \bigcup \mathcal{R}_3} \cap \left(C_2' \cup \overline{\bigcup_j w_{2<j\leqslant k_1-1} \cup \bigcup_{1\leqslant j<m} w_j'} \right) = \emptyset.$$

As in the proof of Theorem 5.3, $\mathcal{R}_j$ contains a chain that covers A_j. Let $w_1 = R_1 \cup \dots \cup R_t$, where $(R_1, \dots, R_t)$ is a chain in $\mathcal{R}_1$ that covers A_1. Let $w_{m+1}' = R_1' \cup \dots \cup R_s'$, where $(R_1', \dots, R_s')$ is a chain in $\mathcal{R}_3$ that covers A_3. So, $(w_1, \dots, w_{k_1})$ is an approximation of $A_1 \cup C_1'$ and $(w_1', \dots, w_{m+1}')$ is an approximation of $C_3' \cup A_3$. Let $(w_{k_1+1}, \dots, w_{k_2})$ be an approximation of C_2'. We can choose this approximation so that the error is small enough so that $(w_1, \dots, w_{k_2}, w_1', \dots, w_m')$ is a simple chain. Let $w_{k_2+j} = w_j'$ when $1 \leqslant j \leqslant m+1$, and let $n = k_2 + m + 1$. It follows that $(w_1, \dots, w_n)$ approximates C. Let $j_1 = k_1 + 1$ and let $j_2 = k_2$.

We can suppose that ϵ is small enough so that if j is not between j_1 and j_2, then $|\phi(t\zeta) - z| > m(s_0, N_0, r_0)$ whenever $1 - s_0 \leqslant t \leqslant 1 - r_0$ and $z \in \overline{w_j}$. We can also suppose that ϵ is small enough so that $\overline{w_j} \subseteq D$ whenever $1 < j \leqslant j_1$ or $j_2 \leqslant j < n$. It follows that $(w_1, \dots, w_n)$ belongs to $\mathcal{A}_{(s_0,r_0,\zeta)}$. □

In order to show that there is an algorithm that enumerates $\mathcal{A}_{(s_0,r_0,\zeta)}$ if ζ and ϕ are computable, we will need the following characterization of $\mathcal{A}_{(s_0,r_0,\zeta)}$. By a *rational polygonal curve* we mean a polygonal curve whose vertices are rational.

LEMMA 5.9. *Suppose that* $(w_1, \dots, w_n)$, j_1, j_2 *satisfy all conditions of Definition* 5.5 *except possibly* (iv) *and* (v). *Then conditions* (iv) *and* (v) *are satisfied if and only if there are rational*

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numbers θ_1, θ_1, open rational rectangles R_1, R_2, and rational polygonal curves P_1, P_2 such that the following hold:

(i) $z_k := \zeta + s_0\zeta e^{i\theta_k} \in \mathbb{D}$;
(ii) *the subarc of $\mathbb{D} \cap \partial D_{s_0}(\zeta)$ from z_1 to z_2 is included in $\phi^{-1}[w_{j_1} \cup \ldots \cup w_{j_2}]$;*
(iii) $R_k \subseteq \phi^{-1}[w_{j_k}]$ *and* $R_k \cap \partial D_{s_0}(\zeta) \neq \emptyset$;
(iv) *one end point of P_1 is in $\phi^{-1}[w_{j_1-1} \cap w_{j_1}]$ and the other is in R_1;*
(v) *one end point of P_2 is in $\phi^{-1}[w_{j_2+1} \cap w_{j_2}]$ and the other is in R_2;*
(vi) $P_k \subseteq D_{s_0}(\zeta) \cap \phi^{-1}[w_{j_k}]$.

Proof. Suppose that conditions (i)–(vi) hold. It follows from Conditions (ii) and (vi) of Definition 5.5 that $1 - s_0$ is between z_1 and z_2 on $\mathbb{D} \cap \partial D_{s_0}(\zeta)$. Let p_1 be the end point of P_1 in $\phi^{-1}[w_{j_1-1} \cap w_{j_1}]$ and let q_1 be the other end point of P_1. Let p_2 be the end point of P_2 in $\phi^{-1}[w_{j_2+1} \cap w_{j_2}]$ and let q_2 be the other end point of P_2. Since $q_k \in D_{s_0}(\zeta)$, $[q_k, z_k] \cap \partial D_{s_0}(\zeta) = \{z_k\}$. Let $G_k = P_k \cup [q_k, z_k]$. Thus, $G_k \cap \partial D_{s_0}(\zeta) = \{z_k\}$. Hence, $\phi[G_k] \cap A_{s_0,\zeta} = \{\phi(z_k)\}$. Let E_k be the component of $\phi(p_k)$ in $w_{j_k} \cap A^+_{s_0,\zeta}$. Since $P_k \subseteq D_{s_0}(\zeta) \cap \phi^{-1}[w_{j_k}]$ and since $R_k \subseteq \phi^{-1}[w_{j_1}]$, it follows that $\phi[G_k] - \{z_k\} \subseteq w_{j_k} \cap A^+_{s_0,\zeta}$. Thus, $\phi(z_k)$ is a boundary point of E_k. Since the subarc of $\mathbb{D} \cap \partial D_{s_0}(\zeta)$ from z_1 to z_2 is contained in $\phi^{-1}[w_{j_1} \cup \ldots \cup w_{j_2}]$, it follows that $\phi(z_k) \in L$. Thus, Conditions (iv) and (v) of Definition 5.5 hold.

Now, suppose that Conditions (iv) and (v) of Definition 5.5 hold. We first show that $L \cap w_{j_k} \cap \partial E_k$ contains a point of the form $\phi(\zeta + s_0\zeta e^{i\theta_k})$, where θ_k is a rational number. Let $q_k \in w_{j_k} \cap L \cap \partial E_k$. Let ϵ be a positive number such that $D_\epsilon(q_k) \subseteq w_{j_k}$ and $D_\epsilon(q_k) \cap A_{s_0,\zeta} \subseteq L$. Let $q'_k \in D_\epsilon(q_k) \cap E_k$. Let E'_k be the component of q'_k in $D_\epsilon(q_k) \cap E_k$. Thus, $E'_k \subseteq E_k$. Let ϵ_1 be a positive number such that $D_{\epsilon_1}(q_k) \cap A_{s_0,\zeta} \subseteq D_\epsilon(q_k)$. By [**21**, Proposition 5.2], $D_{\epsilon_1}(q_k) \cap A_{s_0,\zeta} \subseteq \partial E'_k$. On the other hand, $A_{s_0,\zeta} \cap \partial E'_k \subseteq \partial E_k$. Choose a rational number θ_k so that $\phi(\zeta + s_0\zeta e^{i\theta_k}) \in D_{\epsilon_1}(q_k)$.

Set $z_k = \zeta + s_0\zeta e^{i\theta_k}$. By construction, $\phi(z_k) \in w_{j_k}$. It follows from Conditions (ii) and (vi) of Definition 5.5 that $1 - s_0$ is between z_1 and z_2 on $\mathbb{D} \cap \partial D_{s_0}(\zeta)$. Choose an open rational rectangle R_k so that $\overline{R_k} \subseteq \phi^{-1}[w_{j_k}]$ and $z_k \in R_k$. Since $\phi(z_k) \in \partial E_k$, $z_k \in \partial\phi^{-1}[E_k]$. Thus, R_k contains a point of $\phi^{-1}[E_k]$. Since $R_k \cap \phi^{-1}[E_k]$ is open, it contains a rational point r_k. Since E_1 contains a point of $w_{j_1-1} \cap w_{j_1}$, $\phi^{-1}[E_1]$ contains a point of $\phi^{-1}[w_{j_1-1} \cap w_{j_1}]$. Since this set is open, it contains a rational point p_1. Similarly, $\phi^{-1}[E_2]$ contains a rational point p_2 of $\phi^{-1}[w_{j_2+1} \cap w_{j_2}]$. Since $\phi^{-1}[E_k]$ is open and connected, it contains a rational polygonal curve P_k from p_k to r_k. Hence, $P_k \subseteq D_{s_0}(\zeta) \cap \phi^{-1}[w_{j_k}]$. □

Proof of Theorem 3.7. Suppose that ζ and ϕ are computable. It suffices to exhibit an algorithm that enumerates $\mathcal{A}_{(s_0,r_0,\zeta)}$. Let $(w_1, \ldots, w_n)$ be given as input. By Hertling's effective Riemann mapping theorem (see § 2), D is computably open and its boundary is computably closed. So, there is a search procedure that terminates if and only if $(w_1, \ldots, w_n)$ approximates a crosscut of D. Suppose that this procedure terminates. Fix j_1 and j_2.

We then check that Condition (i) of Definition 5.5 is met. If it is, then we proceed by searching for rational numbers q_1, q_2 so that $\pi/2 < q_1 < \pi < q_2 < 3\pi/2$ and so that $\{w_j\}_{j=j_1}^{j_2}$ approximates the subarc of $A_{s_0,\zeta}$ with end points $\phi(\zeta + s_0\zeta e^{iq_1})$ and $\phi(\zeta + s_0\zeta e^{iq_2})$. Here, we are applying the uniform version of Lemma 5.4. This search terminates if and only if Condition (ii) of Definition 5.5 is met.

Suppose that this search terminates as well. It is well known that if $f : \mathbb{D} \to \mathbb{C}$ is computable and if U is computably open, then $f^{-1}[U]$ is computably open. Furthermore, this result is uniform. It follows that $\phi^{-1}[U]$ is computably open whenever U is a computably open subset of D. The sets w_{j_1}, w_{j_2}, $w_{j_1-1} \cap w_{j_1}$, and $w_{j_2} \cap w_{j_2+1}$ are all computably open. It then follows from Lemma 5.9 that there is a search procedure that terminates if and only if Conditions (iv) and (v) hold.

Suppose that this search terminates. It follows from the effective open mapping theorem (see [**14**]) that $A^+_{s_0,\zeta}$ is computably open. Furthermore, this result is uniform. So, we next search for a finite set of rational rectangles $\mathcal{B}$ so that

$$\bigcup_{1<j<j_1} w_j \cup \bigcup_{j_2<j<n} w_j \subseteq \bigcup \mathcal{B}$$

and so that $\overline{R} \subseteq A^+_{s_0,\zeta}$ whenever $R \in \mathcal{B}$. It follows that this search terminates if and only if Condition (iii) of Definition 5.5 is met. If this search is successful, then we continue by searching for an approximation $(u_1, \ldots, u_s)$ of the arc traced by $\phi(t\zeta)$ as t ranges from $1 - s_0$ to $1 - r_0$ so that

$$d\bigg(\bigcup_j \overline{u_j}, \bigcup_{1\leqslant j\leqslant j_1} \overline{w_j} \cup \bigcup_{j_2\leqslant j\leqslant n} \overline{w_j}\bigg) > m(s_0, N_0, r_0).$$

Here, we are applying the uniform version of Lemma 5.4. It follows that this search is successful if and only if Condition (vi) of Definition 5.5 is met.

If, for some j_1 and j_2, all of these searches terminate, then $(w_1, \ldots, w_n)$ belongs to $\mathcal{A}_{(s_0,r_0,\zeta)}$. Conversely, if $(w_1, \ldots, w_n)$ belongs to $\mathcal{A}_{(s_0,r_0,\zeta)}$, then all of these searches must halt. □

6. *Computability of boundary extensions*

We now prove Theorem 3.8 by means of the following three lemmas. When f is a continuous and complex-valued function on $[0, 1]$, let

$$\|f\|_\infty = \max\{|f(t)| : 0 \leqslant t \leqslant 1\}.$$

LEMMA 6.1. *Let G be a crosscut of $\mathbb{D}$. Suppose that $(w_1, \ldots, w_n)$ approximates $\phi[G]$. Then there is a positive number δ so that $(w_1, \ldots, w_n)$ approximates $\phi[H]$ whenever H is a crosscut of $\mathbb{D}$ such that $\|G - H\|_\infty < \delta$.*

Proof. Let $C = \phi[G]$. Let $C = C_1 + \ldots + C_n$ be a decomposition of C so that $C_j \subseteq w_j$ for all j. Each C_j is compact. So, for each j, there is a positive number ϵ_j so that $z \in w_j$ whenever $|z - p| < \epsilon_j$ for some $p \in C_j$.

Let $G_j = \phi^{-1}[C_j] \cap G$. (It is necessary to take the intersection with G in order to deal with the possibility that one or both end points of C has more than one pre-image.) Then $G = G_1 \cup \ldots \cup G_n$ and each G_n is closed. By compactness, for each j there is a number δ_j so that $|\phi(z_1) - \phi(z_2)| < \epsilon_j$ whenever $z_2 \in G_j$ and $|z_1 - z_2| < \delta_j$. Let δ be the minimum of $\delta_1, \ldots, \delta_n$.

There exist $t_0, \ldots, t_n$ such that $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ and $G_j = G[t_{j-1}, t_j]$. So, $\phi(G(t)) = C_j(t)$ if $t_{j-1} \leqslant t \leqslant t_j$. Suppose $\|H - G\|_\infty < \delta$. Let $H_j = H[t_{j-1}, t_j]$. Then $H = H_1 + \ldots + H_n$. If $t_{j-1} \leqslant t \leqslant t_j$, then $|H(t) - G(t)| < \delta$ and so $|\phi(H(t)) - C_j(t)| < \epsilon_j$. Thus, $\phi[H_j] \subseteq w_j$. In other words, $\phi[H]$ is approximated by $(w_1, \ldots, w_n)$. □

LEMMA 6.2. *Suppose $|\zeta| = 1$ and (s_0, r_0) witnesses that a crosscut C recognizably bounds the value of ϕ on ζ. Suppose that C is approximated by $(w_1, \ldots, w_n)$. Then, whenever ζ' is a unimodular point that is sufficiently close to ζ, $(w_1, \ldots, w_n)$ approximates a crosscut C' such that (s_0, r_0) witnesses that C' recognizably bounds the value of ϕ on ζ'.*

Proof. Let $C_1 = C - \partial D$. Thus, ϕ^{-1} is defined at every point of C_1. Let C^- be the closure of $\phi^{-1}[C_1]$. Hence, $C = \phi[C^-]$. Suppose $|\zeta'| = 1$. When S is a subset of the plane and ξ is a point in the plane, let ξS denote the set of all points of the form ξz such that $z \in S$. Thus,

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because of the structure of C^-, (s_0, r_0) witnesses that $\phi[(\zeta'/\zeta)C^-]$ recognizably bounds the value of ϕ on ζ'. If ζ' is sufficiently close to ζ, then it follows from Lemma 6.1 that $\phi[(\zeta/\zeta')C^-]$ is approximated by $(w_1, \dots, w_n)$. □

LEMMA 6.3. *From $k \in \mathbb{N}$ it is possible to uniformly compute a finite set of open rational rectangles $\mathcal{R}_k$ that covers the unit circle and so that $|\phi(z_1) - \phi(z_2)| < 2^{-k}$ whenever $R \in \mathcal{R}_k$ and $z_1, z_2 \in R \cap \overline{\mathbb{D}}$.*

Proof. Fix k. Compute a positive integer k_0 such that $2^{-(k_0+1)} < |\phi(0) - \phi(\zeta/2)|$ for all unimodular ζ.

For each rational number θ, let $\zeta_\theta = e^{\theta i}$. Thus, the set of all the ζ_θ is dense in the unit circle.

Let $\mathcal{R}'_k$ be the set of all open rational rectangles R for which there exist $s_0, r_0, \theta \in \mathbb{Q}$ and $C \in \mathcal{C}_{s_0, r_0, \zeta_\theta}$ such that $\zeta_\theta \in R \subseteq D_{r_0}(\zeta_\theta)$ and the diameter of C is smaller than $2^{-g(k+k_0+2)}$. It follows from the uniformity of Theorem 3.7 that $\mathcal{R}'_k$ is computably enumerable uniformly in k. It follows from Theorems 3.2 and 3.3 that if $R \in \mathcal{R}'_k$ and if $z_1, z_2 \in R \cap \overline{\mathbb{D}}$, then $|\phi(z_1) - \phi(z_2)| < 2^{-k}$.

We claim that $\mathcal{R}'_k$ covers the unit circle. For, suppose $|\zeta| = 1$. By Theorem 3.4, there is a crosscut C whose diameter is smaller than $2^{-g(k+k_0+2)}$ and that recognizably bounds the value of ϕ on ζ. Let (s_0, r_0) witness that C recognizably bounds the value of ϕ on ζ. Let $(w_1, \dots, w_n)$ be an approximation of C so that the diameter of $\bigcup_j w_j$ is smaller than $2^{-g(k+k_0+2)}$. By Lemma 6.2, there is a closed rational rectangle $R \subseteq D_{r_0}(\zeta)$ such that $\zeta \in R$ and, for all $\zeta' \in R \cap \partial\mathbb{D}$, $(w_1, \dots, w_n)$ approximates a crosscut C' such that (s_0, r_0) witnesses that C' recognizably bounds the value of ϕ on ζ'. The interior of R contains a point of the form ζ_θ for some $\theta \in \mathbb{Q}$. Since R is closed, if ζ_θ is close enough to ζ, then $R \subseteq D_{r_0}(\zeta_\theta)$. Thus, the interior of R belongs to $\mathcal{R}'_k$. Hence, $\partial\mathbb{D} \subseteq \bigcup \mathcal{R}'_k$.

To compute $\mathcal{R}_k$, we enumerate $\mathcal{R}'_k$ just until the unit circle is covered. □

Proof of Theorem 3.8. Let R be given as input. If R contains no point of the unit circle, then do not halt. Otherwise, search for the least k such that $\overline{R} \not\subseteq \bigcup \mathcal{R}_k$. If $k = 0$, then do not halt. Suppose $k > 0$. Then $\overline{R} \subseteq \bigcup \mathcal{R}_{k-1}$. Let $R_1, \dots, R_t$ be all the rectangles in $\mathcal{R}_{k-1}$ that contain a point of $\overline{R}$. For each j, compute a rational point ζ_j in $R_j \cap \mathbb{D}$. Then, for each j, compute a rational point q_j such that $|\phi(\zeta_j) - q_j| < 2^{-k}$. Thus, if $z \in R$, then

$$\phi(z) \in \bigcup_j D_{2^{-k+1}}(q_j).$$

Set

$$\begin{aligned} m_1 &= \min_j \operatorname{Re}(q_j) - 2^{-k+1}, \\ M_1 &= \max_j \operatorname{Re}(q_j) + 2^{-k+1}, \\ m_2 &= \min_j \operatorname{Im}(q_j) - 2^{-k+1}, \\ M_2 &= \max_j \operatorname{Im}(q_j) + 2^{-k+1}. \end{aligned}$$

Then

$$\bigcup_j D_{2^{-k+1}}(q_j) \subseteq (m_1, M_1) \times (m_2, M_2).$$

So, we output $(m_1, M_1) \times (m_2, M_2)$. Thus, the strong correctness criterion of Definition 2.1 is satisfied.

We now verify the convergence criterion. Suppose $z \in R \cap \partial\mathbb{D}$. Set $\delta = \max_j |\phi(\zeta_j) - \phi(z)|$. Let $a = \mathrm{Re}(\phi(z))$ and let $b = \mathrm{Im}(\phi(z))$. Thus,

$$(m_1, M_1) \times (m_2, M_2) \subseteq (a - \delta - 2^{-k+1}, a + \delta + 2^{-k+1}) \times (b - \delta - 2^{-k+1}, b + \delta + 2^{-k+1}).$$

Then $\delta \to 0^+$ as $\mathrm{diam}(R) \to 0^+$. In addition, $k \to \infty$ as $\mathrm{diam}(R) \to 0^+$ (otherwise, ϕ is constant on a neighborhood of z). It follows that the convergence criterion is satisfied. □

7. *Conclusions and questions*

The creation of an algorithm to solve a problem first requires an assessment of the information that must be provided. It was shown in [**20**] that there is a computable conformal map of the unit disk onto a Jordan domain whose boundary extension is incomputable. Thus, the map ϕ by itself does not provide sufficient information for the computation of its boundary extension. We are thus led to consider what additional information must be provided. Here, we have shown that a boundary connectivity function for D provides *sufficient* additional information. In a forthcoming paper [**23**], it is shown that there is a conformal map on the unit disk that has a computable boundary extension even though its range does not have a computable boundary connectivity function. Thus, a boundary connectivity function does not provide *necessary* additional information for the computation of boundary extensions. That is, it provides too much information.

We might then investigate other additional parameters. Since the boundary of D is compact and connected, by the Hahn–Mazurkiewicz theorem (see [**15**, §§ 3–5]) the boundary of D is locally connected if and only if it is the range of a continuous map on the unit interval. Such a map might seem to be a reasonable and perhaps more intuitive additional parameter than a boundary connectivity function. However, it fails to provide sufficient information. For, it is quite easy to show that there is a computable map of the unit interval onto the boundary of the aforementioned example from [**20**]. So, pinning down the precise amount of additional information required to compute boundary extensions is still a question for investigation.

We note that the proof of Theorem 3.9 is uniform in that it produces an algorithm that given as input an algorithm for computing a conformal map ϕ of the unit disk onto a bounded domain D, an algorithm for computing a boundary connectivity function for D, and a rational upper bound on the area of D, produces an algorithm for computing the boundary extension of ϕ. Further uniformity in the format of type-two effectivity [**30**] also holds.

We conclude by proposing two additional and related questions:

(i) What is the complexity of computing $\phi(1)$ from ϕ, g?

(ii) Is there a proof of Pommerenke's theorem in the constructive framework of Bishop?

Acknowledgements. I thank the referee for many insightful and helpful remarks and suggestions. I also thank Elgin Johnston for very helpful comments and my wife Susan for support.

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https://doi.org/10.1112/S1461157014000096 Published online by Cambridge University Press